HANDBOOK OF DIFFERENTIAL EQUATIONS

Stationary Partial Differential Equations

VOLUME 5

Edited by

Michel Chipot

Handbook of Differential Equations

Stationary Partial Differential Equations

Volume V

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STATIONARY PARTIAL DIFFERENTIAL EQUATIONS

Volume V

Edited by

M. CHIPOT
Institute of Mathematics, University of Zürich, Zürich, Switzerland

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Preface

This handbook is the fifth volume in the series devoted to stationary partial differential equations. As the preceding volumes, it is a collection of self-contained, state-of-the-art surveys written by well-known experts in the field.

The topics covered by this volume include in particular semilinear and superlinear elliptic systems, the fibering method for nonlinear variational problems, some nonlinear eigenvalue problems, the studies of the stationary Boltzmann equation and the Gierer–Meinhardt system. I hope that these surveys will be useful for both beginners and experts and help to the diffusion of these recent deep results in mathematical science.

I would like to thank all the contributors for their elegant articles. I also thank Lauren Schultz Yuhasz and Mara Vos-Sarmiento at Elsevier for the excellent editing work of this volume.

M. Chipot

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List of Contributors

de Figueiredo, D.G., *IMECC-Unicamp, C.P. 6065 Campinas, S. Paulo 13081-970, Brazil djairo@ime.unicamp.br* (Ch. 1)

Pohozaev, S.I., *Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina str. 8, 119991 Moscow, Russia* (Ch. 2)

Ruf, B., *Dipartimento de Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano, Italy* (Ch. 3)

Suzuki, T., *Division of Mathematical Science, Department of Systems Innovation, Graduate School of Science, Osaka University, Japan* (Ch. 4)

Takahashi, F., *Department of Mathematics, Faculty of Science, Osaka City University, Japan* (Ch. 4)

Ukai, S., *Department of Mathematics and Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong* (Ch. 5)

Wei, J., *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong* (Ch. 6)

Yang, T., *Department of Mathematics and Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong* (Ch. 5)

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CHAPTER 1

Semilinear Elliptic Systems: Existence, Multiplicity, Symmetry of Solutions

Djairo G. de Figueiredo
IMECC-UNICAMP, C.P. 6065, Campinas, S. Paulo 13081-970, Brazil
E-mail: *djairo@ime.unicamp.br*

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1. Introduction

Semilinear elliptic systems of the type

$$-\Delta u = f(x, u, v), \qquad -\Delta v = g(x, u, v) \quad \text{in } \Omega, \tag{1.1}$$

and the more general one, where the nonlinearities depend also on the gradients, namely

$$-\Delta u = f(x, u, v, \nabla u, \nabla v), \qquad -\Delta v = g(x, u, v, \nabla u, \nabla v) \quad \text{in } \Omega, \tag{1.2}$$

have been object of intensive research recently. In this work we shall discuss some aspects of this research. For the sake of the interested reader we give in Section 10 some references to other topics that are not treated here.

On the above equations u and v are real-valued functions $u, v : \overline{\Omega} \to \mathbb{R}$, where Ω is some domain in R^N, $N \geqslant 3$, and $\overline{\Omega}$ its closure. There is also an extensive literature on the case of $N = 2$, but in the present notes we omit the study of this interesting case; only on Section 4 we make some remarks about this case.

We shall discuss mainly the following questions pertaining to the above systems:

- Existence of solutions for the Dirichlet problem for the above systems, when Ω is some bounded domain in R^N.
- Systems with nonlinearities of arbitrary growth.
- Multiplicity of solutions for problems exhibiting some symmetry.
- Behavior of solutions at ∞ in the case that Ω is the whole of R^N.
- Monotonicity and symmetry of positive solutions.

Although we concentrate in the case of the Laplacian differential operator $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$, many results stated here can be extended to general second order elliptic operators. Of course, there is the problem of Maximum Principles for systems, which poses interesting questions. See some references on Section 10.

The nonlinearity of the problems appears only in the real-valued functions $f, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For that matter, problems involving the p-Laplacian are not studied here. Some references are given in Section 10.

Here we treat only the Dirichlet problem for the above systems. Other boundary value problems like the Neumann and some nonlinear boundary conditions have been also discussed elsewhere, see Section 10.

Some systems of the type (1.1) can be treated by Variational Methods. In Sections 2 and 3 we study two special classes of such systems, the Gradient systems and the Hamiltonian systems. We say that the system (1.1) above is of the *Gradient type* if there exists a function $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$\frac{\partial F}{\partial u} = f, \qquad \frac{\partial F}{\partial v} = g,$$

and it is said to be of the *Hamiltonian type* if there exists a function $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$\frac{\partial H}{\partial v} = f, \qquad \frac{\partial H}{\partial u} = g.$$

In Section 2, associated to Gradient systems we have the functional

$$\Phi(u,v)=\frac{1}{2}\int_\Omega |\nabla u|^2+\frac{1}{2}\int_\Omega |\nabla v|^2-\int_\Omega F(x,u,v), \tag{1.3}$$

which is defined in the Cartesian product $H_0^1(\Omega)\times H_0^1(\Omega)$ provided

$$F(x,u,v)\leqslant |u|^p+|v|^q,\quad \forall x\in\Omega,\ u,v\in\mathbb{R}$$

with $p,q\leqslant\frac{2N}{N-2}$, if the dimension $N\geqslant 3$.

In Section 3, associated to Hamiltonian systems we will first consider the functional

$$\Phi(u,v)=\int_\Omega \nabla u.\nabla v-\int_\Omega H(x,u,v), \tag{1.4}$$

which is defined in the Cartesian product $H_0^1(\Omega)\times H_0^1(\Omega)$ provided again that

$$H(x,u,v)\leqslant |u|^p+|v|^q,\quad \forall x\in\Omega,\ u,v\in\mathbb{R}$$

with $p,q\leqslant\frac{2N}{N-2}$, if the dimension $N\geqslant 3$. However, as we shall see, the restriction on the powers of u and v as above it is too restrictive, in the case of Hamiltonian systems. We shall allow different values of p,q, as observed first in [26] and [84].

In Section 4 we discuss some Hamiltonian systems when one of the nonlinearities may have arbitrary growth following [44].

In Section 5 we present results on the multiplicity of solutions for Hamiltonian systems (1.1) exhibiting some sort of symmetry, we follow [9]. Also in Section 5, following [34], we present a third class of systems which can also be treated by variational methods, namely

$$\begin{aligned}&-\Delta u=H_u(x,u,v)\quad\text{in }\Omega,\qquad -\Delta v=-H_v(x,u,v)\quad\text{in }\Omega,\\ &u(x)=v(x)=0\quad\text{on }\partial\Omega.\end{aligned} \tag{1.5}$$

In this form some supercritical systems can be treated, see [34].

In Section 6 we discuss classes of nonvariational systems, which are then treated by Topological Methods. The difficulty here is obtaining a priori bounds for the solutions. There are several methods to tackle this question. We will comment some of them, including the use of Moving Planes and Hardy type inequalities. However the most successful one in our framework seems to be the blow-up method. Here we follow [42]. This method leads naturally to Liouville-type theorems, that is, theorems asserting that certain systems have no nontrivial solution in the whole space R^N or in a half-space R^N_+.

In Section 7, we present some results on Liouville theorems for systems.

In Section 8, systems defined in the whole of R^N are considered again and the behavior of their solutions is presented.

In Section 9 we discuss symmetry and monotonicity of solutions.

And finally in Section 10 we give references to other topics that are not treated here.

2. Gradient systems

The theory of gradient systems is sort of similar to that of scalar equations

$$-\Delta u = f(x,u) \quad \text{in } \Omega. \tag{2.1}$$

This theory has also been considered by several authors in the framework of p-Laplacians,

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad p > 1.$$

We consider the system of equations

$$-\Delta u = F_u(x,u,v), \qquad -\Delta v = F_v(x,u,v) \tag{2.2}$$

subjected to Dirichlet boundary conditions, that is $u = v = 0$ on $\partial\Omega$. In the context of the Variational Method, here we look for weak solutions, namely solutions in the Sobolev space $H_0^1(\Omega)$. So the variational method consists in looking for such solutions of (2.2) as critical points of the functional

$$\Phi(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(x,u,v), \tag{2.3}$$

whose Euler–Lagrange equations are precisely the weak form of equations (2.2), namely

$$\int_\Omega \nabla u \nabla\phi = \int_\Omega F_u(x,u,v)\phi, \qquad \int_\Omega \nabla v \nabla\phi = \int_\Omega F_v(x,u,v)\phi, \tag{2.4}$$

for all $\phi \in H_0^1(\Omega)$. The functional (2.3) is to be defined in the Cartesian product $E := H_0^1(\Omega) \times H_0^1(\Omega)$. So, due to the Sobolev Embedding Theorem, we require

$$\begin{aligned} & F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{is } C^1 \quad \text{and} \\ \text{(F1)} \quad & \big|F_u(x,u,v)\big| \leqslant C\big(1 + |u|^{2^*-1} + |v|^{2^*-1}\big), \\ & \big|F_v(x,u,v)\big| \leqslant C\big(1 + |v|^{2^*-1} + |u|^{2^*-1}\big), \end{aligned}$$

where $2^* = \frac{2N}{N-2}$, $N \geqslant 3$, and then from the continuous imbedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ it follows that the functional Φ is well defined and of class C^1 in E. In most variational methods some sort of compactness is required, like a Palais–Smale condition (for short, (PS) condition).

DEFINITION. Let X be a Banach space, and $\Phi : X \to \mathbb{R}$ a C^1 functional. We say that Φ satisfies the (*PS*) *condition* if, all sequences (x_n) such that $(\Phi(x_n))$ is bounded and $\Phi'(x_n) \to 0$ contain a convergent subsequence.

In this section we treat only subcritical problems, which means that the powers in the nonlinearity F are strictly less than 2^*. Such a restriction is done viewing some (PS) condition to be obtained later. So we require, due to Sobolev imbeddings, that

$$\text{(F2)} \quad \left|F(x,u,v)\right| \leqslant C\left(1+|u|^r+|v|^s\right),$$

where $0 < r < 2^*$ and $0 < s < 2^*$.

Here, in analogy with the scalar case, a variety of problems have been studied. We single out three noncritical cases, although many other combinations are of interest:

(I) $r, s < 2$ ("sublinear-like"),
(II) $r, s > 2$ ("superlinear-like"),
(III) $r = s = 2$ ("resonant type").

Systems (2.1) satisfying one of the above conditions, as well as other problems, have been discussed in [17,1]. Let us mention three of those results, in order to show the sort of techniques used in this area.

THEOREM 2.1 (The coercive case). *Assume* (F1) *and* (F2) *with r and s as in* (I). *Then Φ achieves a global minimum at some point $(u_0, v_0) \in E$, which is then a weak solution of* (2.1).

This result is an easy consequence of the following theorem on the minimization of coercive weakly lower semi-continuous functionals, see [32,48] for instance.

A THEOREM FROM TOPOLOGY. *Let X be a compact topological space. Let $\Phi : X \to \mathbb{R} \cup +\infty$ be a lower semi-continuous function. Then* (i) *Φ is bounded below, and* (ii) *the infimum of Φ is achieved, i.e., there exists $x_0 \in X$ such that* $\inf_{x\in X} \Phi(x) = \Phi(x_0)$.

Next, if we assume

$$\text{(F3)} \quad F(x,0,0) = F_u(x,0,0) = F_v(x,0,0) = 0, \quad \forall x \in \overline{\Omega},$$

then clearly $u = v = 0$ is a solution of (2.4). And the next result gives conditions for the existence of nontrivial solutions.

THEOREM 2.2 (The coercive case, nontrivial solutions). *Assume* (F1), (F3) *and* (F2) *with r and s as in* (I). *Then Φ achieves a global minimum at a point $(u_0, v_0) \neq (0,0)$, provided that there are positive constants R and $\Theta < 1$, and a continuous function $K : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that*

$$\text{(F4)} \quad F\left(x, t^{\frac{1}{2}}u, t^{\frac{1}{2}}v\right) \geqslant t^{\Theta} K(x,u,v),$$

for $x \in \overline{\Omega}$, $|u|, |v| \leqslant R$ and small $t > 0$.

REMARK. As in Theorem 2.1, Φ achieves its infimum. All we have to do in order to prove Theorem 2.2 is to show that there is a point $(u_1, v_1) \in E$ where $\Phi(u_1, v_1) < 0$. Let φ_1 be a first eigenfunction of the Laplacian, subject to Dirichlet data. The function φ_1 can be taken > 0 in Ω. So we can use $u_1, v_1 = t^{\frac{1}{2}}\varphi_1$, with t positive and small in order to construct (u_1, v_1).

Now let us go to the "superlinear cases." Viewing the need of a Palais–Smale condition we assume, in addition, a condition of the Ambrosetti–Rabinowitz type, namely

(F5) $\quad 0 < F(x, u, v) \leqslant \theta_u u F_u(x, u, v) + \theta_v v F_v(x, u, v),$

for all $x \in \overline{\Omega}$ and $|u|, |v| \geqslant R$, where R is some positive number and

$$\frac{1}{2^*} < \theta_u, \theta_v < \frac{1}{2},$$

THEOREM 2.3. *Assume* (F1), (F3), (F5) *and* (F2) *with r and s as in* (II). *Assume also that there are positive constants C and ε, and numbers $\overline{r}, \overline{s} > 2$ such that*

(F6) $\quad \left|F(x, u, v)\right| \leqslant C\left(|u|^{\overline{r}} + |v|^{\overline{s}}\right),$

for $|u|, |v| \leqslant \varepsilon$, $x \in \overline{\Omega}$. Then Φ has a nontrivial critical point.

The proof goes by an application of the Mountain-Pass Theorem [3,87].

THE MOUNTAIN-PASS THEOREM. *Let X be a Banach space, and $\Phi : X \to \mathbb{R}$ of class C^1 and satisfying the PS condition. Suppose that $\Phi(0) = 0$, and*

(i) *there exists $\rho > 0$ and $\alpha > 0$ such that $\Phi(u) \geqslant \alpha$ for all $u \in X$ with $\|u\| = \rho$,*
(ii) *there exists an $u_1 \in X$ such that $\|u_1\| > \rho$ and $\Phi(u_1) < \alpha$.*

Then Φ has a critical point $u_0 \neq 0$, which is at the level c given by

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma[0,1]} \Phi(u),$$

where $\Gamma := \{\gamma \in C([0, 1], X)$, with $\gamma(0) = 0$, $\gamma(1) = u_1\}$.

The analysis of the resonant case requires the study of some eigenvalue problem for systems, and this can be done even for systems involving p-Laplacians, see [17].

3. Hamiltonian systems

In this section we study elliptic systems of the form

$$-\Delta u = H_v(x, u, v), \qquad -\Delta v = H_u(x, u, v) \quad \text{in } \Omega, \tag{3.1}$$

where $H:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a C^1-function and $\Omega\subset R^N$, $N\geqslant 3$, is a smooth bounded domain.We shall later consider also the case when $\Omega=R^N$, and in this latter case, the system takes the form

$$-\Delta u+u=H_v(x,u,v),\qquad -\Delta v+v=H_u(x,u,v),\tag{3.2}$$

and existence and multiplicity of solutions will be discussed in Section 5.

In the bounded case, we look for solutions of (3.1) subject to Dirichlet boundary conditions, $u=v=0$ on $\partial\Omega$. This has been object of intensive research starting with the work of [26,35,69].

One of the first results on superlinear elliptic Hamiltonian systems appears in [26]. In this work it is discussed the existence of a positive solution for the system below subjected to Dirichlet boundary conditions:

$$-\Delta u=f(v),\qquad -\Delta v=g(u)\quad \text{in }\Omega.\tag{3.3}$$

In this case the Hamiltonian is $H(u,v)=F(v)+G(u)$, where $F(t)=\int_0^t f(s)\,ds$, and similarly G is a primitive of g. However, the treatment given in [26] of system (3.3) was via a Topological argument, using a theorem of Krasnoselskiĭ on Fixed Point Index for compact mappings in cones in Banach spaces, see Krasnoselskiĭ Theorem in Section 6.

The model of a superlinear system as in (3.3) is

$$-\Delta u=|v|^{p-1}v,\qquad -\Delta v=|u|^{q-1}u\quad \text{in }\Omega.\tag{3.4}$$

By analogy with the scalar case one would guess that the subcritical case occurs when $1\leqslant p,q<\frac{N+2}{N-2}$. However, if $p=1$, system (3.4) is equivalent to the biharmonic equation $\Delta^2u=|u|^{q-1}u$, and the Dirichlet problem for the system becomes the Navier problem for the biharmonic, that is $u=\Delta u=0$, on $\partial\Omega$. Since the biharmonic is a fourth order operator the critical exponent is $(N+4)/(N-4)$, which is greater than $(N+2)/(N-2)$. So this raises the suspicion (!) that for systems the notion of criticality should take, very carefully, into consideration the fact that the system is coupled. It appeared in [26] and independently in [84] the notion of the *Critical Hyperbola*, which replaces the notion of the critical exponent of the scalar case:

$$\frac{1}{p+1}+\frac{1}{q+1}=1-\frac{2}{N}.$$

In Figure 1 the higher curve C is the critical hyperbola. The lower parabola is the one obtained by Souto [92], related to the Liouville results, and the two little curves are connected with the work of Busca–Manásevich [18], see Section 7 for an explanation of all these curves.

We call a system (3.4) to be *subcritical* if the powers p,q are the coordinates of a point below the critical hyperbola. And similarly, system (3.3) is subcritical when $f(v)$ grows like v^p as $v\to+\infty$, and $g(u)$ grows like u^q as $u\to+\infty$. A similar definition will be given later for system (3.1).

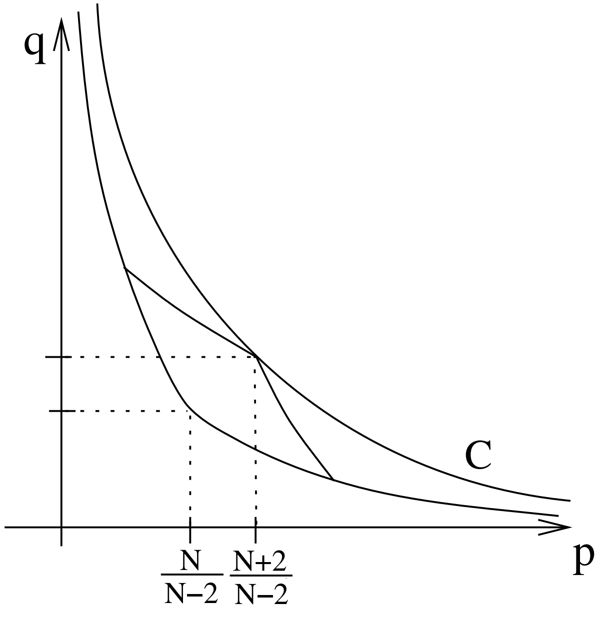

Fig. 1. $\frac{1}{p+1}+\frac{1}{q+1}=1-\frac{2}{N}$.

For $N \geqslant 3$, the "critical hyperbola" plays an important role on the existence of nontrivial solutions. For instance, for the model problem (3.4) with $(p,q) \in \mathbb{R}^2$ on and above this curve, one finds the typical problems of noncompactness, and nonexistence of solutions, as it was proved in [97,26,77], using Pohozaev type arguments.

If the growths of H_u and H_v with respect to u and v as $u, v \to +\infty$ were both less that $(N+2)/(N-2)$ one could consider the functional

$$\Phi(u,v) := \int_\Omega \nabla u \nabla v - \int_\Omega H(x,u,v), \tag{3.5}$$

which is then well defined in the Hilbert space $E = H_0^1 \times H_0^1$. It is easy to see that the critical points of this functional are the weak solutions of system (3.1).

However such a functional is not defined anymore if one of the powers p, q is larger than $(N+2)/(N-2)$. To overcome this difficulty one uses fractional Sobolev spaces. They are defined using Fourier expansions on the eigenfunctions of $(-\Delta, H_0^1(\Omega))$. Let us briefly remind how these spaces can be introduced. It is well known that the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.6}$$

has an increasing sequence of eigenvalues (λ_n) and corresponding eigenfunctions (φ_n), $\varphi_n \in H_0^1(\Omega)$, $\int |\varphi_n|^2 = 1$, with the following properties:

(i) λ_1 is a positive and simple eigenvalue, and $\varphi_1(x) > 0$ for $x \in \Omega$,

(ii) $\lambda_n \to +\infty$,

(iii) $\int \varphi_i \varphi_j = \int \nabla\varphi_i \nabla\varphi_j = 0$, for $i \neq j$.
(iv) $\int |\varphi_i|^2 = 1$, $\int |\nabla\varphi_i|^2 = \lambda_i$, for all i.
It is well known also that (φ_n) is an orthonormal system in $L^2(\Omega)$ and an orthogonal system in $H_0^1(\Omega)$.

Now we define the fractional Sobolev spaces as appropriate subsets of $L^2(\Omega)$:

DEFINITION. For $s \geqslant 0$, we define

$$E^s = \left\{ u = \sum_{n=1}^{\infty} a_n \varphi_n \in L^2(\Omega) \colon \sum_{n=1}^{\infty} \lambda_n^s a_n^2 < \infty \right\}. \tag{3.7}$$

Here $a_n = \int_\Omega u\varphi_n$, $n = 1, 2, \ldots,$ are the Fourier coefficients of u. E^s is a Hilbert space with the inner-product given by

$$\langle u, v\rangle_{E^s} = \sum_{n=1}^{\infty} \lambda_n^s a_n b_n, \quad \text{where } v = \sum_{n=1}^{\infty} b_n \varphi_n. \tag{3.8}$$

Associated with these spaces we have the following maps, which are isometric isomorphisms:

$$\begin{aligned} &A^s : E^s \to L^2, \\ &u = \sum_{n=1}^{\infty} a_n \varphi_n \mapsto A^s u = \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n \varphi_n, \end{aligned} \tag{3.9}$$

$$\int A^s u A^s v = \langle u, v\rangle_{E^s}. \tag{3.10}$$

In this framework, the Sobolev imbedding theorem says that

"$E^s \subset L^{p+1}$ continuously if $\frac{1}{p+1} \geqslant \frac{1}{2} - \frac{s}{N}$, and compactly if the previous inequality is strict."

Instead of the functional (3.5), we have to construct one defined in these fractional Sobolev spaces, which will be chosen depending on the growths of the Hamiltonian. Let us impose the following conditions on the Hamiltonian:

(H.1) $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and $H \geqslant 0$.

(H.2) There exist positive constants p, q and c_1 with

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \quad p, q > 1, \tag{3.11}$$

such that

$$\left|H_u(x, u, v)\right| \leqslant c_1\left(|u|^p + |v|^{\frac{p(q+1)}{p+1}} + 1\right) \tag{3.12}$$

and

$$\left|H_v(x,u,v)\right| \leqslant c_1\left(|v|^q + |u|^{\frac{q(p+1)}{q+1}} + 1\right) \tag{3.13}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

Choose $s, t > 0$, such that $s + t = 2$ and

$$\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}, \qquad \frac{1}{q+1} > \frac{1}{2} - \frac{t}{N}.$$

Thus $E^s \subset L^{p+1}(\Omega)$, and $E^t \subset L^{q+1}(\Omega)$, with compact immersions.

Let now $E = E^s \times E^t$. If $z = (u,v) \in E$, then $H(x,u,v) \in L^1$. So the functional below

$$\Phi(z) = \int_\Omega A^s u A^t v - \int_\Omega H(x,u,v) \tag{3.14}$$

is well defined for $z = (u,v) \in E$ and it is of class C^1. Its derivative is given by the following expression

$$\left\langle \Phi'(z), \eta \right\rangle = \int_\Omega \left(A^s u A^t \psi + A^s \phi A^t v\right) - \int_\Omega (H_u \phi + H_v \psi),$$

where $\eta = (\phi, \psi) \in E$. So the critical points of the functional Φ given by (3.14) are the weak solutions $(u,v) \in E^s \times E^t$ of the system

$$\int_\Omega A^s \phi A^t v = \int_\Omega H_u \phi, \quad \forall \phi \in E^s, \tag{3.15}$$

$$\int_\Omega A^s u A^t \psi = \int_\Omega H_v \psi, \quad \forall \psi \in E^t. \tag{3.16}$$

REMARK 3.1. The following regularity theorem was proved in [35]:

> "these weak solutions (u,v) are indeed $u \in W_0^{1,\frac{p+1}{p}}(\Omega) \cap W^{2,\frac{p+1}{p}}$ and $v \in W_0^{1,\frac{q+1}{q}}(\Omega) \cap W^{2,\frac{q+1}{q}}$, and consequently they are strong solutions of (3.1)."

The following result was also proved in [35]:

THEOREM 3.1. *Assume* (H1), (H2) *with* $p, q > 1$ *satisfying* (3.11). *In addition, assume*:

(H3) *There exists* $R > 0$ *such that*

$$\frac{1}{p+1} H_u(x,u,v)u + \frac{1}{q+1} H_v(x,u,v)v \geqslant H(x,u,v) > 0$$

for all $x \in \overline{\Omega}$ *and* $|(u,v)| \geqslant R$.

(H4) *There exist* $r > 0$ *and* $c > 0$ *such that*

$$\left|H(x, u, v)\right| \leqslant c\left(|u|^{p+1} + |v|^{q+1}\right),$$

for all $x \in \overline{\Omega}$ *and* $|(u, v)| \leqslant r$.
Then, *system* (3.1) *has a strong solution*.

ON THE PROOF OF THEOREM 3.1. The proof consists in obtaining a critical point of the functional (3.14). First we observe that Φ is strongly indefinite. This means that the space E, where the functional Φ is defined, decomposes as $E = E^+ \oplus E^-$, and $E^\pm$ are infinite dimensional subspaces and the quadratic part

$$Q(z) = \int_\Omega A^s u A^t v, \quad \text{for } z = (u, v)$$

is positive definite in E^+ and negative definite in E^-. This fact and (H4) induce a geometry on the functional Φ that calls for the use of a linking theorem of Benci–Rabinowitz [11] in a version due to Felmer [54]. Conditions (H2), and (H3) are used to prove a Palais–Smale condition.

REMARK 3.2. Condition (H4) in the previous theorem excludes cases when H_u and H_v have linear terms. Indeed, on the one hand, the superlinearity condition (3.11) gives $pq > 1$, and, on the other hand, linear terms would imply that (H4) cannot be satisfied with $p = q = 1$. Let us now treat this case.

Suppose now that H has a quadratic part, namely $\frac{1}{2}cu^2 + \frac{1}{2}bv^2 + auv$. In this case the system becomes

$$-\Delta u = au + bv + H_v, \qquad -\Delta v = cu + av + H_u, \tag{3.17}$$

where H satisfies part of the assumptions of the previous theorem. This situation has been studied in special cases by Hulshof–van der Vorst [69] and de Figueiredo–Magalhães [36]. The result we present below extends the previous ones, and it is due to de Figueiredo–Ramos [38].

We replace conditions (H3) and (H4) of the previous theorem by the following ones. In [38] we consider more general conditions.

(H3′) There exist $R > 0$ and a positive constant C such that

$$\frac{1}{2}H_u(x, u, v)u + \frac{1}{2}H_v(x, u, v)v - H(x, u, v) \geqslant C\left(|u|^{p+1}|v|^{q+1}\right)$$

for all $x \in \overline{\Omega}$ and $|(u, v)| \geqslant R$.

(H4′) $$\lim_{|u|+|v|\to 0} \frac{H(x, u, v)}{|u|^2| + |v|^2} = 0, \quad \text{uniformly in } x \in \overline{\Omega}.$$

And finally instead of having that $H(x, u, v) \geqslant 0$, we assume

(H5) there exists $r > 0$ such that either

$$H(x,u,v) \geqslant 0, \quad \forall x \in \overline{\Omega},\ \forall |u| + |v| \leqslant r \quad \text{or} \tag{3.18}$$

$$H(x,u,v) \leqslant 0, \quad \forall x \in \overline{\Omega},\ \forall |u| + |v| \leqslant r \tag{3.19}$$

THEOREM 3.2. *Let a, b, c be real constants and p, q as in* (3.11). *Assume that H satisfies* (H1), (H2), (H3′), (H4′) *and* (H5). *Then system* (3.17) *admits a nonzero strong solution*.

ON THE PROOF OF THEOREM 3.2. The proof relies on a Linking Theorem for strongly indefinite functionals, see [74,75,90]. In order to state this result we need two further concepts.

DEFINITION. We say that a functional $\Phi : E \to \mathbb{R}$ defined in a Banach space E has a *local linking at the origin* if there is a splitting of the space $E = E^+ \oplus E^-$, and a $r > 0$ such that

$$\Phi(z) \leqslant 0, \quad \forall z \in E^-,\ \|z\|_E \leqslant r \quad \text{and} \quad \Phi(z) \geqslant 0, \quad \forall z \in E^+,\ \|z\|_E \leqslant r$$

DEFINITION. Let (E_n^+) be a sequence of finite dimensional subspaces such that

$$E_n^+ \subset E_{n+1}^+ \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} E_n^+} = E^+.$$

Similarly, we have

$$E_n^- \subset E_{n+1}^- \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} E_n^-} = E^-.$$

Let $E_n = E_n^+ \oplus E_n^-$. We say that a C^1 functional $\Phi : E \to \mathbb{R}$ defined in a Banach space E satisfies a (PS)* *condition* if, every sequence (z_n), $z_n \in E_n$, such that

$$\left|\Phi(z_n)\right| \leqslant \text{const}, \quad \text{and} \quad \left|\Phi'(z_n)\eta\right| \leqslant \varepsilon_n \|\eta\|_E,\ \forall \eta \in E_n,\ \varepsilon_n \to 0,$$

contains a convergent subsequence.

Now we state the theorem used to prove Theorem 3.2.

THEOREM 3 (Li–Liu–Willem). *Let $\Phi : E \to \mathbb{R}$ be a C^1 functional defined in a Banach space E which satisfies a* (PS)* *condition and has a local linking at the origin. Assume that Φ maps bounded sets into bounded sets. Suppose further that the following holds*

$$\Phi(z) \to -\infty, \quad \textit{as } \|z\|_E \to \infty,\ z \in E_n^+ \oplus E^-.$$

Then Φ has a nontrivial critical point.

4. Nonlinearities of arbitrary growth

In the previous section we discussed subcritical systems that include the simpler one below

$$\begin{cases} -\Delta u = g(v), & \text{in } \Omega, \\ -\Delta v = f(u), & \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases} \tag{4.1}$$

with

$$f(s) \sim s^q, \quad q > 1, \quad \text{and} \quad g(s) \sim s^p, \quad p > 1 \quad \text{as } s \to +\infty.$$

Existence was proved under the condition (3.11).

We should observe that under the hypothesis $p, q > 1$ we are leaving out a region below the critical hyperbola and still in the first quadrant, that is $p, q > 0$. One may guess that existence of solutions should still persist in this case. This section is devoted to this question, see [44].

- The case $N = 2$.

For $N = 2$ any $(p, q) \in \mathbb{R}^+$ satisfies the inequality (3.11). Actually, a higher growth than polynomial is admitted: by the inequality of Trudinger–Moser, see [96,83], *subcritical growth* for a single equation is given by the condition (see [46])

$$\lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = 0, \quad \forall \alpha > 0.$$

In [46] it is proved that system (4.1) has a nontrivial solution for nonlinearities f and g which have this type of subcritical growth and satisfying an Ambrosetti–Rabinowitz condition. Also existence results for certain nonlinearities with critical growth are given in [46]. Here, we consider a different type of extension of these known results: we show that if one nonlinearity, say g, has polynomial growth (of any order), then, to prove existence of solutions, *no growth restriction* is required on the other nonlinearity f (other than a Ambrosetti–Rabinowitz condition).

- The case $N = 3$.

For $N = 3$ the critical hyperbola has the asymptotes $p_\infty = 2$ and $q_\infty = 2$. In particular, if $g(s) = s^p$ with $1 < p < 2$, then the cited existence results tell us that there exists a solution (u, v) for the system (4.1) with $f(s) = s^q$, for any $q > 1$. Also in this case, existence of solutions can be proved requiring *no growth restriction* on the nonlinearity f (other than a Ambrosetti–Rabinowitz condition).

- The case $N \geqslant 4$.

For $N \geqslant 4$ the asymptotes of the critical hyperbola are the values $p_\infty = \frac{2}{N-2} \leqslant 1$ and $q_\infty = \frac{2}{N-2} \leqslant 1$. Note that for an exponent $p < 1$, the corresponding equation in the system is *sublinear*, i.e. we have a system with one sublinear and one superlinear equation. In this situation, the approach used in previous sections is no longer applicable. However, in this case a reduction of the system to a single equation is possible (see [56]), which allows to prove again a result of the same form; moreover this approach also allows to extend to the whole range the cases $N = 2$ and $N = 3$, that is for $N = 2$: $0 < p < +\infty$, and for $N = 3$: $0 < p < 2$.

We have the following theorems:

THEOREM 4.1. *Suppose that*

(1) $g(s) = s^p$, *with* $\begin{cases} 0 < p, & \text{if } N = 2, \\ 0 < p < \frac{2}{N-2}, & \text{if } N \geqslant 3; \end{cases}$

(2) $f \in C(\mathbb{R})$, *and set* $F(s) = \int_0^s f(t)\,dt$;
 – *there exist constants*

$$\theta > \begin{cases} 2, & \text{if } p \geqslant 1, \\ 1 + \frac{1}{p}, & \text{if } p < 1 \end{cases}$$

and $s_0 \geqslant 0$ *such that* $\theta F(s) \leqslant f(s)s$, $\forall\, |s| \geqslant s_0$;
 – *and for* s *near* 0:

$$f(s) = \begin{cases} o(s), & \text{if } p \geqslant 1, \\ o(s^{1/p}), & \text{if } p < 1. \end{cases}$$

Then the system

$$\begin{cases} -\Delta u = v^p, & \text{in } \Omega, \\ -\Delta v = f(u), & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

has a nontrivial (*strong*) *solution*.

THEOREM 4.2. *Suppose that*

(1) (p, q) *satisfy* $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}$, *and* $\frac{2}{N-2} \leqslant p \leqslant 1$.

(2) $f \in C(\mathbb{R})$, *and there exist constants* $\theta > \frac{p+1}{p}$ *and* $s_0 \geqslant 0$ *such that*

$$\theta F(s) := \theta \int_0^s f(t)\,dt \leqslant f(s)s, \quad \forall |s| \geqslant s_0,$$

and

$$\bigl|f(s)\bigr| \leqslant c|s|^q + d, \quad \text{for some constants } c, d > 0.$$

Then the system

$$\begin{cases} -\Delta u = v^p, & \text{in } \Omega, \\ -\Delta v = f(u), & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

has a nontrivial (*strong*) *solution*.

The proofs are variational and use fractional Sobolev spaces as in Section 3. For the case when $p > 1$ one use the Linking Theorem, Theorem 3 of Section 3. For the case of $p \leqslant 1$ we apply the Mountain-Pass Theorem to the functional

$$I(u) = \frac{p}{p+1} \int_\Omega |\Delta u|^{\frac{p+1}{p}} - \int_\Omega F(u) \geqslant \frac{p}{p+1} \|u\|_C^{\frac{p+1}{p}} - o\big(\|u\|_C^{\frac{p+1}{p}}\big)$$

which is a C^1-functional on the space

$$E = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega).$$

Observe that since $p < \frac{2}{N-2}$, it follows $\frac{p+1}{p} > 1 + \frac{N-2}{2} > \frac{N}{2}$, and thus

$$W^{2,\frac{p+1}{p}}(\Omega) \Subset C(\Omega),$$

which implies that the second term of the functional I is defined if F is continuous, and so no growth restriction on F is necessary!

5. Multiplicity of solutions for elliptic systems

In this section we discuss the multiplicity of solutions for elliptic systems of the form studied previously. Namely

$$\begin{cases} -\Delta u = H_v(x, u, v) & \text{in } \Omega, \\ -\Delta v = H_u(x, u, v) & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset R^N$, $N \geqslant 3$, is a smooth bounded domain and $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function. We shall also consider here the case when $\Omega = R^N$, and in this case the system takes the form

$$\begin{cases} -\Delta u + u = H_v(x, u, v) & \text{in } R^N, \\ -\Delta v + v = H_u(x, u, v) & \text{in } R^N. \end{cases} \tag{5.2}$$

As before we look for solutions of (5.1) subjected to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$. In the case when $\Omega = R^N$ we assume that some symmetry with respect to x holds; for instance, that the x-dependence of H is radial, or that H is invariant with respect to certain subgroups of $O(N)$ acting on R^N. We shall obtain both radial and nonradial solutions in the radial symmetric case, thus observing a symmetry breaking effect. The results presented next are due to Bartsch–de Figueiredo [9]. Related results have been obtained also by Felmer–Wang [57], see also [60,53,51].

Let us start with the case when Ω is bounded. In such a case, the following set of hypotheses is assumed.

(H$_1$) $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and $H \geqslant 0$.

(H$_2$) There exist constants $p, q > 1$ and $c_1 > 0$ with

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N} \tag{5.3}$$

such that

$$\left|H_u(x,u,v)\right| \leqslant c_1\left(|u|^p + |v|^{p(q+1)/p+1} + 1\right), \tag{5.4}$$

$$\left|H_v(x,u,v)\right| \leqslant c_1\left(|v|^q + |u|^{q(p+1)/q+1} + 1\right) \tag{5.5}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

The next condition is the *nonquadraticity condition at infinity* introduced by Costa–Magalhães [29]. It is related to the so-called Ambrosetti–Rabinowitz condition and it is devised to get some sort of Palais–Smale condition for the functionals involved. In fact, here we obtain a condition which is related to the so-called Cerami condition, see condition $(\mathrm{PS})_c^{\mathcal{F}}$ below.

(H$_3$) There exist $1 < \alpha < p+1$ and $1 < \beta < q+1$ with

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \text{such that} \tag{5.6}$$

$$\frac{1}{\alpha} H_u(x,u,v)u + \frac{1}{\beta} H_v(x,u,v)v - H(x,u,v) \geqslant a\left(|u|^\mu + |v|^\nu - 1\right) \tag{5.7}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Here a, μ, ν are positive constants satisfying

$$\mu > \frac{(p+1)N}{2} \max\left\{\frac{1}{2} - \frac{1}{p+1}, 1 - \frac{1}{p+1} - \frac{1}{q+1}\right\}$$

and

$$\nu > \frac{(q+1)N}{2} \max\left\{\frac{1}{2} - \frac{1}{q+1}, 1 - \frac{1}{p+1} - \frac{1}{q+1}\right\}.$$

It follows from condition (H$_3$) that

$$H(x,u,v) \geqslant c\left(|u|^\alpha + |v|^\beta - 1\right). \tag{5.8}$$

The next condition provides the symmetry assumed here.

(H$_4$) $H(x,-u,-v) = H(x,u,v)$ for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

Now we are prepared to state the result in the case of Ω bounded. For that matter we introduce a nonincreasing sequence of constants δ_n, $n \in \mathbb{N}$, with $\delta_n \to 0$, which will be appropriately defined later, and which depend only on p, q, α and β.

THEOREM 5.1. *Suppose that* (H_1)–(H_4) *hold. Then there is a* $k_0 \in \mathbb{N}$ *such that, if*

$$\liminf_{|(u,v)|\to\infty} \frac{2H(x,u,v)}{|u|^\alpha + |v|^\beta} > \frac{1}{\delta_K} \tag{5.9}$$

holds for $K \geqslant k_0$, *system* (5.1), *subjected to Dirichlet boundary conditions, has* $K - k_0 + 1$ *pairs of nontrivial solutions.*

Moreover, if

$$\lim_{|(u,v)|\to\infty} \frac{H(x,u,v)}{|u|^\alpha + |v|^\beta} = +\infty,$$

(*in particular, if* H *is superquadratic*) *then system* (5.1), *subjected to Dirichlet boundary conditions has infinitely many solutions.*

As already observed in Section 3, the solutions obtained in Theorem 5.1 are strong solutions in the sense that $u \in W^{2,p+1/p}(\Omega) \cap W_0^{1,p+1/p}(\Omega)$ and $v \in W^{2,q+1/q}(\Omega) \cap W_0^{1,q+1/q}(\Omega)$.

Let us now state a result for the case of system (5.2) considered in the whole of R^N. We shall need a distinct set of hypotheses. Namely

(H_1') $H: R^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1, $H \geqslant 0$, $H(x,u,v) > 0$ for $|(u,v)| > 0$ and H is radial in the variable x.

(H_2') There exist positive constants p, q, a, b and c_1 with

$$p, q > 1, \qquad \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \qquad 1 < a < p, \qquad 1 < b < q, \tag{5.10}$$

such that

$$\left|H_u(x,u,v)\right| \leqslant c_1\left(|u|^p + |v|^{p(q+1)/p+1} + |u|^a\right) \tag{5.11}$$

and

$$\left|H_v(x,u,v)\right| \leqslant c_1\left(|v|^q + |u|^{q(p+1)/q+1} + |v|^b\right) \tag{5.12}$$

for all $(x,u,v) \in R^N \times \mathbb{R} \times \mathbb{R}$.

(H_3') There exist $1 < \alpha < p+1$ and $1 < \beta < q+1$ with $\alpha^{-1} + \beta^{-1} < 1$ and such that

$$\frac{1}{\alpha} H_u(x,u,v)u + \frac{1}{\beta} H_v(x,u,v)v \geqslant H(x,u,v)$$

for all $(x,u,v) \in R^N \times \mathbb{R} \times \mathbb{R}$.

(H_4') There are positive constants c and r such that

$$H(x,u,v) \geqslant c\big(|u|^{p+1} + |v|^{q+1}\big) \quad \text{for } x \in R^N \text{ and } \big|(u,v)\big| \leqslant r.$$

(H_5') $H(x,u,v) = H(x,-u,-v)$ for all $(x,u,v) \in R^N \times \mathbb{R} \times \mathbb{R}$.

REMARK 5.1. It follows from (H_3') that there are positive constants c and R such that

$$H(x,u,v) \geqslant c\big(|u|^{p+1} + |v|^{q+1}\big) \quad \text{for } \big|(u,v)\big| \geqslant R. \tag{5.13}$$

Then (5.13) and assumption (H_4') imply that

$$H(x,u,v) \geqslant c\big(|u|^{p+1} + |v|^{q+1}\big) \quad \text{for all } (x,u,v) \in R^N \times \mathbb{R} \times \mathbb{R}. \tag{5.14}$$

THEOREM 5.2. *Assume that the Hamiltonian H satisfies the hypotheses* (H_1')–(H_5'). *Then system* (5.2) *has infinitely many radial solutions.*

As in the previous results, the solutions obtained in Theorem 5.2 are strong solutions in the sense that they satisfy $u \in W^{2,p+1/p}_{\mathrm{loc}}(R^N)$ and $v \in W^{2,q+1/q}_{\mathrm{loc}}(R^N)$.

The next result exhibits the breaking of symmetry in certain dimensions. Let us just mention an interesting result coming from [9], which in the scalar case was proved by Bartsch–Willem [10].

THEOREM 5.3. *Suppose that* (H_1')–(H_5') *holds. If $N = 4$ or $N \geqslant 6$ then system* (5.2) *has infinitely many nonradial solutions.*

5.1. *Some abstract critical point theory*

The two previous theorems are proved by Variational Methods, using some abstract critical point theory that we explain next. It is then applied to functionals associated to the systems as in Section 3.

We consider a Hilbert space E and a functional $\Phi \in C^1(E,\mathbb{R})$. Given a sequence $\mathcal{F} = (X_n)$ of finite dimensional subspaces $X_n \subset X_{n+1}$, with $\overline{\bigcup X_n} = E$, we say that Φ satisfies $(\mathrm{PS})_c^{\mathcal{F}}$, at level $c \in \mathbb{R}$, if every sequence (z_j), $j \in \mathbb{N}$, with $z_j \in X_{n_j}$, $n_j \to \infty$, such that

$$\Phi(z_j) \to c \quad \text{and} \quad \big(1 + \|z_j\|\big)(\Phi|_{X_{n_j}})'(z_j) \to 0 \tag{5.15}$$

has a subsequence which converges to a critical point of Φ. In the case when $X_n = E$ for all $n \in \mathbb{N}$ this form of the Palais–Smale condition is due to Cerami [23]. It is closely related to the standard Palais–Smale condition and to the $(\mathrm{PS})^*$ condition that we defined in Section 3. It also yields a deformation lemma. In the present form $(\mathrm{PS})_c^{\mathcal{F}}$ was introduced in Bartsch–Clapp [8].

Now suppose that E splits as a direct sum $E = E^+ \oplus E^-$. Let $E_1^\pm \subset E_2^\pm \subset \cdots$ be a strictly increasing sequence of finite dimensional subspaces of $E^\pm$ such that $\overline{\bigcup_{n=1}^\infty E_n^\pm} = E^\pm$. Setting $E_n = E_n^+ \oplus E_n^-$ we can formulate the hypotheses on Φ which are needed for our first abstract theorem.

(Φ_1) $\Phi \in C^1(E,\mathbb{R})$ and satisfies $(\mathrm{PS})_c^{\mathcal{F}}$ for $\mathcal{F} = (E_n)_{n\in\mathbb{N}}$ and $c > 0$.

(Φ_2) For some $k \geqslant 2$ and some $r > 0$ one has

$$b_k := \inf\bigl\{\Phi(z)\colon\ z \in E^+,\ z \perp E_{k-1},\ \|z\| = r\bigr\} > 0. \tag{5.16}$$

(Φ_3) There exists an isomorphism $T : E \to E$ with $T(E_n) = E_n$, for all $n \in \mathbb{N}$, and there exist $K \geqslant k$ and $R > 0$ such that

$$\text{for} \quad z = z^+ + z^- \in E_K^+ \oplus E^- \quad \text{with} \quad \max\bigl\{\|z^+\|, \|z^-\|\bigr\} = R$$

one has

$$\|Tz\| > r \quad \text{and} \quad \Phi(Tz) \leqslant 0,$$

where k and r are the constants introduced in (Φ_2).

(Φ_4) $\quad d_K := \sup\bigl\{\Phi \circ T(z^+ + z^-)\colon\ z^+ \in E_K^+,\ z^- \in E^-,\ \|z^+\|, \|z^-\| \leqslant R\bigr\} < \infty.$

(Φ_5) Φ is even, i.e. $\Phi(-z) = \Phi(z)$.

A stronger condition that implies (Φ_4) and holds in our application is:

(Φ_6) Φ maps bounded sets to bounded sets.

THEOREM 5.4. *Assume (Φ_1)–(Φ_5). Then, for every $b < b_k$, Φ has at least $K - k + 1$ pairs $\pm z_i$ of critical points with critical values in $[b, d_K]$.*

ON THE PROOF OF THEOREM 5.4. The proof relies heavily on properties of the equivariant limit category defined in [8]. For a complete proof see [9].

As an immediate corollary of Theorem 5.4, we obtain the Fountain Theorem, which we state below. Let us first introduce the following set of conditions.

(Φ_2') There exists a sequence $r_k > 0$, $k \in \mathbb{N}$, such that $b_k \to +\infty$ as $k \to \infty$. (Here b_k is defined as in (Φ_2) with r_k instead of r.)

(Φ_3') There exists a sequence of isomorphisms $T_k : E \to E$, $k \in \mathbb{N}$, with $T_k(E_n) = E_n$ for all k and n, and there exists a sequence $R_k > 0$, $k \in \mathbb{N}$, such that, for $z = z^+ + z^- \in E_k^+ \oplus E^-$ with $\max\{\|z^+\|, \|z^-\|\} = R_k$, one has

$$\|T_k z\| > r_k \quad \text{and} \quad \Phi(T_k z) < 0,$$

where r_k is given in (Φ_2').

(Φ_4') $\quad d_k := \sup\bigl\{\Phi\bigl(T_k(z^+ + z^-)\bigr)\colon\ z^+ \in E_k^+,\ z^- \in E^-,\ \|z^+\|, \|z^-\| \leqslant R_k\bigr\} < \infty.$

THEOREM 5.5 (Fountain theorem). *Suppose that* (Φ_1), (Φ_2')–(Φ_4'), (Φ_5) *hold. Then* Φ *has an unbounded sequence of critical values.*

5.2. *A second class of Hamiltonian systems*

To conclude this section we remark that Hamiltonian systems of the type

$$\begin{aligned} &-\Delta u = H_u(x,u,v) \quad \text{in } \Omega, \qquad -\Delta v = -H_v(x,u,v) \quad \text{in } \Omega, \\ &u(x) = v(x) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.17}$$

where $\Omega \subset R^N$, $N \geqslant 3$, is a smooth bounded domain and $H \in C^1(\overline{\Omega} \times R^2, \mathbb{R})$, can also be treated by variational methods. We assume that the nonlinear term satisfies

$$H(x,u,v) \sim |u|^p + |v|^q + R(x,u,v) \quad \text{with} \quad \lim_{|(u,v)| \to \infty} \frac{R(x,u,v)}{|u|^p + |v|^q} = 0,$$

where $p \in (1, 2^*)$, with $2^* := 2N/(N-2)$, and $q \in (1, \infty)$. So, we see that some supercritical systems are included. General results about such systems can be seen in [34].

6. Nonvariational elliptic systems

In this section we study some systems of the general form (1.1) that do not fall in the categories of the systems studied in the previous sections. That is, they are not variational systems. So we shall treat them by Topological Methods. We will discuss here the existence of positive solutions. The main tool is the following result, due to Krasnoselskiĭ [72], see also [2,12,31,39].

THEOREM 6.1 (Krasnoselskiĭ). *Let* $\mathcal{C}$ *be a cone in a Banach space* X *and* $T : \mathcal{C} \to \mathcal{C}$ *a compact mapping such that* $T(0) = 0$. *Assume that there are real numbers* $0 < r < R$ *and* $t > 0$ *such that*

(i) $x \neq tTx$ *for* $0 \leqslant t \leqslant 1$ *and* $x \in \mathcal{C}$, $\|x\| = r$, *and*

(ii) *there exists a compact mapping* $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ (*where* $B_\rho = \{x \in \mathcal{C}\colon \|x\| < \rho\}$) *such that*

  (a) $H(x, 0) = Tx$ *for* $\|x\| = R$,

  (b) $H(x, t) \neq x$ *for* $\|x\| = R$ *and* $t \geqslant 0$,

  (c) $H(x, t) = x$ *has no solution* $x \in \overline{B}_R$ *for* $t \geqslant t_0$.

Then

$$i_c(T, B_r) = 1, \qquad i_c(T, B_R) = 0, \qquad i_c(T, U) = -1,$$

where $U = \{x \in \mathcal{C}\colon r < \|x\| < R\}$, *and* i_c *denotes the Leray–Schauder index. As a consequence* T *has a fixed point in* U.

When applying this result the main difficulty arises in the verification of condition (b), which is nothing more than an *a priori bound* on the solutions of the system. It is well known that the existence of a priori bounds depends on the growth of the functions f and g as u and v go to infinity. We have seen when treating the variational systems that the nonlinearities were restricted to have polynomial growth. This was a requirement in order to get the associated functional defined, as well as a Palais–Smale condition. Here similar restrictions appear in order to get a priori bounds.

A priori bounds for positive solutions of superlinear elliptic equations (the scalar case), namely

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{6.1}$$

was first considered by Brézis–Turner [21] using an inequality due to Hardy. The same technique was used in [27] to obtain a priori bounds for solutions of systems

$$\begin{aligned} &-\Delta u = f(x,u,v,\nabla u,\nabla v), \\ &-\Delta v = g(x,u,v,\nabla u,\nabla v) \quad \text{in } \Omega, \\ &u = v = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6.2}$$

under the following set of conditions:

(f_1) $f,g:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\times R^N\times R^N\to\mathbb{R}$ is continuous,

(f_2) $\liminf_{t\to\infty}\frac{f(x,s,t,\xi,\eta)}{t}>\lambda_1$ uniformly in $(x,s,t,\xi,\eta)\in\Omega\times\mathbb{R}\times\mathbb{R}\times R^N\times R^N$,

(f_3) there exist $p\geqslant 1$ and $\sigma\geqslant 0$ such that

$$\bigl|f(x,s,t,\xi,\eta)\bigr|\leqslant C\bigl(|t|^p+|s|^{p\sigma}+1\bigr),$$

(g_2) $\liminf_{t\to\infty}\frac{g(x,s,t,\xi,\eta)}{t}>\lambda_1$ uniformly in $(x,s,t,\xi,\eta)\in\Omega\times\mathbb{R}\times\mathbb{R}\times R^N\times R^N$,

(g_3) there exist $q\geqslant 1$ and $\sigma'\geqslant 0$ such that

$$\bigl|g(x,s,t,\xi,\eta)\bigr|\leqslant C\bigl(|s|^q+|t|^{q\sigma'}+1\bigr).$$

In the work [27], instead of the critical hyperbola, two other hyperbolas appeared, due to the limitations coming from the method. This is precisely like in the scalar case in [21] where the exponent $\frac{N+1}{N-1}$ appeared instead of $\frac{N+2}{N-2}$. Observe that the intersection of the two hyperbolas below is the Brézis–Turner exponent $\frac{N+1}{N-1}$:

$$\begin{aligned} &\frac{1}{p+1}+\frac{N-1}{N+1}\frac{1}{q+1}=\frac{N-1}{N+1}, \\ &\frac{N-1}{N+1}\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-1}{N+1}. \end{aligned}$$

THEOREM 6.2. *Let Ω be a smooth bounded domain in R^N, with $N \geqslant 4$. Assume that the conditions f_1, f_2, f_3, g_2, g_3 hold with p, q being the coordinates of a point below both of the above hyperbolas. Suppose that σ, σ' are given by*

$$\sigma = \frac{L}{\max(L, K)}, \qquad \sigma' = \frac{K}{\max(L, K)},$$

where

$$K = \frac{p}{p+1} - \frac{2}{N} > 0, \qquad L = \frac{q}{q+1} - \frac{2}{N} > 0.$$

Then the positive solutions of the system (6.2) *are bounded in L^∞.*

REMARKS ON THE PROOF OF THEOREM 6.2. As said above the proof relies on an inequality of Hardy, namely

$$\left\| \frac{u}{\varphi_1} \right\|_{L^q} \leqslant C \|Du\|_{L^q}, \quad \forall u \in W_0^{1,q}.$$

Here $q > 1$ and as before φ_1 is the eigenfunction associated to the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In [21] they proceed an interpolation of the Hardy inequality ($q = 2$) with Sobolev inequality

$$\|u\|_{2^*} \leqslant C \|Du\|_{L^2}, \quad \forall u \in H_0^1,$$

obtaining the inequality below

$$\left\| \frac{u}{\varphi_1^\tau} \right\|_{L^q} \leqslant C \|Du\|_{L^2}, \quad \forall u \in H_0^1,$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1-\tau}{N}$. For the purpose of proving Theorem 6.2 one needs a Hardy-type inequality which follows from the inequalities above, namely

PROPOSITION 6.1. *Let $r_0 \in (1, \infty]$, $r_1 \in [1, \infty)$ and $u \in L^{r_0}(\Omega) \cap W_0^{1,r_1}$. Then for all $\tau \in [0, 1]$ we have*

$$\frac{u}{\varphi_1^\tau} \in L^r(\Omega), \quad \textit{where } \frac{1}{r} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}.$$

Moreover

$$\left\| \frac{u}{\varphi_1^\tau} \right\|_{L^r} \leqslant C \|u\|_{L^{r_0}}^{1-\tau} \|u\|_{W^{1,r_1}}^\tau,$$

where the constant C depends only on τ, r_0 and r_1.

In [39], moving planes techniques and Pohozaev type identities were used to obtain a priori bounds for positive solutions of the scalar equation (6.1). This method was extended by Clement–de Figueiredo–Mitidieri [26] to Hamiltonian systems of the type (3.3). Although one obtains the right growth for the nonlinear terms, namely $f(s) \sim s^q$, $g(s) \sim s^p$ with any p, q below the critical hyperbola, the method does not generalizes for other second order elliptic operators, and there are restrictions on the type of regions Ω.

Another interesting approach to obtaining a priori bounds can be seen in Quittner–Souplet [82] using weighted Lebesgue spaces.

6.1. *The blow-up method*

The other technique used to obtain a priori bounds for solutions of systems is the *blow-up method*, first used in [64] to treat scalar equations as (6.1). Since there is some symmetry regarding the assumptions on the behavior of the nonlinearities with respect to the unknowns u, v, we change henceforth in this section the notations of these variables, and use u_1, u_2. So, let us consider the system in the form:

$$\begin{cases} -\Delta u_1 = f(x, u_1, u_2) & \text{in } \Omega, \\ -\Delta u_2 = g(x, u_1, u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

where u_1, u_2 are consequently real-valued functions defined on a smooth bounded domain Ω in R^N, $N \geqslant 3$, and f and g are real-valued functions defined in $\overline{\Omega} \times R \times R$.

We then write the system as follows, assuming that the leading parts of f and g involve just pure powers of u_1 and u_2.

$$\begin{cases} -\Delta u_1 = a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + h_1(x, u_1, u_2), \\ -\Delta u_2 = c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + h_2(x, u_1, u_2), \end{cases} \tag{6.4}$$

where the α's are nonnegative real numbers, $a(x), b(x), c(x), d(x)$ are nonnegative continuous functions on $\overline{\Omega}$, and h_1, h_2 are locally bounded functions (the lower order terms) such that

$$\begin{cases} \lim_{|(u_1,u_2)|\to\infty} (a(x)u_1^{\alpha_{11}} + b(x)u_3^{\alpha_{12}})^{-1} |h_1(x, u_1, u_2)| = 0, \\ \lim_{|(u_1,u_2)|\to\infty} (c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}})^{-1} |h_2(x, u_1, u_2)| = 0. \end{cases} \tag{6.5}$$

Now we treat the system (6.4) using the blow-up method. So let us assume, by contradiction, that there exists a sequence $(u_{1,n}, u_{2,n})$ of positive solutions of (6.4) such that at least one of the sequences $u_{1,n}$ and $u_{2,n}$ tends to infinity in the L^∞-norm. Without loss of generality we may suppose that

$$\|u_{1,n}\|_\infty^{\beta_2} \geqslant \|u_{2,n}\|_\infty^{\beta_1},$$

where β_1, β_2 are positive constants to be chosen later. Let $x_n \in \Omega$ be a point where $u_{1,n}$ assumes its maximum: $u_{1,n}(x_n) = \max_{x\in\Omega} u_{1,n}(x)$. Then the sequence $\lambda_n = u_{1,n}(x_n)^{-\frac{1}{\beta_1}}$ is such that $\lambda_n \to 0$. The functions

$$v_{i,n}(x) = \lambda_n^{\beta_i} u_{i,n}(\lambda_n x + x_n),$$

satisfy $v_{1,n}(0) = 1$, $0 \leqslant v_{i,n} \leqslant 1$ in Ω. One also verifies that the functions $v_{1,n}$ and $v_{2,n}$ satisfy

$$\begin{cases} -\Delta v_{1,n} = a(\cdot)\lambda_n^{\beta_1+2-\beta_1\alpha_{11}} v_{1,n}^{\alpha_{11}} + b(\cdot)\lambda_n^{\beta_1+2-\beta_2\alpha_{12}} v_{2,n}^{\alpha_{12}} + \widetilde{h_1}(\cdot), \\ -\Delta v_{2,n} = c(\cdot)\lambda_n^{\beta_2+2-\beta_1\alpha_{21}} v_{1,n}^{\alpha_{21}} + d(\cdot)\lambda_n^{\beta_2+2-\beta_2\alpha_{22}} v_{2,n}^{\alpha_{22}} + \widetilde{h_2}(\cdot), \end{cases} \tag{6.6}$$

in the domain $\Omega_n = \frac{1}{\lambda_n}(\Omega - x_n)$, where the dot stands for $\lambda_n x + x_n$.

The idea of the method is then to pass to the limit as $n \to \infty$ in (6.6) and obtain a system either in R^N or in R_+^N, which can be proved that it has only the trivial solution. This would contradict the fact that the limit of $v_{1,n}$ has value 1 at the origin. By compactness the sequence (x_n) or a subsequence of it converges to a point x_0. We observe that the limiting system is defined in R^N or in R_+^N, accordingly to this limit point (x_0) being a point in Ω or in $\partial\Omega$. In the next proposition we make precise these statements.

PROPOSITION 6.2. *The sequences* $(v_{1,n})$ *and* $(v_{2,n})$ *converge in* $W^{2,p}_{\text{loc}}$, *with* $2 \leqslant p < \infty$ *to functions* $v_1, v_2 \in C^2(G) \cap C^0(\overline{G})$, *satisfying the limiting system of* (6.6) *in* $G = R^N$ *or in* $G = R_+^N$, *provided all the powers of* λ_n *in* (6.6) *are nonnegative. This limiting system is obtained by removing the terms in* (6.6) *where the powers of* λ_n *are strictly positive, the terms where the coefficients vanishes at* x_0, *and the lower order terms.*

In [79] and [41] two special classes of systems were studied, (i) weakly coupled and (ii) strongly coupled. The terminology is explained by the type of system obtained after the passage to the limit. We next analyze these two classes, and later we present more general results obtained recently in [42]. See also [2,8,45,100].

DEFINITION 1. System (6.4) is *weakly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{aligned} &\beta_1 + 2 - \beta_1\alpha_{11} = 0, \qquad \beta_1 + 2 - \beta_2\alpha_{12} > 0, \\ &\beta_2 + 2 - \beta_1\alpha_{21} > 0, \qquad \beta_2 + 2 - \beta_2\alpha_{22} = 0. \end{aligned} \tag{6.7}$$

DEFINITION 2. System (6.4) is *strongly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{aligned} &\beta_1 + 2 - \beta_1\alpha_{11} > 0, \qquad \beta_1 + 2 - \beta_2\alpha_{12} = 0, \\ &\beta_2 + 2 - \beta_1\alpha_{21} = 0, \qquad \beta_2 + 2 - \beta_2\alpha_{22} > 0. \end{aligned} \tag{6.8}$$

REMARK 6.1. It follows that if the system (6.4) is weakly coupled then necessarily we should have

$$\beta_1 = \frac{2}{\alpha_{11} - 1} \quad \text{and} \quad \beta_2 = \frac{2}{\alpha_{22} - 1} \tag{6.9}$$

which requires that $\alpha_{11} > 1, \alpha_{22} > 1$ and

$$\alpha_{12} < \frac{\alpha_{22} - 1}{\alpha_{11} - 1}\alpha_{11} \quad \text{and} \quad \alpha_{21} < \frac{\alpha_{11} - 1}{\alpha_{22} - 1}\alpha_{22}. \tag{6.10}$$

REMARK 6.2. If the system (6.4) is strongly coupled then

$$\beta_1 = \frac{2(\alpha_{12} + 1)}{\alpha_{12}\alpha_{21} - 1} \quad \text{and} \quad \beta_2 = \frac{2(\alpha_{21} + 1)}{\alpha_{12}\alpha_{21} - 1} \tag{6.11}$$

which requires that $\alpha_{12}\alpha_{21} > 1$ and

$$\alpha_{11} < \frac{\alpha_{21} + 1}{\alpha_{12} + 1}\alpha_{12} \quad \text{and} \quad \alpha_{22} < \frac{\alpha_{12} + 1}{\alpha_{21} + 1}\alpha_{21}. \tag{6.12}$$

REMARK 6.3. We observe that the requirements that $\alpha_{11}, \alpha_{22} > 1$ and $\alpha_{12}\alpha_{21} > 1$ are known as super-linearity conditions.

WEAKLY COUPLED SYSTEM. After the blow-up, the limiting system becomes, using a scaling of the solutions v_1, v_2:

$$\begin{aligned} &-\Delta w_1 = w_1^{\alpha_{11}}, \\ &-\Delta w_2 = w_2^{\alpha_{22}} \quad \text{in } R^N, \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} &-\Delta w_1 = w_1^{\alpha_{11}}, \\ &-\Delta w_2 = w_2^{\alpha_{22}} \quad \text{in } R^N_+, \\ &w_1 = w_2 = 0 \quad \text{on } x_N = 0. \end{aligned} \tag{6.14}$$

The existence or not of positive solutions for such systems is the object of the so-called Liouville-type theorems. They will be discussed in the next section. For the time being we anticipate that

(i) the equations in system (6.13) have only the trivial solution if $0 < \alpha_{11}, \alpha_{22} < \frac{N+2}{N-2}$,
(ii) the equations in system (6.14) have only the trivial solution if $1 < \alpha_{11}, \alpha_{22} < \frac{N+1}{N-3}$, if the dimension $N > 3$, see Section 7.

So the following result holds.

THEOREM 6.3. *Let* (6.4) *be a weakly coupled system with continuous coefficients* a, b, c, d, *exponents* $\alpha' s \geqslant 0$, *and such that* $a(x), d(x) \geqslant c_0 > 0$ *for* $x \in \overline{\Omega}$. *Assume also that* $0 < \alpha_{11}, \alpha_{22} < (N+2)/(N-2)$. *Then there is a constant* $C > 0$ *such that*

$$\|u_1\|_{L^\infty}, \|u_2\|_{L^\infty} \leqslant C$$

for all positive solutions $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *of system* (6.4).

STRONGLY COUPLED SYSTEM. As in the case of a weakly coupled system, the limiting systems are

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \qquad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \quad \text{in } R^N \tag{6.15}$$

and

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \qquad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \quad \text{in } \left(R^N\right)^+ \tag{6.16}$$

with

$$w_1 = w_2 = 0 \quad \text{on } x_N = 0.$$

So a contradiction comes if the exponents are such that (6.15) and (6.16) have only the trivial solution $\omega_1 = \omega_2 \equiv 0$. In summary, the following result holds.

THEOREM 6.4. *Let* (6.4) *be a strongly coupled system with continuous coefficients* a, b, c, d, *exponents* $\alpha' s \geqslant 0$, *and such that* $b(x), c(x) \geqslant c_0 > 0$ *for* $x \in \overline{\Omega}$. *Assume that the following conditions hold*:

(L1) *The exponents* α_{12} *and* α_{21} *are such that the only nonnegative solution of*

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \qquad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \quad \textit{in } R^N$$

is $w_1 = \omega_2 \equiv 0$.

(L2) *The only nonnegative solution of*

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \qquad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \quad \textit{in } R^N_+$$

with $\omega_1(x', 0) = \omega_2(x', 0) = 0$ *is* $\omega_1 = \omega_2 \equiv 0$. *Then there is a constant* $C > 0$ *such that*

$$\|u_1\|_{L^\infty}, \|u_2\|_{L^\infty} \leqslant C$$

for all nonnegative solutions (u_1, u_2) *of system* (6.4).

REMARK 6.4. Which conditions should be required on the exponents α_{12} and α_{21} in such a way that (L1) and (L2) holds? Again these are Liouville-type theorems for systems, which will be described in the next section.

6.2. *A more complete analysis of the blow-up process*

Now we proceed to do a more complete analysis of the system (6.4) using the blow-up explained above and considering other types of limiting systems. We follow [42].

By looking at system (6.6), we see that in order to arrive at some system in R^N or R^N_+, by using the blow up method, it is necessary that all the exponents in the λ_n are greater than or equal to 0. When one of these exponents is positive, then the corresponding term vanishes in the limit (as $n \to \infty$), while if the exponent is zero than that term remains after the limiting process. The many possibilities would be better understood by the analysis of the figure below, Figure 2.

In the (β_1, β_2)-plane we denote $\vec{\beta} = (\beta_1, \beta_2) \in R^2_+$, and introduce the lines, whose expressions come from the exponents of λ_n in (6.6),

$$l_1 = \{\vec{\beta} \mid \beta_1 + 2 - \beta_1\alpha_{11} = 0\}, \qquad l_2 = \{\vec{\beta} \mid \beta_2 + 2 - \beta_2\alpha_{22} = 0\},$$
$$l_3 = \{\vec{\beta} \mid \beta_1 + 2 - \beta_2\alpha_{12} = 0\}, \qquad l_4 = \{\vec{\beta} \mid \beta_2 + 2 - \beta_1\alpha_{21} = 0\}.$$

In order to have exponents of the λ_n greater or equal to 0, we have to consider points $(\beta_1, \beta_2) \in R^2_+$, which are to the left of or on l_1, below or on l_2 (note that l_1 and l_2 can be empty, and then they introduce no restriction), below or on l_3, and above or on l_4. Those points are called *admissible*. We divide the systems studied in three classes, which are determined by the exponents $\alpha_{i,j}$:

CASE A. *The intersection of l_1 and l_2 is admissible*. Then we set $(\beta_1, \beta_2) = l_1 \cap l_2$. In this case we shall assume that the functions $a(x)$ and $d(x)$ are bounded below on $\overline{\Omega}$ by a positive constant.

CASE B. *The intersection of l_3 and l_4 is admissible*. Then we set $(\beta_1, \beta_2) = l_3 \cap l_4$. In this case we shall assume that the functions $b(x)$ and $c(x)$ are bounded below on $\overline{\Omega}$ by

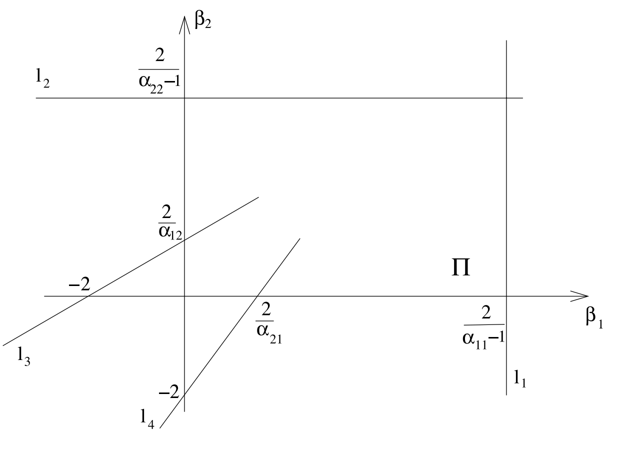

Fig. 2. Admissible couples (β_1, β_2) lie to the left of or on l_1, below or on l_2, below or on l_3, and above or on l_4.

a positive constant. Further, we have to assume that $\alpha_{12}, \alpha_{21} > 1$. (This last requirement comes from a restriction in some Liouville theorems.)

CASE C. *None of* $l_1 \cap l_2$ *and* $l_3 \cap l_4$ *is admissible*. Then either $l_1 \cap l_3$ or $l_2 \cap l_4$ is admissible and we take this intersection point to be our (β_1, β_2). In this case we shall assume that the function $b(x)$ (respectively $c(x)$) is bounded below on $\overline{\Omega}$ by a positive constant.

Now we can state the main result of this section:

THEOREM 6.5. *Assume that system* (6.4) *satisfies the conditions above and that the pair* (β_1, β_2), *which corresponds to the type of the system* (*A, B or C*), *satisfies the condition*

$$\min\{\beta_1, \beta_2\} \geqslant \frac{N-2}{2}. \tag{6.17}$$

Then all positive solutions of the system (5.4) *are bounded in* L^∞.

REMARKS ON THE PROOF OF THEOREM 6.5. The proof in all three cases consists in verifying that the limiting systems have only the trivial, coming then to a contradiction. In the sequel we use G to denote either R^N or R^N_+.

In Case A we choose $(\beta_1, \beta_2) = l_1 \cap l_2$, that is

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \qquad \beta_2 = \frac{2}{\alpha_{22} - 1}.$$

For the limiting systems there are three possibilities:

(a) if none of the lines l_3 and l_4 passes through $l_1 \cap l_2$ we get

$$\begin{aligned} -\Delta w_1 &= w_1^{\alpha_{11}}, \\ -\Delta w_2 &= w_2^{\alpha_{22}} \quad \text{in } G, \end{aligned} \tag{6.18}$$

and as a consequence of hypothesis (6.17) we obtain

$$\max\{\alpha_{11}, \alpha_{22}\} < \frac{N+2}{N-2},$$

which then implies that system (6.18) has only the trivial solution. (This is precisely the weakly coupled case discussed before.)

(b) If exactly one of the lines l_3 and l_4 (say l_3) passes through $l_1 \cap l_2$ we get

$$\begin{aligned} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}}, \\ -\Delta v_2 &= d_0 v_2^{\alpha_{22}} \quad \text{in } G, \end{aligned} \tag{6.19}$$

where $a_0 > 0, b_0 \geqslant 0, d_0 > 0$. So as above system (6.19) has only the trivial solution.

(c) If all four lines meet (this also contains one of the possibilities of Case B) we get

$$-\Delta v_1 = a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}},$$
$$-\Delta v_2 = c_0 v^{\alpha_{21}} + d_0 v_2^{\alpha_{22}} \quad \text{in } G, \tag{6.20}$$

with all four coefficients positive. In order to see that such a system under the assumption (6.17) has only the trivial solution, one requires a new Liouville theorem, which is proved in [42]. As in [18] our proof uses a change to polar coordinates and some monotonicity argument.

In Case B we choose $(\beta_1, \beta_2) = l_3 \cap l_4$, that is,

$$\beta_1 = \frac{2(1+\alpha_{12})}{\alpha_{12}\alpha_{21} - 1}, \qquad \beta_2 = \frac{2(1+\alpha_{21})}{\alpha_{12}\alpha_{21} - 1}.$$

For the limiting systems there are two possibilities:

(a) if none of the lines l_1 and l_2 passes through $l_3 \cap l_4$ we get (after scaling)

$$-\Delta v_1 = v_2^{\alpha_{12}},$$
$$-\Delta v_2 = v_1^{\alpha_{21}} \quad \text{in } G, \tag{6.21}$$

From results of the next section it follows that system (5.21), under the hypothesis (6.17), has only the trivial solution. (This is precisely the strongly coupled case discussed above.)

(b) If one of the lines l_1 and l_2 (say l_1) passes through $l_3 \cap l_4$ we get

$$-\Delta v_1 = a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}},$$
$$-\Delta v_2 = c_0 v_1^{\alpha_{21}} \quad \text{in } G, \tag{6.22}$$

where $b_0, c_0 > 0$, and $a_0 \geqslant 0$, and the results on Section 7 give that $v_1 = v_2 \equiv 0$.

In Case C there are two possibilities: (a) the line l_3 meets l_1 in a point above l_4, (b) the line l_4 meets l_2 in a point below l_3. In both cases (β_1, β_2) is chosen as this point of intersection. In case (a) we take

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \qquad \beta_2 = \frac{2\alpha_{11}}{\alpha_{12}(\alpha_{11} - 1)}.$$

The limiting system is

$$-\Delta v_1 = a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}},$$
$$-\Delta v_2 = 0 \quad \text{in } G, \tag{6.23}$$

which is easily treated by the Liouville results of the next section.

7. Liouville theorems

The classical Liouville theorem from Function Theory says that every bounded entire function is constant. In terms of a differential equation one has: if $(\partial/\partial\overline{z})f(z)=0$ and $|f(z)|\leqslant C$ for all $z\in\mathbb{C}$ then $f(z)=\text{const}$. Hence results with a similar contents are nowadays called Liouville theorems. For instance, a superharmonic function defined in the whole plane R^2, which is bounded below, is constant. Also, all results discussed in this section have this nature. For completeness, we survey also results on a single equation, namely

$$-\Delta u=u^p. \tag{7.1}$$

If the equation is considered in R^2, then a nonnegative solution of (7.1) is necessarily identically zero. The case when R^N, $N\geqslant 3$, is quite distinct. We discuss this case next.

THEOREM 7.1. *Let u be a nonnegative C^2 function defined in the whole of R^N, such that* (7.1) *holds in R^N. If $0<p<(N+2)/(N-2)$, then $u\equiv 0$.*

This result was proved by Gidas–Spruck [65] in the case $1<p<(N+2)/(N-2)$. A simpler proof using the method of moving parallel planes was given by Chen–Li [25], and it is valid in the whole range of p. A very elementary proof valid for $p\in[1,\frac{N}{N-2})$ was given by Souto [92]. In fact, his proof is valid for the case of u being a nonnegative supersolution, i.e.

$$-\Delta u\geqslant u^p \quad \text{in } R^N, \tag{7.2}$$

with p in the same restricted range.

THEOREM 7.2. *Let $u\in C^2(R^N_+)\cap C^0(R^N_+)$ be a nonnegative function such that*

$$\begin{cases} -\Delta u=u^p & \text{in } R^N_+,\\ u(x',0)=0. \end{cases} \tag{7.3}$$

If $1<p\leqslant(N+2)/(N-2)$ then $u\equiv 0$.

REMARK 7.1. This is Theorem 1.3 of [64], plus Remark 2 on page 895 of the same paper. It is remarkable that in the case of the half-space the exponent $(N+2)/(N-2)$ is not the right one for theorems of Liouville type. Indeed, Dancer [30] has proved the following result.

THEOREM 7.3. *Let $u\in C^2(R^N_+)\cap C^0(R^N_+)$ be a nonnegative bounded solution of* (7.3). *If $1<p<(N+1)/(N-3)$ for $N\geqslant 4$ and $1<p<\infty$ for $N=3$, then $u\equiv 0$.*

REMARK 7.2. If $p=(N+2)/(N-2)$, $N\geqslant 3$, then (7.1) has a two-parameter family of bounded positive solutions:

$$U_{\varepsilon,x_0}(x)=\left[\frac{\varepsilon\sqrt{N(N-2)}}{\varepsilon^2+|x-x_0|^2}\right]^{\frac{N-2}{2}},$$

which are called *instantons*.

Next we state some results on supersolutions still in the scalar case.

THEOREM 7.4. *Let $u\in C^2(R^N)$ be a nonnegative supersolution of* (7.2). *If* $1\leqslant p\leqslant\frac{N}{N-2}$, *then* $u\equiv 0$.

This result is proved in Gidas [62] for $1<p\leqslant N/(N-2)$. The case $p=1$ is included in Souto [92]. See [7] and [73] for Liouville theorems for equations defined in cones.

7.1. *Liouville for systems defined in the whole of R^N*

We start considering systems of the form

$$-\Delta u=v^p,\qquad -\Delta v=u^q. \tag{7.4}$$

In analogy with the scalar case just discussed, here the dividing line between existence and nonexistence of positive solutions (u,v) defined in the whole of R^N should be the *critical hyperbola*, [26,84], already introduced. Such hyperbola associated to problems of the form (7.4) is defined by

$$\frac{1}{p+1}+\frac{1}{q+1}=1-\frac{2}{N},\quad p,q>0. \tag{7.5}$$

Continuing the analogy with the scalar case, one may *conjecture* that (7.4) has no bounded positive solutions defined in the whole of R^N if

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2}{N},\quad p,q>0. \tag{7.6}$$

To our knowledge, this conjecture has not been settled in full so far. Why such a conjecture? In answering it, let us remind some facts, already contained in the previous sections. The critical hyperbola appeared in the study of existence of positive solutions for superlinear elliptic systems of the form

$$-\Delta u=g(v),\qquad -\Delta v=f(u) \tag{7.7}$$

subject to Dirichlet boundary conditions in a bounded domain Ω of R^N. If $g(v)\sim v^p$ and $f(u)\sim u^q$ as $u,v\to\infty$, then system (7.7) is said to be subcritical if p,q satisfy (7.6).

For such systems [in analogy with subcritical scalar equations, $-\Delta u = f(u)$, $f(u) \sim u^p$ and $1 < p < (N+2)/(N-2)$] one can establish a priori bounds of positive solutions, prove a Palais–Smale condition and put through an existence theory by a topological or a variational method. This sort of work initiated in [26] and [84] has been continued. We have surveyed some of this work in the previous sections. Recall that the problem in the critical scalar case (that is, $-\Delta u = |u|^{2^*-2}u$ in Ω, $u = 0$ on $\partial\Omega$) has no solution $u \neq 0$ if Ω is a star-shaped bounded domain in R^N, $N \geqslant 3$. In the case of systems, the critical hyperbola appears in the statement: if Ω is a bounded star-shaped domain in R^N, $N \geqslant 3$, the Dirichlet problem for the system below has no nontrivial solution:

$$-\Delta u = |v|^{p-1}v, \qquad -\Delta v = |u|^{q-1}u$$

if, p, q satisfy (7.5). This follows from an identity of Pohozaev-type, see Mitidieri [77]; also Pucci–Serrin [86] for general forms of Pohozaev-type identities.

Next we describe several Liouville-type theorem for systems.

THEOREM 7.5. *Let $p, q > 0$ satisfying* (7.6). *Then system* (7.4) *has no nontrivial radial positive solutions of class $C^2(R^N)$.*

REMARK 7.3. This result settles the conjecture in the class of *radial* functions. It was proved in [77] for $p, q > 1$, and for p, q in the full range by Serrin–Zou [89]. The proof explores the fact that eventual positive radial solutions of (7.4) have a definite decay at ∞; this follows from an interesting observation (cf. Lemma 6.1 in [77]): If $u \in C^2(R^N)$ is a positive radial superharmonic function, then

$$ru'(r) + (N-2)u(r) \geqslant 0, \quad \text{for all } r > 0.$$

Theorem 7.5 is sharp as far as the critical hyperbola is concerned. Indeed, there is the following existence result of Serrin–Zou [89].

THEOREM 7.6. *Suppose that $p, q > 0$ and that*

$$\frac{1}{p+1} + \frac{1}{q+1} \leqslant 1 - \frac{2}{N}. \tag{7.8}$$

Then there exist infinitely many values $\xi = (\xi_1, \xi_2) \in R^+ \times R^+$ such that system (7.4) *admits a positive radial solution (u, v) with central values $u(0) = \xi_1$, $v(0) = \xi_2$. Moreover $u, v \to 0$ as $|x| \to \infty$. So the solution is in fact a ground state for* (7.4).

Let us now mention some results on the nonexistence of positive solutions (or super-solutions) of (7.4), without the assumption of being radial.

THEOREM 7.7. *Let $u, v \in C^2(R^N)$ be nonnegative super-solutions of* (7.4), *that is*

$$-\Delta u \geqslant v^p, \qquad -\Delta v \geqslant u^q \quad \text{in } R^N, \tag{7.9}$$

where $p, q > 0$ *and*

$$\frac{1}{p+1} + \frac{1}{q+1} \geqslant \frac{N-2}{N-1}. \tag{7.10}$$

Then $u = v \equiv 0$.

This result is due to Souto [92]. The idea of his interesting proof is to reduce the problem to a question concerning a scalar equation. Suppose, by contradiction, that u and v are positive solutions of (7.9) in R^N. Introduce a function $\omega = uv$. So

$$\Delta\omega \leqslant 2\nabla u \nabla v - u^{q+1} - v^{p+1}. \tag{7.11}$$

Using the inequality

$$a \cdot b \leqslant \frac{1}{4}|a+b|^2, \quad a, b \in R^N$$

we get that

$$2\nabla u \nabla v \leqslant \frac{1}{2}\omega^{-1}|\nabla\omega|^2.$$

On the other hand, choose $r > 0$ such that $\frac{1}{r} = \frac{1}{p+1} + \frac{1}{q+1}$. Then by Young's inequality

$$\omega^r = u^r v^r \leqslant \frac{r}{q+1}u^{q+1} + \frac{r}{p+1}v^{p+1} \leqslant u^{q+1} + u^{p+1}.$$

So

$$\Delta\omega \leqslant \frac{1}{2}\omega^{-1}|\nabla\omega|^2 - \omega^r. \tag{7.12}$$

Replacing ω by f^2 in (7.12) one obtains

$$-\Delta f \geqslant \frac{1}{2} f^{2r-1} \quad \text{in } R^N,$$

with $f > 0$ in R^N. Using Theorem 7.4, we see that this is a contradiction, since $2r - 1 \leqslant N/(N-2)$. It is of interest to observe that Souto's hyperbola intersects the bisector of the first quadrant precisely at the Serrin exponent $\frac{N}{N-2}$.

In order to state the next results, we assume that $pq > 1$ and introduce the following notations

$$\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}. \tag{7.13}$$

THEOREM 7.8. *Suppose that $p, q > 1$ and*

$$\max\{\alpha, \beta\} \geqslant N - 2. \tag{7.14}$$

Then system (7.9) *has no nontrivial super-solution of class $C^2(R^N)$.*

REMARK. The above result is Corollary 2.1 in [78]. Under hypothesis (6.13), it is proved in [88] that system (7.4) has no nontrivial solution, provided a weaker condition than $p, q > 1$ holds, namely $pq > 1$. In [88] it is also proved that (7.4) has no nontrivial solution in R^N, if $pq \leqslant 1$.

In order to illustrate some useful technique, let us comment briefly the proof given in [78], which uses spherical means. Let $v \in C(R^N)$, then the *spherical mean* of v at x of radius ρ is

$$M(v; x, \rho) = \frac{1}{\operatorname{meas}[\partial B_\rho(x)]} \int_{\partial B_\rho(x)} v(y)\, d\sigma(y).$$

Changing coordinates we obtain

$$M(v; x, \rho) = \frac{1}{\omega_N} \int_{|\nu|=1} v(x + \rho\nu)\, d\omega, \tag{7.15}$$

where ω_N denotes the surface area of the unit sphere of R^N and ν ranges over this unit sphere. Then, one has Darboux formula

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right) M(v; x, \rho) = \Delta_x M(v; x, \rho). \tag{7.16}$$

Now let us use these ideas for the functions u and v in system (7.9):

$$\Delta_x M(u; x, \rho) = \frac{1}{\omega_N} \int_{|\nu|=1} \Delta_x u(x + \rho\nu)\, d\omega \leqslant -\frac{1}{\omega_N} \int_{|\nu|=1} \big[v(x + \rho\nu)\big]^p\, d\omega$$

and using Jensen's inequality we obtain

$$\Delta_x M(u; x, \rho) \leqslant -\big[M(v; x, \rho)\big]^p. \tag{7.17}$$

Using the notation

$$M\big(u(x); x, \rho\big) = u^\#(\rho), \qquad M\big(v(x); x, \rho\big) = v^\#(\rho)$$

we obtain

$$-\Delta_\rho u^\# \geqslant (v^\#)^p, \qquad -\Delta_\rho v^\# \geqslant (u^\#)^q,$$
$$\text{where, } \Delta_\rho = \left(\frac{\partial^2}{\partial\rho^2} + \frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right). \tag{7.18}$$

The proof of Theorem 7.8 will be concluded by the use of the following results, see [78].

PROPOSITION 7.1. *If $v \in C^2(R^N)$, then $M(v; x, \rho)$ is also $C^2(R^N)$ in the variable x and $C^2([0,\infty))$ in the variable ρ. Moreover,*

$$\left(\frac{d}{d\rho} v^{\#}\right)(0) = 0 \quad \textit{and} \quad \left(\frac{d}{d\rho} v^{\#}\right)(\rho) \leqslant 0,$$

that is $v^{\#}(\rho)$ is nonincreasing.

PROPOSITION 7.2. *If $u \in C^2(R^N)$ is a positive radial superharmonic function, then*

$$ru'(r) + (N-2)u(r) \geqslant 0 \quad \textit{for } r > 0. \tag{7.19}$$

PROPOSITION 7.3. *Let $u(\rho)$, $v(\rho)$ be two C^2 functions defined and nonincreasing in $[0,\infty)$, such that $u'(0) = v'(0) = 0$ and*

$$-\Delta_\rho u \geqslant v^p, \qquad -\Delta_\rho v \geqslant u^q. \tag{7.20}$$

Suppose that $p, q > 1$ and that (6.13) holds. Then $u = v \equiv 0$.

The next result extends, as compared with the previous results, the region under the critical hyperbola where the Liouville theorem holds.

THEOREM 7.9.
(A) *If $p > 0$ and $q > 0$ are such that $p, q \leqslant (N+2)/(N-2)$, but not both equal to $(N+2)/(N-2)$, then the only nonnegative solution of (7.4) is $u = v = 0$.*
(B) *If $\alpha = \beta = (N+2)/(N-2)$, then u and v are radially symmetric with respect to some point of R^N.*

This theorem is due to de Figueiredo–Felmer [40]. The proof uses the method of Moving Planes. A good basic reference of this method is [14]. The idea in the proof of the above theorem is to use Kelvin transform in the solutions u, v of (7.4), which a priori have no known (or prescribed) behavior at infinite. By means of Kelvin's u and v are transformed in new unknowns w and z satisfying

$$\begin{aligned} -\Delta w &= \frac{1}{|x|^{N+2-p(N-2)}} z^p(x), \\ -\Delta z &= \frac{1}{|x|^{N+2-q(N-2)}} w^q(x) \end{aligned} \tag{7.21}$$

which now have a definite decay at ∞, provided (p, q) satisfy the conditions of Theorem 7.9. It is precisely at this point that we cannot take $p > \frac{N+2}{N-2}$, because then one would loose the right type of monotonicity of the coefficients necessary to put the moving plane method to work. So having this correct monotonicity of the coefficients the method of moving planes can start. This result has been extended by Felmer [55] to systems with more than two equations.

The next result is due to Busca–Mánasevich [18] and extends further, as compared with Theorem 7.9, the region of values of p, q where the Liouville theorem for system (7.4) holds

THEOREM 7.10. *Suppose that* $p, q > 1$ *and*

$$\min\{\alpha, \beta\} > \frac{N-2}{2}. \tag{7.22}$$

Then system (7.4) *has no nontrivial solution of class* $C^2(R^N)$.

If some behavior of u and v at ∞ is known, the Liouville theorem can be established for all (p, q) below the critical hyperbola, as in the next result.

THEOREM 7.11. *Let* $p > 0$ *and* $q > 0$ *satisfying* (7.6) *then there are no positive solutions of* (7.4) *satisfying*

$$u(x) = o\big(|x|^{-\frac{N}{q+1}}\big), \qquad v(x) = o\big(|x|^{-\frac{N}{p+1}}\big), \quad \text{as } |x| \to \infty. \tag{7.23}$$

The above result is due to Serrin–Zou [89], where the next result is also proved.

THEOREM 7.12. *Let* $N = 3$, *and* $p, q > 0$ *satisfying* (7.6). *Then there are no positive solutions of* (7.4) *for which either* u *or* v *has at most algebraic growth at infinity.*

REMARK 7.4. Observe that Theorem 7.11 extends Theorem 7.5, since radial positive solutions have a decay at infinity.

The proof of Theorem 7.11 is based on an interesting L^2 estimate of the gradient of a superharmonic function, namely,

LEMMA 7.1. *Let* $\omega \in C^2(R^N)$ *be positive, superharmonic* (*i.e.* $-\Delta\omega \geqslant 0$ *in* R^N) *and*

$$\omega(x) = o\big(|x|^{-\gamma}\big) \quad \text{as } |x| \to \infty. \tag{7.24}$$

Then

$$\int_{B_{2R}\setminus B_R} |\nabla\omega|^2 = o\big(R^{N-2-2\gamma}\big) \quad \text{as } R \to \infty, \tag{7.25}$$

where B_R *is the ball of radius* R *in* R^N *centered at the origin.*

Another basic ingredient in the proof of Theorem 7.11 is an identity of Pohozaev-type, a special case of a general identity in [86], namely, LEMMA 7.2. *Let* (u, v) *be a positive solution of* (7.4) *and let* a_1 *and* a_2 *be constants such that* $a_1 + a_2 = N - 2$. *Then*

$$\int_{B_\rho} \left\{ \left(\frac{N}{p+1} - a_1 \right) v^{p+1} + \left(\frac{N}{q+1} - a_2 \right) u^{q+1} \right\}$$
$$= \int_{\partial B} \left\{ \frac{v^{p+1}}{p+1} + \frac{u^{q+1}}{q+1} \right\} + \int_{\partial B} \left(2 \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} - \nabla u \cdot \nabla v \right)$$
$$+ \int_{\partial B} \left(a_1 \frac{\partial u}{\partial r} v + a_2 u \frac{\partial v}{\partial r} \right). \tag{7.26}$$

PROOF OF THEOREM 7.11. Using these two lemmas. Choose a_1 and a_2 in such a way that

$$\frac{N}{p+1} - a_1 = \frac{N}{q+1} - a_2 = \delta, \quad a_1 + a_2 = N - 2.$$

Next, dividing (7.26) by ρ and integrating with respect to ρ between some R and $2R$ and estimating we get

$$\delta \ln 2 \int_{B_R} (u^{q+1} + v^{p+1})$$
$$\leqslant \int_{B_{2R} \setminus B_R} \left(\frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1} \right)$$
$$+ \int_{B_{2R} \setminus B_R} |\nabla u . \nabla v| + R^{-1} \int_{B_{2R} \setminus B_R} \big(v|\nabla u| + u|\nabla v| \big). \tag{7.27}$$

Now using the hypothesis (7.23), we see that the first integral in the right side of (7.27) is $o(1)$. Next one uses Lemma 7.1 with $\omega = u$, $\gamma = \frac{N}{q+1}$ and $\omega = v$, $\gamma = \frac{N}{p+1}$. With that we can estimate the second and third integrals using Cauchy–Schwarz and get that they are $o(R^{N-2-\frac{N}{p+1}-\frac{N}{q+1}})$ which is $o(1)$. This contradicts (7.27) as $R \to \infty$. □

7.2. *Liouville theorems for systems defined in half-spaces*

Now we look at the system below and state some results on the nonexistence of nontrivial solutions and also of supersolutions.

$$\begin{cases} -\Delta u = v^p & \text{in } R^N_+, \\ -\Delta v = u^q & \text{in } R^N_+, \\ u, v \geqslant 0 & \text{in } R^N_+, \\ u, v = 0 & \text{on } \partial R^N_+. \end{cases} \tag{7.28}$$

THEOREM 7.13. *Let $p, q > 1$ satisfying*

$$\max(\alpha, \beta) \geqslant N - 3. \tag{7.29}$$

Then the system (7.28) *has only the trivial solution.*

REMARK 7.5. This result is due to Birindelli–Mitidieri [16], where, instead of a half-space, more general cones are considered. It is also proved there that system (7.28) has no supersolutions if

$$\max(\alpha, \beta) \geqslant N - 1,$$

where α, β are defined in (7.13).

7.3. *A Liouville theorem for a full system*

Now we consider the following system

$$\begin{aligned} -\Delta u_1 &= u_1^{\alpha_{11}} + u_2^{\alpha_{12}}, \\ -\Delta u_2 &= u_1^{\alpha_{21}} + u_2^{\alpha_{22}} \quad \text{in } R^N, \end{aligned} \tag{7.30}$$

In order to state the next result we use the following notation, in analogy with (7.13):

$$\alpha = \frac{2(\alpha_{12}+1)}{\alpha_{12}\alpha_{21}-1}, \qquad \beta = \frac{2(\alpha_{21}+1)}{\alpha_{12}\alpha_{21}-1}.$$

THEOREM 7.14. *System* (7.30) *has only the trivial solution if the following conditions hold*:

$$\alpha_{11}, \alpha_{22} < \frac{N+2}{N-2}, \quad \min\{\alpha, \beta\} > \frac{N-2}{2}. \tag{7.31}$$

This result is due to de Figueiredo–Sirakov [42], and it relies on the following three theorems, which are also proved in [42]. The first one is an extension of a result by Dancer [30], proved for the scalar case. The third one is an extension of a result in [18].We state those results in the special case of system (7.30), although they are valid for more general functions in the right-hand sides of the system.

THEOREM 7.15. *Suppose that u_1, u_2 is a nonnegative bounded classical solution of system* (7.30) *in R_+^N such that $u_1 = u_2 = 0$ on ∂R_+^N. Then*

$$\frac{\partial u_i}{\partial x_N} > 0 \quad \text{in } R_+^N, \text{ for } i = 1, 2.$$

THEOREM 7.16. *Suppose that system* (7.30) *has a nontrivial nonnegative bounded classical solution defined in* R^N_+, *such that* $u_1 = u_2 = 0$ *on* ∂R^N_+. *Then the same problem has a positive solution in* R^{N-1} (*the limit as* $x_N \to \infty$ *in* R^N_+).

THEOREM 7.17. *Let* $u_i(t,\theta)$, $i = 1,2$, *be a* C^2-*function defined in* $\mathbb{R} \times S^{N-1}$ *satisfying*

$$\frac{\partial^2 u_i}{\partial t^2} + \Delta_\theta u_i - \delta_i \frac{\partial u_i}{\partial t} - \nu_i u_i + u_1^{\alpha_{i1}} + u_2^{\alpha_{i2}} = 0,$$

in $\mathbb{R} \times S^{N-1}$, *with* $u_i \to 0$ *as* $t \to -\infty$. *Suppose that* $\delta_i \geqslant 0, \max\{\delta_1, \delta_2\} > 0$, $\nu_i > 0$, $i = 1,2$, *are constants. Assume also that there exists* $t_0 \in \mathbb{R}$ *such that* $\frac{\partial u_i}{\partial t} > 0$ *in* $(-\infty, t_0) \times S^{N-1}$, $i = 1,2$. *Then* $\frac{\partial u_i}{\partial t} > 0$ *in* $\mathbb{R} \times S^{N-1}$, *for* $i = 1,2$.

7.4. *Remarks on the proof of Theorem 7.14*

We follow [42]. Assume first $G = R^N_+$. Then it follows from Theorem 7.16 that if $(u_1,u_2) \neq (0,0)$ then there exists a nontrivial solution of system (7.30) in R^{N-1}. So if we prove that system (7.30) has only the trivial solution in R^N under hypothesis (7.31), then it has no nontrivial solution in R^N_+, under the hypothesis $\min\{\beta_1,\beta_2\} > \frac{N-3}{2}$, which is a consequence of (7.31).

So from now on we suppose $G = R^N$ and distinguish two cases,

$$\max\{\beta_1,\beta_2\} \geqslant N-2 \quad \text{(Case 1)} \quad \text{and} \quad \max\{\beta_1,\beta_2\} < N-2 \quad \text{(Case 2)}.$$

In Case 1 (say $\beta_1 \geqslant N-2$) we have $\alpha_{11} \leqslant \frac{N}{N-2}$. But the first equality in system (7.30) implies $-\Delta u_1 \geqslant a_0 u_1^{\alpha_{11}}$ in R^N, so $u_1 \equiv 0$ in R^N, by the results about nonexistence of supersolutions for scalar equations. Then the second equation in (7.30) becomes $-\Delta u_2 = d_0 u_2^{\alpha_{22}}$ in R^N. So $u_2 \equiv 0$ in R^N, because $\alpha_{22} < \frac{N+2}{N-2}$.

In Case 2 we write system (7.30) in polar coordinates $(r,\theta) \in \mathbb{R} \times S^{N-1}$ and make the change of variables, as in [18],

$$t = \ln|x| \in \mathbb{R}, \qquad \theta = \frac{x}{|x|} \in S^{N-1},$$

and set

$$w_i(t,\theta) = e^{\beta_i t} u_i\big(e^t, \theta\big).$$

Then system (7.30) transforms into

$$\begin{cases} -L_1 w_1 = a_0 e^{(\beta_1 + 2 - \alpha_{11}\beta_1)t} w_1^{\alpha_{11}} + b_0 e^{(\beta_1 + 2 - \alpha_{12}\beta_2)t} w_2^{\alpha_{12}}, \\ -L_2 w_2 = c_0 e^{(\beta_2 + 2 - \alpha_{21}\beta_1)t} w_1^{\alpha_{21}} + d_0 e^{(\beta_2 + 2 - \alpha_{22}\beta_2)t} w_2^{\alpha_{22}}, \end{cases} \tag{7.32}$$

in $\mathbb{R} \times S^{N-1}$, where

$$L_i = \frac{\partial^2}{\partial t^2} + \Delta_\theta - \delta_i \frac{\partial}{\partial t} - \nu_i, \quad i = 1, 2,$$

and

$$\delta_i = 2\beta_i - (N-2), \qquad \nu_i = \beta_i (N - 2 - \beta_i), \quad i = 1, 2.$$

Using a Harnack inequality, see [42], we obtain that u_1 and u_2 are strictly positive in R^N. Then using Theorem 7.17 above we get that $\frac{\partial w_i}{\partial t} > 0$ in $\mathbb{R} \times S^{N-1}$, which gives

$$\beta_i u_i + \frac{\partial u_i}{\partial t} > 0.$$

Finally using an argument of [18] we get

$$\beta_i u_i(x) + \nabla u_i(x).(x - x_0) > 0$$

for all $x, x_0 \in R^N$. This leads to a contradiction.

7.5. *Final remarks on Liouville theorem for systems*

(i) The conjecture on the validity of a Liouville theorem in the whole of R^N for all p and q below the critical hyperbola, and $p, q > 0$, seems to be unsettled at this moment. In dimension $N = 3$, the conjecture has been proved in [88], see Theorem 7.12 above, provided one supposes that u or v has at most algebraic growth.

(ii) Liouville theorems for systems of inequalities in the whole of R^N are given in Theorems 7.7 and 7.8. Is inequality

$$\max\{\alpha, \beta\} \geqslant N - 2$$

in (7.14) sharp? Observe that if $p = q$, (7.14) yields $p \leqslant N/(N-2)$, which is the value obtained in Theorem 7.4.

(iii) Observe that a Liouville theorem for a system of inequalities in R_+^N is stated in the remark right after Theorem 7.13. Compare this result with the following theorem of [92].

THEOREM 7.18. *Let $u, v \in C^2(R_+^N) \cap C^0(\overline{R_+^N})$ be nonnegative solutions of* (7.28) *with $u = v = 0$ on ∂R_+^N. If $1 \leqslant p, q \leqslant \frac{N+2}{N-2}$ then $u = v \equiv 0$.*

(iv) Liouville-type theorems for systems of p-Laplacians have been studied recently by Mitidieri–Pohozaev.

(v) Liouville theorems for equations with a weight have been considered in Berestycki–Capuzzo-Dolcetta–Nirenberg [13].

8. Decay at infinite

In this section we consider solutions of Hamiltonian systems and study their behavior as $|x| \to \infty$. Let us consider the system

$$\begin{cases} -\Delta u + u = g(x, v) & \text{in } R^N, \\ -\Delta v + v = f(x, u) & \text{in } R^N. \end{cases} \tag{8.1}$$

The functions f, g satisfy the following conditions:

(H1) $f, g : R^N \times \mathbb{R} \to \mathbb{R}$ are continuous, with the property that there is an $\varepsilon > 0$ such that

$$tf(x,t) \geqslant 0, \qquad tg(x,t) \geqslant 0, \quad \text{for all } |t| < \varepsilon.$$

(H2) There exists a constant $c_1 > 0$ such that

$$\left|f(x,t)\right| \leqslant c_1\left(|t|^p + 1\right), \qquad \left|g(x,t)\right| \leqslant c_1\left(|t|^p + 1\right) \quad \text{for all } t,$$

where $p, q > 1$ and they are below the critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \quad N \geqslant 3, \tag{8.2}$$

and

$$p, q \leqslant (N+4)/(N-4), \quad \text{if } N \geqslant 5.$$

(H3) There are constants $\alpha, \beta > 2$ and $c_0 > 0$

$$c_0 t^\alpha \leqslant \alpha F(x,t) \leqslant tf(x,t), \qquad c_0 t^\beta \leqslant \beta G(x,t) \leqslant tg(x,t) \quad \text{for all } t.$$

(H4) There are real numbers $a, b \geqslant 1$ and positive constants c_2 and r such that

$$\left|f(x,t)\right| \leqslant c_2|t|^a, \qquad \left|g(x,t)\right| \leqslant c_2|t|^b \quad \text{for } |t| \leqslant r.$$

EXAMPLE. $f(t) = (t^+)^p$ and $g(t) = (t^+)^q$, with p, q as above and $\alpha = p+1$, $\beta = q+1$, $a = p$ and $b = q$.

The following results appear in de Figueiredo–Yang [43]. By a strong solution we mean $u \in W^{2,\frac{p+1}{p}}$ and $v \in W^{2,\frac{q+1}{q}}$.

THEOREM 8.1. *Assume conditions* (H1), (H4) *with* $a = b = 1$ *and* (H2) *with* $p, q < \frac{N+2}{N-2}$. *Then the strong solutions* u, v *of system* (8.1) *are such that*

$$\lim_{|x|\to\infty} u(x) = 0, \qquad \lim_{|x|\to\infty} v(x) = 0,$$

$$\lim_{|x|\to\infty} \left|\nabla u(x)\right| = 0, \qquad \lim_{|x|\to\infty} \left|\nabla v(x)\right| = 0.$$

THEOREM 8.2. *Assume* (H1), (H2) *with* $p, q < (N+2)/(N-2)$ *and* (H4) *with* $a, b > 1$. *Then the strong positive solutions of system* (8.1) *have the following asymptotic behavior*:

$$\lim_{|x|\to\infty} u(x)e^{\theta|x|} = 0, \qquad \lim_{|x|\to\infty} v(x)e^{\theta|x|} = 0$$

for $0 < \theta < 1$.

REMARK. Condition (H3) is used to prove the existence of solutions, a question that we have studied before, even under more general conditions. If the nonlinearities do not depend on x we can prove also the decay of the derivatives of u and v. More general results have been obtained in [91].

8.1. *Remarks on the proof of the above theorems*

A crucial result in the proofs is the following result:

LEMMA 8.1. *Let* u *and* v *be as in Theorem* 8.1. *Then they belong to* L^γ *for* $\gamma \in [2, \infty]$.

Now the proof of Theorem 8.1 goes as follows. For Theorem 8.2 see [43]. Let $B_2 = B_{2R}(x_0)$ be a ball of radius $2R$ centered at x_0. From the assumptions we have

$$\big|g(x,v)\big| \leqslant c\big(|v|^p + |v|\big)$$

which implies

$$\big\|g(x,v)\big\|_{L^\gamma(B_2)} \leqslant C\big(\|v\|^p_{L^{\gamma p}(B_2)} + \|v\|_{L^\gamma(B_2)}\big).$$

Using the above lemma we obtain $\Delta u \in L^\gamma(B_2)$ for all $\gamma \geqslant 2$. By the Calderon–Zygmund inequality (see Theorem 9.9 in [66]) we conclude that $u \in W^{2,\gamma}(B_2)$. Using the interior L^p-estimates we have

$$\|u\|_{W^{2,\gamma}(B_1)} \leqslant C\big(\|u\|_{L^\gamma(B_2)} + \big\|g(x,v)\big\|_{L^\gamma(B_2)}\big).$$

Taking $\gamma > N$ and using the Sobolev imbedding theorem we obtain

$$\|u\|_{C^{1,\lambda}(B_1)} \leqslant C\big(\|u\|_{L^\gamma(B_2)} + \|v\|^p_{L^{\gamma p}(B_2)} + \|v\|_{L^\gamma(B_2)}\big).$$

Letting $|x_0| \to \infty$ we conclude that $\|u\|_{C^{1,\lambda}(B_R(x_0))} \to 0$ and similarly for v.

9. Symmetry properties of the solutions

In [63], Gidas–Ni–Nirenberg proved the symmetry of positive solutions for positive solutions of

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

The question for systems was considered by Troy in [95]. Here we review the results obtained in [33]. Some more general results have been proved in [19,52]. Although the results are valid for general linear second order elliptic operators and systems with more than two equations, we consider the simpler case

$$\begin{cases} -\Delta u_1 = f_1(u_1, u_2), \\ -\Delta u_2 = f_2(u_1, u_2) \quad \text{in } \Omega, \end{cases} \tag{9.1}$$

where Ω is some domain in R^N and we consider nonnegative solutions which vanish on $\partial\Omega$. A crucial hypothesis, which is used in order to apply maximum principles for systems, is the so-called cooperativeness, namely

$$\frac{\partial f_1}{\partial u_2} \geqslant 0, \qquad \frac{\partial f_2}{\partial u_1} \geqslant 0.$$

Fix a direction γ, that is, $\gamma \in R^N$, $|\gamma| = 1$. We suppose that there is $a > -\infty$ such that $\gamma.x > a$, $\forall x \in \Omega$, that is Ω is at one side of some hyperplane normal to γ. For simplicity suppose that $\gamma = (1, 0, \dots, 0)$. Writing $x = (x_1, y)$, let us use the notations

$$T_\lambda = \{x \in R^N\colon\ \gamma.x = \lambda\},$$
$$\Sigma(\lambda) = \{x \in \Omega\colon\ \gamma.x < \lambda\},$$
$$\Sigma'(\lambda) = \big\{x \in R^N\colon\ x^\lambda \in \Sigma(\lambda)\big\},$$

where x^λ is the reflection of x with respect to T_λ, that is, if $x = (x_1, y)$ then $x^\lambda = (2\lambda - x_1, y)$.

Let $\lambda_0 = \sup\{\lambda\colon\ T_\lambda \cap \Omega = \emptyset\}$. Now we make the following assumption on the domain Ω:

$$\exists \varepsilon > 0 \text{ so that for } \lambda_0 < \lambda < \lambda_0 + \varepsilon,\ \ \Sigma'(\lambda) \subset \Omega,\ \text{ and } \Sigma(\lambda) \text{ is bounded.}$$

We remark that this assumption is satisfied if Ω is bounded and $\partial\Omega$ is of class C^2. Now define

$$\overline{\lambda} = \sup\big\{\lambda\colon\ \Sigma'(\mu) \subset \Omega,\ \Sigma(\mu) \text{ bounded } \forall \mu < \lambda\big\}.$$

THEOREM 9.1. *Let* $(u_1, u_2) \in W^{2,N}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega})$ *be a solution of the system* (8.1) *with* $(u_1, u_2) = 0$ *on* $\partial\Omega$. *Then each* u_i *is monotonically increasing with respect to* x_1 *for* $x \in \Sigma(\overline{\lambda})$.

The main tool used in the proof is the technique of parallel moving planes, as in [63]. The proof uses also an interesting Maximum Principle for operators of the form $-\Delta + c(x)$ with no assumption on the sign of $c(x)$; such a condition is replaced by some smallness of the domain, see [14].

10. Some references to other questions

As mentioned in the Introduction there has been recently an ever-increasing interest in systems of nonlinear elliptic equations. Many aspects of this recent research has not been discussed above. For the benefit of the reader we give some references to other questions not considered here. Of course these references are not exhausting, and eventually some important work has been overlooked.

(1) Systems involving p-Laplacians: [6,49,81,58,67].
(2) For singular equations and solutions, see [61,15,99,80,71].
(3) Other boundary value problems, like Neumann, can be seen in [93,22,5], or some nonlinear boundary conditions in [59].
(4) Maximum Principles for systems: [85,37,20].
(5) Ambrosetti–Prodi problems for systems: [76,24,4,47].
(6) Critical Hamiltonian Systems: [70,22,68].

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CHAPTER 2

Nonlinear Variational Problems via the Fibering Method

Stanislav I. Pohozaev

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina str. 8, 119991 Moscow, Russia

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
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Introduction

In this chapter we consider the method of fibering proposed in [42] for investigating nonlinear boundary value problems.

Let X and Y be Banach spaces, and let A be an operator (nonlinear in general) acting from X to Y.

We consider the equation

$$A(u) = h.$$

In the case when the operator is “globally nice”, for example, monotone, demicontinuous, and coercive for $Y = X^*$, methods for investigating such equations have been sufficiently developed and are widely used in various problems in mathematical physics [33]. In the case when A is “locally bad”, for example, not invertible in a neighborhood of some point, the Lyapunov–Schmidt method is used, which is based on the representation of the desired solution in the form $u = u_1 + u_2$, where u_1 is an element in some (as a rule) finite-dimensional subspace of X and u_2 is in the corresponding complement. This approach is applicable in local nonlinear analysis of nonlinear equations and global analysis of equations close to being linear [37].

To investigate a class of noncoercive problems and to find solutions of such problems, wide use has been made of the Ambrosetti–Rabinowitz method, which is based on the “mountain pass” lemma [5], and its generalizations, such as the linking method.

The fibering method presented here has some definite advantages over the known methods. In particular, new nonlinear problems were investigated in [7,39] and [44–47] with the use of the fibering method.

We present the method of fibering Banach spaces for the solution of variational problems.

As an application of the general theorems, we consider boundary value problems for equations of Emden–Fowler type and go through analysis of the application of this method to a linear problem.

We also present a scheme for using the fibering method to derive conditions for the nonexistence of solutions of nonlinear boundary value problems, and we give an example.

The main contents of this survey are based on joint results of the author with Pavel Drábek, Alberto Tesei, and Laurent Véron. We also included results by Yuri Bozhkov and Enzo Mitidieri concerning the application of the fibering method to the problem of multiple solvability for some quasilinear equations and systems.

Note also the works by my student Yavdat Il’yasov [28–31], who contributed to further development of the fibering method and considered its applications to nonlinear problems involving the analysis of some geometrical questions, see for instance [35].

The application of the fibering method to the problem of existence of periodic solutions to certain classes of nonlinear hyperbolic equations can be found in [39,43,48].

Let us mention an interesting fact that the bifurcation equation in the fibering approach generates a Nehari manifold. Thus, the Nehari manifold can be considered as a bifurcation manifold from the fibering point of view.

In conclusion note that the transition from the one-parameter fibering method presented in this chapter to the multiparameter fibering method widens the possibilities of the method [45].

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1. Simple examples

1.1. *Nonlinear Fredholm alternative*

We begin this chapter with an example. Consider the following boundary value problem:

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = h(x) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.1}$$

Here Δ_p is the p-Laplacian:

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)$$

with $p = 1$, $\lambda \in \mathbb{R}$ and $h \in W_{p'}^{-1}(\Omega) \equiv (W_0^{1,p}(\Omega))^*$, $\frac{1}{p} + \frac{1}{p'} = 1$, where Ω is a bounded domain in $\mathbb{R}^N$. Introduce the definition of the spectrum

$$\sigma_p := \big\{\lambda \in \mathbb{R} \;\big|\; \exists u \in W_0^{1,p}(\Omega) \setminus \{0\}\colon\ \Delta_p u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega\big\}.$$

For $p = 2$ we have the classical linear boundary value problem and the classical definition of the spectrum of the Laplace operator. Note that, for λ greater than the first eigenvalue $\lambda_1 > 0$, the appropriate nonlinear operator

$$A_p(u) \equiv \Delta_p u + \lambda |u|^{p-2} u$$

is not coercive. Thus the classical theory of nonlinear monotone coercive operators, , developed by M. Vishik, F. Browder, G. Minty and other mathematicians (see for instance [33]) is not applicable to boundary value problem (1.1.1) with $\lambda > \lambda_1$. In order to overcome this lack in 1967 the “Nonlinear Fredholm Alternative” was developed [40].

Let X be a reflexive Banach space with basis, and denote by X^* the conjugate space. Let A and T be operators acting from X into X^*. Consider the abstract nonlinear equation

$$A(u) + \lambda T(u) = h$$

with a scalar parameter λ. The following assumptions are made:

(A1) A and T are odd positive homogeneous (in principal part) continuous operators;

(A2) A is a strictly closed operator;

(A3) T is a compact operator.

Define the spectrum of the *pair* (A, T) as

$$\sigma(A, T) := \left\{\lambda \in \mathbb{R} \,\middle|\, \exists v \neq 0\colon\ A(v) + \lambda T(v) = 0\right\}.$$

Then we have the following statement.

THEOREM 1.1.1 (Nonlinear Fredholm Alternative). *Let A and T satisfy assumptions* (A1)–(A3). *Then the equation*

$$A(u) + \lambda T(u) = h$$

for any $h \in X^*$ admits a solution $u \in X$ if

$$\lambda \notin \sigma(A, T).$$

EXAMPLE 1.1.2. If we apply this general abstract result to boundary value problem (1.1.1), we obtain that for any $\lambda \notin \sigma(\Delta_p)$ there exists $u \in W_0^{1,p}(\Omega)$ which is a solution of (1.1.1).

REMARK 1.1.3. If $p = 2$ then Δ_p reduces to the Laplacian; in this case the Nonlinear Fredholm Alternative gives the same result as the classical (linear) Fredholm Alternative.

EXAMPLE 1.1.4. Consider the problem

$$\begin{cases} A(u) = h(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$A(u) \equiv -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^3 + c\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^3 + u^3 + a\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 u.$$

Then the Nonlinear Fredholm Alternative implies that for any $a, c \in \mathbb{R}$ there exists a solution $u \in W_0^{1,4}(\Omega)$.

REMARK 1.1.5. Note that the operator A is not coercive for a suitable choice of a and c. We can take for instance $c = 0$, and $a < 0$ sufficiently large such that for a fixed $\phi \in W_0^{1,4}(\Omega)$:

$$-a\int_\Omega |\nabla\phi|^2\phi^2 > \int_\Omega |\nabla\phi|^4 + \int_\Omega \phi^4;$$

in this case we have $\langle A(t\phi), t\phi\rangle \to -\infty$ as $t \to +\infty$; thus A is not coercive, and then the classical theory of nonlinear monotone coercive operators cannot be applied.

In order to develop this approach to noncoercive nonlinear equations we used some elements of the fibering method. Now we give a short description of the main underlying ideas.

(1) The first idea is the *extension* of the nonlinear operator: that is, instead of equation

$$A(u) = h, \tag{1.1.2}$$

where A is acting between Banach spaces X and Y, we consider wider spaces $\tilde{X}$ and $\tilde{Y}$ and

$$\tilde{A} : \tilde{X} \to \tilde{Y} \quad \text{with } \tilde{A}|_X = A;$$

then we get the extended nonlinear equation

$$\tilde{A}(\tilde{u}) = \tilde{h}.$$

(2) The second idea is equipping $\tilde{X}$ with *nonlinear* structure associated with the nonlinear operator A.

A further development of this approach consists in constructing, for a given triple (X, A, Y), a corresponding triple (ξ, α, η), where ξ and η are fibrations of the spaces X and Y respectively, and α is a morphism of ξ into η; the correspondence

$$(X, A, Y) \mapsto (\xi, \alpha, \eta)$$

is determined by the initial triple and, for given spaces X and Y, by the operator A from X to Y. If we take

$$\tilde{X} = \mathbb{R}^k \times X,$$

we obtain the k-parametric fibering method. We begin with the simplest case, namely when $k = 1$: in this case we get the so-called one-parametric fibering method. The one-parametric fibering method is based on representation of solutions for equation (1.1.2) in the form

$$u = tv, \tag{1.1.3}$$

where t is a real parameter ($t \neq 0$ in some open set $J \subset \mathbb{R}$), and v is a nonzero element of the Banach space X satisfying the *fibering constraint*

$$H(t, v) = c. \tag{1.1.4}$$

Roughly speaking, any functional satisfying a sufficiently general condition (see Subsection 1.2) can be taken as the "fibering functional" $H(t, v)$. In particular, we can take the norm $H(t, v) \equiv \|v\|$; then condition (1.1.4) reduces to $\|v\| = 1$, realizing a so-called spherical fibering. Here a solution $u \neq 0$ of (1.1.2) is sought in the form (1.1.3), where $t \in \mathbb{R}$ and $v \in S = \{w \in X\colon \ \|w\| = 1\}$.

Thus, the essence of the one-parametric fibering method consists in embedding space X of the original problem (1.1.2) into the larger space $\tilde{X} = \mathbb{R} \times X$ and investigating the new problem of conditional solvability under condition (1.1.4). This method makes it possible to get both new solvability theorems and new theorems on the absence of solutions for nonlinear boundary value problems. Further, in the investigation of solvability of boundary value problems this method makes it possible to separate the algebraic factors of the problem that affect the number of solutions from the topological ones.

1.2. *The one-parameter fibering method*

Let X be a real Banach space with a norm $\|w\|_X$ that is differentiable for $w \neq 0$, and let f be a functional on X of class $C^1(X \setminus \{0\})$. We associate with f a functional $\tilde{f}$ defined on $\mathbb{R} \times X$ by

$$\tilde{f}(t, v) = f(tv), \tag{1.2.1}$$

where $(t, v) \in J \times S$; here J is an arbitrary nonempty set in $\mathbb{R}$, and S is the unit sphere in X.

THEOREM 1.2.1. *Let X be a real Banach space with a norm differentiable on $X \setminus \{0\}$, and let $(t, v) \in (J \setminus \{0\}) \times S$ be a conditionally critical point of the functional $\tilde{f}$ considered on $J \times S$. Then the vector $u = tv$ is a stationary* (*critical*) *point of the functional f, that is, $f'(u) = 0$.*

PROOF. At the conditionally critical point (t, v) we have

$$\lambda \tilde{f}'_v(t, v) = \mu \|v\|', \tag{1.2.2}$$

$$\tilde{f}'_t(t, v) = 0 \tag{1.2.3}$$

with $\lambda^2 + \mu^2 \neq 0$. Here the prime and the subscript mean the derivative with respect to the corresponding variable (the derivative with respect to v is understood as the value of the derivative with respect to v in the space X for $v \in S$). By (1.2.2), we have

$$\lambda \big\langle \tilde{f}'_v(t, v), v \big\rangle = \mu \big\langle \|v\|', v \big\rangle,$$

where $\langle w^*, u \rangle$ is the value of the functional w^* in the dual space X^* on an element u in X. Then from this and the equalities

$$\big\langle \tilde{f}'_v(t, v), v \big\rangle = \big\langle f'(tv), v \big\rangle = t \tilde{f}'_t(t, v)$$

and

$$\big\langle \|v\|', v \big\rangle = 1 \quad \text{for } v \in S$$

we get $\lambda t \tilde{f}_t'(t, v) = \mu$; from this equality and (1.2.3) follows that $\mu = 0$. Then $\lambda \neq 0$ and by (1.2.2)

$$t f'(u) = \tilde{f}_v'(t, v) = 0$$

for $u = tv$, $t \neq 0$. Consequently, $f'(u) = 0$ and the theorem is proven. □

Now we consider a more general fibering: for this we introduce a fibering functional $H(t, v)$ defined on $\mathbb{R} \times X$ and we consider the functional $\tilde{f}(t, v)$ under condition (1.1.4). In general, we can take as $H(t, v)$ an arbitrary functional that is differentiable under this condition and satisfies

$$\langle H_v', v\rangle \neq t H_t' \quad \text{for } H(t, v) = c; \tag{1.2.4}$$

we will call (1.2.4) the *nondegeneracy condition*.

THEOREM 1.2.2. *Let H be a functional from the described class. Let $(t, v) \in J \times X$ with $tv \neq 0$ be a conditionally critical point of the functional $\tilde{f}(t, v)$ under condition* (1.1.4). *Then the vector $u = tv$ is a nonzero critical point of the original functional f, i.e. $f'(u) = 0$.*

PROOF. At the conditionally critical point (t, v) we have

$$\mu \tilde{f}_v'(t, v) = \lambda H_v'(t, v), \qquad \mu \tilde{f}_t'(t, v) = \lambda H_t'(t, v) \tag{1.2.5}$$

with $\lambda^2 + \mu^2 \neq 0$. On the other hand,

$$\tilde{f}_v'(t, v) = t f'(tv), \qquad \tilde{f}_t'(t, v) = \langle f'(tv), v\rangle.$$

Then from (1.2.5) we get

$$\mu t f'(tv) = \lambda H_v'(t, v), \qquad \mu \langle f'(tv), v\rangle = \lambda H_t'(t, v). \tag{1.2.6}$$

From this we obtain

$$\mu t \langle f'(tv), v\rangle = \lambda \langle H_v'(t, v), v\rangle,$$

$$\mu t \langle f'(tv), v\rangle = \lambda t H_t'(t, v)$$

and consequently

$$\lambda \langle H_v'(t, v), v\rangle = \lambda t H_t'(t, v)$$

for $t \neq 0$ and $H(t, v) = c$. Then, by condition (1.2.4), we get $\lambda = 0$ and hence $\mu \neq 0$. As a result, the first equation in (1.2.6) takes the form $f'(u) = 0$ with $u = tv \neq 0$. □

1.3. *Comparison with the Lyapunov–Schmidt approach*

Now we compare the fibering method with the classical Lyapunov–Schmidt one. We restrict our comparison to spherical fibering: in this case

$$H(t, v) \equiv \|v\|$$

and condition (1.2.4) with $c = 1$ takes the form

$$\langle H'_v, v \rangle = \|v\| = 1 \neq t H'_t \equiv 0.$$

Due to the Lyapunov–Schmidt approach we seek a solution of (1.2.2) in the form $u = u_1 + u_2$, where u_1 is an element of a (usually finite-dimensional) subspace X_1 of X, and u_2 lies in a suitable "good" complement. Then from (1.1.2) we get the system of equations

$$\begin{cases} A_1(u_1, u_2) = h_1, \\ A_2(u_1, u_2) = h_2, \end{cases} \tag{1.3.1}$$

where the second equation for a fixed $u_1 \in X_1$ is a well-posed equation with a unique solution

$$u_2 = T(u_1, h_2).$$

By substituting this expression into the first equation of the system, we derive the so-called Lyapunov–Schmidt bifurcation equation

$$\mathcal{A}(u_1, h_2) = h_1, \tag{1.3.2}$$

where

$$\mathcal{A}(u_1, h_2) = A_1\big(u_1, T(u_1, h_2)\big).$$

Following the spherical fibering method, we seek a solution of the variational problem

$$f'(u) = 0$$

in the form $u = tv \neq 0$ with $(t, v) \in \mathbb{R} \times S$. Then the original variational problem is equivalent to the system

$$\begin{cases} \langle f'(tv), v \rangle = 0, \\ f'_\tau(tv) = 0 \quad \text{for } v \in S \end{cases} \tag{1.3.3}$$

(here f'_τ is the tangential derivative of f on the unit sphere S). The first equation of (1.3.3), namely,

$$\langle f'(tv), v \rangle = 0 \tag{1.3.4}$$

plays the same role as the bifurcation equation in the Lyapunov–Schmidt approach; therefore we will refer to it as to the *bifurcation equation* in the fibering method. Indeed, if we have a solution $t = t(v)$ of this equation, then we get the induced functional

$$\hat{f}(v) := f\big(t(v)v\big).$$

The conditionally critical point $v_c \in S$ of $\hat{f}$ with $t_c = t(v_c) \neq 0$ generates a critical point $u_c = t_c v_c$ of the original functional f.

From a geometrical point of view:

- in the Lyapunov–Schmidt approach, the representation $u = u_1 + u_2$ corresponds to introduction of Cartesian coordinates;
- in the spherical fibering method, the representation $u = tv$ with $\|v\| = 1$ corresponds to introduction of curvilinear (spherical) coordinates.

1.4. *Simple examples of known problems*

In these examples the bifurcation equation (1.3.4) admits an explicit smooth solution $t = t(v)$ for $v \in S$: this makes it possible to use a parameter-free realization of the spherical fibering method. In all examples below, Ω is a bounded domain in $\mathbb{R}^N$ with a locally Lipschitz boundary $\partial\Omega$. The solutions of the problems are considered in the Sobolev space $W_0^{1,2}(\Omega)$, the dual space of which is denoted by $W_2^{-1}(\Omega)$.

EXAMPLE 1.4.1. Consider the eigenfunction problem

$$\begin{cases} \Delta u + \lambda |u|^{p-2}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.1}$$

Here $2 < p < 2^*$, where

$$2^* := \begin{cases} \frac{2N}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N = 2. \end{cases}$$

The Euler functional f for this problem has the form

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

According to the fibering method, we set $u = tv$. Then the functional f takes the form

$$f(tv) = \frac{t^2}{2}\int_\Omega |\nabla v|^2 - \frac{|t|^p}{p}\int_\Omega |v|^p.$$

In the spherical fibering

$$\|v\| = \left(\int_\Omega |\nabla v|^2\right)^{1/2} = 1$$

the functional f reduces to

$$\tilde{f}(t, v) = \frac{t^2}{2} - \frac{|t|^p}{p} \int_\Omega |v|^p.$$

From the bifurcation equation $\tilde{f}_t'(t, v) = 0$, i.e.

$$t - |t|^{p-2} t \int_\Omega |v|^p = 0,$$

we find explicitly the real nonzero solutions

$$t = \pm \left(\int_\Omega |v|^p \right)^{\frac{1}{2-p}}.$$

Then the functional $\hat{f}(v) := \tilde{f}(t(v), v)$ takes the form

$$\hat{f}(v) = \frac{p-2}{2p} \left(\int_\Omega |v|^p \right)^{-\frac{2}{p-2}}.$$

Considering this functional on the unit sphere $S \subset W_0^{1,2}(\Omega)$, we can apply to it the well-known Lyusternik–Shnirel'man theory, in view of which $\hat{f}$ has a countable set of geometrically distinct conditionally critical points $v_1, v_2, v_3, \ldots$ on S with $\hat{f}(v_m) \to \infty$ (and hence $\int_\Omega |v_m|^p \to 0$) as $m \to \infty$. Thus we obtain that problem (1.4.1) has a countable set of geometrically distinct solutions $\pm u_1, \pm u_2, \ldots, \pm u_m, \ldots$ with

$$u_m(x) = \frac{v_m(x)}{(\int_\Omega |v_m|^p)^{\frac{1}{p-2}}}$$

and $\|u_m\| \to \infty$ as $m \to \infty$.

REMARK 1.4.2. If we start from the astrophysical meaning of the Emden–Fowler equation (1.4.1), and consider a solution u_m as a "star" in the Sobolev space $W_0^{1,2}(\Omega)$, then the set of all solutions of (1.4.1) looks like an "expanding Universe": indeed, since $\|u_m\| \to \infty$ as $m \to \infty$, for any $R > 0$ there exists a "star" u_m such that $\|u_m\| > R$. From the mathematical point of view it means that the boundary value problem

$$\begin{cases} \Delta u + \lambda |u|^{p-2} u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $2 < p < 2^*$ does not admit a priori estimates. Consequently, the Leray–Schauder method cannot be applied to this problem.

EXAMPLE 1.4.3. Consider the linear Dirichlet problem for the Poisson equation

$$\begin{cases} \Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4.2}$$

with $h \in W_2^{-1}(\Omega) \setminus \{0\}$. The functional associated with (1.4.2) is

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu.$$

Following the fibering method, we set $u = tv$. Then the functional f takes the form

$$f(tv) = -\frac{t^2}{2}\int_\Omega |\nabla v|^2 - t\int_\Omega hv.$$

In the spherical fibering

$$\|v\|^2 = \int_\Omega |\nabla v|^2 = 1$$

the functional $\tilde{f}$ equals

$$\tilde{f}(t, v) = -\frac{t^2}{2} - t\int_\Omega hv, \tag{1.4.3}$$

and then from the bifurcation equation

$$\tilde{f}_t'(t, v) = -t - \int_\Omega hv = 0$$

we find $t = -\int_\Omega hv$, and then

$$\hat{f}(v) := \tilde{f}\big(t(v), v\big) = \frac{1}{2}\left(\int_\Omega hv\right)^2. \tag{1.4.4}$$

Note that in this case the minimax realization of the fibering method—see Subsection 1.5—would give rise to the same functional $\hat{f}$. We now consider the critical points of this even functional $\hat{f}$ on the unit sphere S. Obviously, there exists an infinite set of conditionally critical points of $\hat{f}$ on the unit sphere. In this set there are only two *regular* conditionally critical points v_1 and $v_2 = -v_1$, i.e. conditionally critical points such that $t = t(v_1) \neq 0$ and $t_2 = t(v_2) \neq 0$: these are the points where $\hat{f}(v)$ attains the maximum on the closed unit ball B (v_1 and v_2 cannot lie in the interior part of B because $\hat{f}$ is homogeneous on v; see also the maximum principle expressed by Corollary 1.5.4). Then $u_1 = t_1 v_1$ and $u_2 = t_2 v_2$ are solutions of the Dirichlet problem (1.4.2); note in particular that, since $t_1 = -t_2$ and $v_1 = -v_2$, we actually obtain $u_1 = u_2$, that is, the two nonzero solutions coincide.

REMARK 1.4.4. Example 1.4.2 can be treated as an application of Theorems 1.5.1 and 1.6.3 that will be stated in the next subsections. To clear up the essence of the fibering method, we verify assumptions of these theorems in this example. We know $v_1 \in S$ is a maximum point of $\hat{f}$ on the unit sphere S. Then, by the Lagrange rule, at this point one has

$$h \int_\Omega h v_1 = -\nu \Delta v_1, \quad v_1 \in W_0^{1,2}(\Omega).$$

From this we find for $\int_\Omega |\nabla v_1|^2 = 1$ that $\nu = (\int_\Omega h v_1)^2$, and $\nu \neq 0$, because $\max_{v \in S} \hat{f}(v) > 0$ for $h \neq 0$. Then

$$h \int_\Omega h v_1 = -\left(\int_\Omega h v_1\right)^2 \Delta v_1$$

or, setting $t_1 = -\int_\Omega h v_1 \neq 0$ and $u_1 = t_1 v_1$,

$$\Delta u_1 = h, \quad u_1 \in W_0^{1,2}(\Omega),$$

i.e. u_1 is a solution of problem (1.4.2). We see similarly that u_2 is a solution of this problem, and $u_1 = u_2$.

EXAMPLE 1.4.5. In the above we have considered some applications to the global analysis of certain nonlinear boundary value problems. It is clear that the fibering method can be applied to the local analysis of certain nonlinear variational problems. We restrict ourselves to a simple example.

Let us consider the following boundary value problem:

$$\begin{cases} \Delta u + u^3 = h(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4.5}$$

with $N < 4$. The Euler functional that corresponds to this boundary value problem is

$$E(u) = -\frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4} \int_\Omega u^4 - \int_\Omega hu.$$

Due to spherical fibering we have $u(x) = tv(x)$ with

$$\|v\|^2 = \int_\Omega |\nabla v|^2 = 1 \quad \text{for } v \in W_0^{1,2}(\Omega).$$

Then the Euler functional E generates

$$\tilde{E}(t, v) = -\frac{t^2}{2} + \frac{t^4}{4} \int_\Omega v^4 - t \int_\Omega hv,$$

and the bifurcation equation takes the form

$$\frac{d\tilde{E}}{dt} \equiv -t + t^3 \int_\Omega v^4 - \int_\Omega hv = 0.$$

An elementary calculation shows that if the inequality

$$\sup_{v\in S}\left\{\left|\int_\Omega hv\right|\left(\int_\Omega v^4\right)^{1/2}\right\} < \frac{2}{3\sqrt{3}}$$

is satisfied, then the bifurcation equation possesses three isolated smooth branches of solutions: $t_1 = t_1(v,h)$, $t_2 = t_2(v,h)$, and $t_3 = t_3(v,h)$. By substituting them into $\tilde{E}$ we get three induced functionals

$$\hat{E}_i(v) = \tilde{E}\big(t_i(v,h), v\big) = -\frac{1}{2}t_i^2(v,h) + \frac{1}{4}t_i^4(v,h)\int_\Omega v^4 - t_i(v,h)\int_\Omega hv$$

for $i = 1,2,3$. These functionals are distinct and smooth on S. Every functional $\hat{E}_i(v)$ has a critical point v_i on S. Hence, the original Euler functional E under our conditions on h possesses three distinct critical points, i.e. solutions of (1.4.5):

$$u_1(x) = t_1 v_1(x), \qquad u_2(x) = t_2 v_2(x), \quad \text{and} \quad u_3(x) = t_3 v_3(x)$$

in $W_0^{1,2}(\Omega)$ such that

$$\int_\Omega hu_1 \leqslant 0, \qquad \int_\Omega hu_2 \leqslant 0, \qquad \int_\Omega hu_3 \geqslant 0,$$

and $\|u_1\| < \|u_2\|$.

REMARK 1.4.6. For sufficiently small h, the existence of a first solution u_1 for (1.4.5) can be proved via the contraction mapping principle. The existence of a second solution u_2 can be obtained from

$$\begin{cases} \Delta w + (w+u_1)^3 - u_1^3 = 0, \\ w = 0 \quad \text{on } \partial\Omega \end{cases}$$

by means of the theory of eigenfunctions for nonlinear elliptic problems. However, we are not aware of any way to prove the existence of a third distinct solution u_3 without using the fibering method.

1.5. *Minimax realization of the fibering method*

The fibering method admits various ways of realization. Here we consider some of these.

Let X be a real Banach space with norm differentiable on $X \setminus \{0\}$, let f be a functional on X belonging to the class $C^1(X \setminus \{0\})$, denote by S the unit sphere of X, and let J be a nonempty open subset of $\mathbb{R}$. Then the following result holds.

THEOREM 1.5.1. *Suppose that for any $v \in S$ the quantity*

$$\hat{f}(v) = \max_{t \in J} f(tv) \tag{1.5.1}$$

exists, and $\hat{f}(v) > f(0)$ if $0 \in J$. Assume that $\hat{f}$ is differentiable on the unit sphere S. Then to each conditionally stationary point v_c of the functional $\hat{f}$, considered on S, there corresponds a stationary point $u_c = t_c v_c$ of f with $t_c \in J \setminus \{0\}$ such that $f(u_c) = \hat{f}(v_c)$.

PROOF. Assume the statement is false, and hence $t_c f'(u_c) \neq 0$. Then there exists $w_0 \in X$ such that

$$t_c \langle f'(u_c), w_0 \rangle > 0. \tag{1.5.2}$$

Since $v_c \in S$ is a conditionally stationary point of the functional f, which is differentiable on S, it follows that

$$f\left(\frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) = \hat{f}(v_c) + \zeta \epsilon(\zeta) = f(t_c v_c) + \zeta \epsilon(\zeta) \tag{1.5.3}$$

for sufficiently small ζ, with $\epsilon(\zeta) \to 0$ as $\zeta \to 0$. On the other hand, by (1.5.1),

$$f\left(t \frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) \leqslant \hat{f}\left(\frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) \quad \forall t \in J. \tag{1.5.4}$$

By the assumptions of the theorem, $\max_{t \in J} f(tv_c) = f(t_c v_c)$ is attained on the open set $J \setminus \{0\}$. Hence, $t_c \|v_c + \zeta w_0\| \in J \setminus \{0\}$ for sufficiently small ζ, because $\|v_c\| = 1$. Then, by setting $t = t_c \|v_c + \zeta w_0\|$ in (1.5.4) for sufficiently small ζ, we get by (1.5.3) that

$$f(t_c v_c + \zeta t_c w_0) \leqslant f(t_c v_c) + \zeta \epsilon(\zeta). \tag{1.5.5}$$

Since f is differentiable,

$$f(t_c v_c + \zeta t_c w_0) = f(u_c) + \zeta t_c \langle f'(u_c), w_0 \rangle + \zeta \epsilon_1(\zeta), \quad \epsilon_1(\zeta) \to 0 \text{ as } \zeta \to 0.$$

Then it follows from (1.5.5) that

$$\zeta t_c \langle f'(u_c), w_0 \rangle \leqslant \zeta \epsilon_2(\zeta), \quad \epsilon_2(\zeta) \to 0 \text{ as } \zeta \to 0.$$

From this last inequality, for sufficiently small $\zeta > 0$, we get

$$t_c \langle f'(u_c), w_0 \rangle \leqslant 0.$$

In view of (1.5.2), this contradicts our assumption. The theorem is proven. □

REMARK 1.5.2. Let J be a nonempty open set in $\mathbb{R}$, symmetric with respect to zero. If the functional $\hat{f}$ defined by (1.5.1) exists, then it is *even:* this makes it possible to use the Lyusternik–Shnirel'man theory for certain functionals that are not even, and obtain theorems on the existence of many geometrically different stationary points.

EXAMPLE 1.5.3. Consider again the linear Dirichlet problem (1.4.2)

$$\begin{cases} \Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Euler functional associated with this problem is

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu.$$

In the spherical fibering

$$u = tv, \quad \|v\|^2 = \int_\Omega |\nabla v|^2 = 1$$

the functional f reduces to

$$\tilde{f}(t, v) = -\frac{t^2}{2} - t\int_\Omega hv.$$

Then, the minimax realization gives rise to the functional

$$\hat{f}(v) = \max_{t\in\mathbb{R}} \tilde{f}(t, v) = \frac{1}{2}\left(\int_\Omega hv\right)^2, \quad t_{\max} = -\int_\Omega hv,$$

which is the same as the $\hat{f}$ defined by (1.4.4). So the original *not even* Euler functional f generates by the minimax realization of the fibering method an *even* functional $\hat{f}$.

REMARK 1.5.4 (to Example 1.8.5). From the Lyusternik–Shnirel'man theory we know that each even weakly continuous functional $\hat{f}$ possesses at least a countable set of critical points on S. Thanks to the fibering method we have to expect a countable set of solutions for boundary value problem (1.4.2), but we know that this problem has only one solution; what is the matter? Let us consider this "contradiction" in more detail. The even functional

$$\hat{f} = \frac{1}{2}\left(\int_\Omega hv\right)^2$$

has actually a *continuous* set of critical points on S. Indeed, any $v \in S$, such that $\int_\Omega hv = 0$, is a critical point of $\hat{f}$, because in this case

$$\hat{f}'(v) = v\int_\Omega hv = 0.$$

Thus the equator

$$\mathcal{E}_0 := S \cap \{h\}^{\perp} = \left\{ v \in S\colon \int_{\Omega} hv = 0 \right\}$$

is the critical set for $\hat{f}$. But for $v \in \mathcal{E}_0$ we have

$$t = t(v) = -\int_{\Omega} hv = 0,$$

and consequently $u = tv = 0$; that is, these critical points are *invisible* with respect to f. On the other hand, as we have already seen, $\hat{f}$ also admits a pair of conditionally critical points

$$v_+\colon \quad -t(v_+) = \int_{\Omega} hv_+ = \max_{v \in S} \int_{\Omega} hv > 0,$$

$$v_-\colon \quad -t(v_-) = \int_{\Omega} hv_- = \min_{v \in S} \int_{\Omega} hv < 0,$$

that give rise to a *visible* "double" solution $u_+ = u_-$. Thus, we can reasonably expect that a perturbed nonlinear boundary value problem may admit a countable set of visible solutions. Indeed, the boundary value problem

$$\begin{cases} \Delta u + \epsilon |u|^{\delta} u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for any sufficiently small $\epsilon, \delta > 0$ possesses a countable set of solutions

$$u_1, u_2, \ldots, u_k, \ldots,$$

with $\|u_k\| \to +\infty$ as $k \to \infty$.

Before the next example we point out an immediate corollary of Theorem 1.5.1:

THEOREM 1.5.5. *Let X be an infinite-dimensional reflexive Banach space. Let the functional*

$$\hat{f} = \max_{t \in J} f(tv) > 0$$

(*or* $1/\hat{f}$) *satisfy the Lyusternik–Shnirel'man conditions* (*in any version of this theory*). *Then the functional f admits at least a countable set of distinct critical points.*

EXAMPLE 1.5.6. Consider the boundary value problem

$$\begin{cases} -\Delta u - u^3 + (\int_{\Omega} hv)^2 h = 0 & \text{in } \Omega \subset \mathbb{R}^N,\ N \leqslant 3, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.6}$$

Though the Euler functional

$$f(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 - \frac{1}{4}\int_{\Omega} u^4 + \frac{1}{3}\left(\int_{\Omega} hu\right)^3$$

is not even, by means of minimax realization we obtain an even functional

$$\hat{f}(v) = \max_{t\in\mathbb{R}} \tilde{f}(t, v),$$

where

$$\tilde{f}(t, v) = \frac{t^2}{2}\int_{\Omega} |\nabla v|^2 - \frac{t^4}{4}\int_{\Omega} v^4 + \frac{t^3}{3}\left(\int_{\Omega} hv\right)^3.$$

Then by Theorem 1.5.5 we obtain that problem (1.5.6) admits a countable set of solutions in $W_0^{1,2}(\Omega)$ for each $h \in W_2^{-1}(\Omega)$, since:

- $\hat{f} \in C^1(S)$;
- $\hat{f}(-v) = \hat{f}(v)$;
- $\hat{f} > 0$ on S;
- $\hat{f}$ is weakly continuous on S.

1.6. *The choice of the fibering functional*

Realization of the fibering method depends evidently on the choice of the fibering functional $H(t, v)$ satisfying condition (1.2.4). As the simplest of such functionals we can take the norm in the Banach space, provided that it is differentiable away from zero, but this choice is not unique. Here we propose as a fibering functional the functional naturally generated by the problem itself (i.e. by the Euler functional f). We remark, however, that in many cases this choice is not necessary. The scalar equation

$$\langle f'(tv), v\rangle = 0 \tag{1.6.1}$$

in the scalar parameter $t = t(v)$ is the defining bifurcation equation in the fibering method. To separate algebraically different solutions of this equation, it seems natural to take the following as a fibering functional H in the case of a functional f of class $C^3(X \setminus \{0\})$:

$$H(t, v) \equiv \langle f''(tv)v, v\rangle. \tag{1.6.2}$$

In this case X satisfies the relation

$$\langle H'_v, v\rangle - tH'_t = 2H. \tag{1.6.3}$$

In fact, by the equalities

$$\begin{aligned}
&\langle H_v', v\rangle = \frac{d}{d\zeta} H(t, \zeta v)\big|_{\zeta=1},\\
&tH_t' = \frac{d}{d\zeta} H(\zeta t, v)\big|_{\zeta=1},\\
&H(t, \zeta v) = \zeta^2 \langle f''(\zeta t v)v, v\rangle
\end{aligned}$$

we get

$$\begin{aligned}
\langle H_v', v\rangle &= \frac{d}{d\zeta} H(t, \zeta v)\big|_{\zeta=1}\\
&= 2\langle f''(\zeta t v)v, v\rangle + \frac{d}{d\zeta}\langle f''(tv)v, v\rangle\big|_{\zeta=1}\\
&= 2H + tH_t'.
\end{aligned}$$

By (1.6.3) the nondegeneracy condition (1.2.4) turns out to hold always for the functional H defined by (1.6.2). When such a functional is chosen, the solution $t = t(v)$ of (1.6.1) a priori inherits the smoothness of the original functional, with loss of one derivative. Accordingly, the problem of finding critical points for $f \in C^3(X \setminus \{0\})$ reduces to the problem of finding conditionally critical points for $f(tv)$ under the condition

$$\langle f''(tv)v, v\rangle = c \neq 0, \quad t \in J.$$

Set $k := \sqrt{|c|} \neq 0$; then, by substituting t/k and kv in place of t and v, respectively, the last equality can be re-written as

$$\langle f''(tv)v, v\rangle = c_0, \quad t \in kJ, \tag{1.6.4}$$

where c_0 is either $+1$ or -1, and $kJ := \{kt\colon t \in J\}$.

THEOREM 1.6.1. *Let f be a functional on X of the class $C^3(X \setminus \{0\})$, and let $(t, v) \in kJ \times X$ with $tv \neq 0$ be a conditionally critical point of the functional $f(tv)$ under condition* (1.6.4). *Then the point $u = tv$ is a nonzero critical point of f.*

The proof follows immediately from Theorem 1.2.2.

REMARK 1.6.2. If we apply the general formula (1.6.2) to the linear problem

$$\begin{cases} \Delta u = h & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we obtain

$$H(t,v) = \frac{d^2}{dt^2} f(tv) = \frac{d^2}{dt^2}\left(-\frac{t^2}{2}\int_\Omega |\nabla v|^2 - t\int_\Omega hv\right) = -\int_\Omega |\nabla v|^2,$$

and then our general condition (1.6.4) takes the form

$$\int_\Omega |\nabla v|^2 = 1.$$

That is, in this case the general constructive formula (1.6.2) leads to *spherical fibering*.

REMARK 1.6.3. The fibering functional H defined by (1.6.2) enables us to separate convex nonlinearities from concave ones, since formula (1.6.2) involves the second derivative of f:

- convex nonlinearities correspond to $H(t,v) = +1$;
- concave nonlinearities correspond to $H(t,v) = -1$.

1.7. *A parameter-free realization of the fibering method*

In the general case the fibering method reduces the original variational problem to a parametric variational one and to the investigation of its conditionally critical points. However, when the fibering functional H is defined by (1.5.2), it is possible to eliminate the parameter t in the new variational problem. Indeed, condition (1.5.4) for a functional f of class $C^3(X \setminus \{0\})$ means that

$$\frac{d}{dt}\langle f'(tv)v, v\rangle = \langle f''(tv)v, v\rangle \neq 0$$

on the set defined by (1.5.4). Thus, condition (1.5.4) for a functional f of class $C^3(X \setminus \{0\})$ enables us to single out the algebraically distinct smooth solutions $t_i(v)$ (for $i = 1, \ldots, m$) in the bifurcation equation (1.5.1), when they exist, and to eliminate the scalar parameter t. The problem of finding the nonzero critical points of $f \in C^3(X \setminus \{0\})$ reduces to the problem of finding conditionally critical points of the functionals

$$\begin{cases} F_i(v) := f(t_i(v)v) \\ \text{under condition} \\ H_i(v) := \langle f''(t_i(v)v)v, v\rangle = c_0 \end{cases} \tag{1.7.1}$$

with $c_0 = \pm 1$. Here $t_i(v)$ is the corresponding solution of (1.5.1) for $i = 1, \ldots, m$.

THEOREM 1.7.1. *Let f be a functional defined on X, $f \in C^3(X \setminus \{0\})$. Let $t_i(v)$ be the solution of* (1.5.1) *under condition* (1.5.4)*, and let v_i be a conditionally critical point of problem* (1.7.1) *with $t_i(v_i) \neq 0$. Then the point $u_i = t_i(v_i)v_i$ is a nonzero critical point of f.*

The proof follows from Theorem 1.5.1, since the pair $(t_i, v_i) \in kJ \times X$ satisfies the conditions of this theorem.

1.8. *Fibering functionals of norm type*

In studying a variational problem it is sometimes convenient, when the principal part of the original Euler functional is of norm type, to take this principal part as the fibering functional H. Then, if there is a complementary weakly continuous functional, the original problem reduces to that of investigating conditionally critical points of a continuous functional on a sphere-type surface (it is simpler to investigate a variational problem in a closed ball, because it is a convex set). We present a class of variational problems, in which this approach can be implemented. Suppose that on a Banach space X, with norm differentiable away from zero, a given functional $f \in C^1(X)$ has the form $f(u) = f_0(u) + f_1(u)$, where $f_0(u)$ generates the norm of X. For definiteness, we assume $f_0(u) = \|u\|^p$ for some $p > 1$, and $f_1 \in C^1(X)$. Then we choose the fibering functional $H(v) = f_0(v) = \|v\|^p$, so that condition (1.1.4) takes the form $\|v\| = 1$ (i.e., we use the method of spherical fibering). The functional $f(tv)$ takes the form

$$\tilde{f}(t, v) = |t|^p + f_1(tv) \quad \text{for } v \in S,$$

and the problem

$$f_0'(u) + f_1'(u) = 0 \tag{1.8.1}$$

is then equivalent to the system

$$p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0, \quad t \neq 0, \tag{1.8.2}$$

$$tf_1'(tv) = \nu\|v\|', \quad v \in S. \tag{1.8.3}$$

Since the functional $\langle f_1'(tv), v\rangle$ is defined for all $v \in X$, the first scalar equation (1.8.2) in t can be considered for $v \in B = \{w \in X\colon\ \|w\| \leqslant 1\}$. Suppose that this equation has solutions $t_i(w)$ for $i = 1, \ldots, N$. Let

$$F_i(w) = |t_i|^p + f_1\big(t_i(w)w\big)$$

and consider these functionals on the closed unit ball B.

DEFINITION 1.8.1. A point $w \in B$ is a *critical point* of the differentiable functional $F_i(w)$ in the closed unit ball B if one of the following conditions holds:

(1) w lies in the interior part of B and is an ordinary critical point of F_i;
(2) w lies on the boundary $\partial B = S$ and is a conditionally critical point of F_i on the sphere S.

DEFINITION 1.8.2. A critical point $w_i \in B$ of the differentiable functional $F_i(w)$ is a *regular critical point* of F_i if $w_i \neq 0$, $t_i(w_i) \neq 0$, and the functional t_i is differentiable at w_i. Here $t_i(w)$ is a solution of (1.8.2) for $v = w \in B$.

THEOREM 1.8.3. *Let $w_i \in B$ be a regular critical point of $F_i(w)$. Then $w_i \in \partial B$, and $u_i = t_i(w_i)w_i$ is a nonzero solution of* (1.8.1).

PROOF. Suppose by contradiction that the regular critical point $w_i \in B$ of F_i is in the interior part of the unit ball B. We study the behavior of F_i along the ray ζw_i as $\zeta \to 1$. By differentiability of f and differentiability of t_i at w_i, one has

$$\frac{dF_i(\zeta w_i)}{d\zeta}\bigg|_{\zeta=1} = \left[p|t_i|^{p-2}t_i + \langle f_1'(t_i w_i), w_i\rangle\right]\frac{dt_i}{d\zeta}\bigg|_{\zeta=1} + \langle f_1'(t_i w_i), w_i\rangle t_i,$$

where $t_i = t_i(w_i)$. Hence, in view of (1.8.2) with $v = w_i$,

$$\frac{dF_i(\zeta w_i)}{d\zeta}\bigg|_{\zeta=1} = t_i\langle f_1'(t_i w_i), w_i\rangle = -p|t_i|^p \neq 0,$$

which contradicts the fact that w_i is a (regular) critical point. Therefore, $w_i \in \partial B$. Moreover, by Theorem 1.2.1 the point $u_i = t_i w_i$ is a nonzero solution of (1.8.1). □

Following Definitions 1.8.1 and 1.8.2, we introduce the concept of a regular extremal point $w \in B$ for the functional F_i by replacing in those definitions the word "critical" by the word "extremal". Then from Theorem 1.8.3 we get

COROLLARY 1.8.4 (The Maximum Principle). *Let $w_i \in B$ be a regular extremal point of $F_i(w)$. Then $w_i \in \partial B$, and $u_i = t_i(w_i)w_i$ is a nonzero solution of* (1.8.1).

EXAMPLE 1.8.5. Here we demonstrate the above "maximum principle" in the simplest situation. Consider the linear problem (1.4.2)

$$\begin{cases} \Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and the corresponding Euler functional

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu.$$

Then the reduced functional is

$$\hat{f}(v) = \left(\int_\Omega hv \right)^2, \quad v \in W_0^{1,2}(\Omega), \ \|v\| = 1.$$

Now we consider B instead of S, i.e. $\|v\| \leqslant 1$ instead of $\|v\| = 1$. In this case we know from classical results that the functional $\hat{f}$ admits a maximum point v_0 in the convex bounded domain B. Actually, v_0 is on the boundary $S = \partial B$: in fact, if we suppose $\|v_0\| < 1$ then for sufficiently small $\varepsilon > 0$ one has still $(1+\epsilon)v_0 \in B$ and

$$\hat{f}\big((1+\epsilon)v_0\big) = (1+\epsilon)^2 \hat{f}(v_0) > \hat{f}(v_0),$$

which contradicts the fact that v_0 is a maximum point. Therefore $v_0 \in S$.

1.9. *Critical and conditionally critical points*

In the preceding subsections we used the fibering method to establish a connection between critical points and conditionally critical points of functionals. Now we consider this connection from a somewhat different point of view. Let l be a differentiable mapping (which can also be a nonzero constant) from a real Banach space X into X itself. We associate with any functional f twice differentiable (in the Gâteaux sense) on X the functional f_l defined by

$$f_l(u) := \big\langle f'(u), l(u) \big\rangle.$$

Clearly, every critical point u of f satisfies $f_l(u) = 0$. Hence, it is a conditionally critical point of f, considered under the condition $f_l(u) = 0$. The following simple theorem gives a condition for the validity of the converse.

THEOREM 1.9.1. *Let f be a twice differentiable functional* (*in the Gâteaux sense*) *on a real Banach space X. Suppose that there exists a differentiable mapping l from X to X such that at a conditionally critical point u_0 of $f(u)$, considered under the constraint $f_l(u) = 0$, one has*

$$\big\langle f_l'(u_0), l(u_0) \big\rangle \neq 0. \tag{1.9.1}$$

Then the conditionally critical point u_0 is actually an unconditionally critical point of f.

PROOF. Indeed, at u_0 one has

$$\lambda f'(u_0) = \mu f_l'(u_0), \quad \lambda^2 + \mu^2 \neq 0.$$

By this equality and (1.9.1) we get $\mu = 0$, and then $\lambda \neq 0$, since

$$\mu \big\langle f_l'(u_0), l(u_0) \big\rangle = \lambda \big\langle f'(u_0), l(u_0) \big\rangle = \lambda f_l(u_0) = 0.$$

Therefore $f'(u_0) = 0$. The theorem is proven. □

REMARK 1.9.2. Let $l(u) \equiv u$. Then we obtain the Nehari functional

$$f_N(u) = \langle f'(u), u \rangle.$$

2. Application of bifurcation equations

2.1. *The algebraic factor*

In this section we demonstrate the application of the bifurcation equations to various nonlinear boundary value problems in the simplest cases.

The bifurcation equation enables us to extract the algebraic factor of nonlinearities. Consider the bifurcation-fibering equation

$$\langle f'(tv), v \rangle = 0. \tag{2.1.1}$$

Let $t_i(v)$ (for $i = 1, 2, \ldots, k$) be the algebraic solution of (2.1.1) under condition

$$\|v\| = 1.$$

Then we obtain k functionals $f_1, \ldots, f_k$ defined by

$$f_1(v) := f\big(t_1(v)v\big),$$
$$f_2(v) := f\big(t_2(v)v\big),$$
$$\vdots$$
$$f_k(v) := f\big(t_k(v)v\big),$$

for which the following result holds (cf. Theorem 1.5.1).

THEOREM 2.1.1. *Let $f_1, \ldots, f_k \in C^1(S)$. Let v_i be a conditionally critical point of $f_i(v)$ with $t_i(v_i) \neq 0$. Then the point*

$$u_i = t_i(v_i)v_i$$

is a nonzero critical point of $f(u)$.

2.2. *The problem of nontrivial solutions*

Let Ω be a bounded domain in $\mathbb{R}^N$ with locally Lipschitz continuous boundary $\partial\Omega$. We consider the question of existence of nontrivial solutions for the boundary value problem

$$\begin{cases} \Delta u + g_1(x, u)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2.1}$$

The conditions on the function g_1 are as follows.

(C1) $g_1(x,0) \equiv 0$, and g_1 is a Carathéodory function on $\Omega \times \mathbb{R}$, i.e. it is measurable with respect to x for all $u \in \mathbb{R}$ and continuous with respect to u for almost all $x \in \Omega$.

(C2) For $N \geqslant 2$ there exist positive constants A and B such that, for all $x \in \Omega$ and all $u \in \mathbb{R}$:

$$\text{for } N > 2, \quad \left|g_1(x,u)\right| \leqslant A + B|u|^m, \quad \text{where } 0 \leqslant m < \frac{4}{N-2};$$

$$\text{for } N = 2, \quad \left|g_1(x,u)\right| \leqslant A + Be^{|u|^\alpha}, \quad \text{where } 0 \leqslant \alpha < 2.$$

(C3) For any function $v \in W_0^{1,2}(\Omega)$ with $\int_\Omega |\nabla v|^2 = 1$, i.e. for any v on the unit sphere S, the equation

$$\int_\Omega g_1\big(x, tv(x)\big)v^2(x)\,dx = 1 \tag{2.2.2}$$

in $t \in \mathbb{R}$ has a solution $t = t(v)$, and $t(v) \in C^1(S)$.

Let $t = t(v)$ be a solution of class $C^1(S)$. We consider the functional

$$F(v) = -\frac{t^2(v)}{2} + \int_\Omega G\big(x, t(v)v\big)\,dx, \tag{2.2.3}$$

where $G(x,s) = \int_0^s g_1(x,y)y\,dy$.

THEOREM 2.2.1. *Assume conditions* (C1), (C2), *and* (C3). *Suppose that the weakly continuous functional F defined in* (2.2.3) *admits a conditionally critical point v on the sphere S. Then $u = t(v)v$ is a nontrivial solution of problem* (2.2.2).

PROOF. The proof follows directly from Theorem 1.8.3, since the regularity of a conditionally critical point of F on S follows from conditions (C1) and (C3). □

EXAMPLE 2.2.2. Consider for $N \leqslant 3$ the boundary value problem

$$\begin{cases} \Delta u + a(x)u^2 + u^3 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2.4}$$

with $a \in L^q(\Omega)$, where $q = 1$ for $N = 1$, $q > 1$ for $N = 2$, $q > 2$ for $N = 3$. Then by Theorem 2.2.1 this boundary value problem has a nontrivial solution $u \in W_0^{1,2}(\Omega)$.

Notice that the Euler functional associated to (2.2.4)

$$E(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega au^3 + \frac{1}{4}\int_\Omega u^4$$

is not even. On the other hand, from (2.2.2) it follows that F is even *anyway:* in fact, if to any $v_1 \in S$ there corresponds a solution $t_1 = t(v_1)$, then to $v_2 = -v_1$ there corresponds $t_2 = t(-v_1) = -t_1$; if the even functional F is smooth, then the Lyusternik–Shnirel'man theory can be applied to it, under appropriate conditions.

THEOREM 2.2.3. *Assume conditions* (C1), (C2), *and* (C3). *Suppose that the even functional F defined by* (2.2.3) *satisfies on S the Lyusternik–Shnirel'man conditions, in any version of this theory. Then the boundary value problem* (2.2.1) *has a countable set of geometrically distinct solutions*.

PROOF. The existence of a countable set of geometrically distinct conditionally critical points for the even weakly continuous functional F on the unit sphere S follows from the Lyusternik–Shnirel'man theory. The regularity of each conditionally critical point of F on S follows from conditions (C1) and (C3), since a solution of (2.2.2) at a conditionally critical point of F on S is nonzero and differentiable at this point. Then we get Theorem 2.2.3 from Theorem 1.8.3. □

EXAMPLE 2.2.4. Consider the boundary value problem

$$\begin{cases} \Delta u + \lambda(x)u + \mu(x)|u|^{m-1}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < m < \frac{N+2}{N-2}$ for $N > 2$ and $m > 1$ for $N = 1, 2$. Denote by λ_1 the first eigenvalue of the Laplace operator in the domain Ω with Dirichlet boundary conditions. We get by Theorem 2.2.3 that for any functions $\lambda, \mu \in C(\overline{\Omega})$, with $\lambda(x) < \lambda_1$ and $\mu(x) > 0$ in $\overline{\Omega}$, this problem has a countable set of geometrically distinct solutions in the Sobolev space $W_0^{1,2}(\Omega)$.

EXAMPLE 2.2.5. We consider for $N \leqslant 3$ the boundary value problem

$$\begin{cases} \Delta u + a(x)|u|^{\alpha-1}u + u^3 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < \alpha < 3$ and $a \in L^q(\Omega)$ (without any assumptions on the *sign* of $a(x)$), where $q = 1$ for $N = 1$, $q > 1$ for $N = 2$, $q > \frac{6}{5-\alpha}$ for $N = 3$. Then by Theorem 2.2.3 this problem admits a countable set of geometrically different solutions in $W_0^{1,2}(\Omega)$.

REMARK 2.2.6 (to Example 2.2.5). We note that the above problem with $0 < \alpha < 1$ was already considered, by using another method called "linking", in [11].

2.3. *A problem with even nonlinearity*

We consider an application of Theorem 1.8.3 to the following boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \leqslant 5$ and with smooth boundary $\partial\Omega$:

$$\begin{cases} \Delta\Phi + \Phi^2 = \phi(x) & \text{in } \Omega, \\ \Phi = h_0(x) & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

where $\phi \in W_2^{-1}(\Omega) = (W_0^{1,2}(\Omega))^*$ and $h_0 \in W^{1/2,2}(\partial\Omega)$. Note that by virtue of the substitution $\Phi \to \Phi_1 = -\Phi$ the equation being considered is equivalent to the equation $\Delta\Phi_1 = \Phi_1^2 + \phi$.

Let h be a harmonic function in $W^{1,2}(\Omega)$ such that $\Delta h = 0$ in Ω and $h = h_0$ on $\partial\Omega$. Then the original boundary value problem is equivalent to the following one, by simply setting $\Phi = u + h$:

$$\begin{cases} \Delta u + (u+h)^2 = \phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3.2}$$

The Euler functional associated with (2.3.2) on the Sobolev space $W_0^{1,2}(\Omega)$ is $E = \frac{1}{2}H + G$, where

$$H(u) := \int_\Omega |\nabla u|^2 = \|u\|^2, \qquad G(u) := \int_\Omega \left(-\frac{1}{3}(u+h)^3 + \phi u + \frac{1}{3}h^3 \right)$$

(we choose H as the fibering functional). The bifurcation equation for $t(v)$ in this case takes the form

$$a(v)t^2 - b(v)t - c(v) = 0,$$

where

$$\begin{aligned} a(v) &= \int_\Omega v^3, \\ b(v) &= 1 - 2\int_\Omega hv^2, \\ c(v) &= \int_\Omega \left(\phi - h^2\right)v. \end{aligned}$$

From this we get, for $a(v) = \int_\Omega v^3 \neq 0$,

$$\begin{aligned} t_\pm(v) &= \frac{b(v) \pm \sqrt{b^2(v) + 4a(v)c(v)}}{2a(v)} \\ &= \left\{ 1 - 2\int_\Omega hv^2 \pm \left[\left(1 - 2\int_\Omega hv^2 \right)^2 \right.\right. \end{aligned}$$

$$+4\Big(\int_\Omega v^3\Big)\int_\Omega(\phi-h^2)v\Big]^{1/2}\Big\}\Big(2\int_\Omega v^3\Big)^{-1} \tag{2.3.3}$$

and, accordingly,

$$\begin{aligned}F_\pm(v) &= E\big(t_\pm(v)v\big)\\ &= \frac{b(v)}{6}t_\pm^2(v)+\frac{2c(v)}{3}t_\pm(v)\\ &= \frac{1}{12a^2(v)}\big(b^3(v)+6a(v)b(v)c(v)\\ &\quad \pm\big(b^2(v)+4a(v)c(v)\big)^{3/2}\big).\end{aligned} \tag{2.3.4}$$

We assume $b(v)>0$. Then the functional F_- is defined for all v in the closed unit ball B of the space $W_0^{1,2}(\Omega)$ (we obtain $F_-(v)=-c^2(v)/2b(v)$ if $a(v)=0$), while F_+ is defined for all $v\in B$ such that $a(v)\neq 0$. Let us consider the behavior of these functionals in the unit ball B as $\|v\|\to 0$ for $a(v)\neq 0$. We have

$$a(v)\to 0,\qquad b(v)\to 1,\qquad c(v)\to 0.$$

By Taylor's formula, up to the second order in $\zeta := \frac{a(v)c(v)}{b^2(v)}$, one has

$$F_\pm=\Big(\frac{b^3}{12a^2}+\frac{bc}{2a}\Big)(1\pm 1)\pm\Big[\frac{c^2}{2b}-\frac{ac^3}{3b^3}\big(1+\sigma(\zeta)\big)\Big]$$

for $a(v),b(v)\neq 0$, with $\sigma(\zeta)\to 0$ as $\zeta\to 0$. Concerning the boundary function $h_0\in W^{1/2,2}(\partial\Omega)$ we assume that there exists a constant $C_0>0$ such that the corresponding harmonic function h satisfies for any $v\in B$

$$b(v)=1-2\int_\Omega hv^2\geqslant C_0.$$

This holds, in particular, if $h(x)\leqslant 0$. Concerning the function $\phi\in W_2^{-1}(\Omega)$ we assume that there exists a constant $C_1>0$ such that for any $v\in B$

$$b^2(v)+4a(v)c(v)=\Big(1-2\int_\Omega hv^2\Big)^2+4\Big(\int_\Omega v^3\Big)\int_\Omega(\phi-h^2)v\geqslant C_1.$$

This holds, in particular, if $\phi-h^2$ is sufficiently small in the norm of the dual space $W_2^{-1}(\Omega)$. Under our assumptions,

$$\sup_{v\in B}F_+(v)=\sup_{v\in S}F_+(v)=+\infty,$$

$$\inf_{v\in B}F_+(v)>-\infty,\qquad \inf_{v\in B}F_-(v)>-\infty.$$

We mention that in the case where $\phi = h^2$ a.e. in Ω the trivial solution $u \equiv 0$ is one of the solutions of problem (2.3.2). Therefore, it is assumed below that

$$\|\phi - h^2\|_{W_2^{-1}} \neq 0,$$

and then we get

$$\inf_{v \in B} F_-(v) < 0.$$

Further, the corresponding minimum points v_- and v_+ exist for the functionals F_- and F_+ in the unit ball $B \subset W_0^{1,2}(\Omega)$. For the functional F_- at the point $v_- \in B$ we have

$$F_-(v_-) = \inf_{v \in B} F_-(v) < 0.$$

Then from representation (2.3.4) we get $v_- \neq 0$ and $t_-(v_-) \neq 0$. The functional F_- is differentiable at the point v_-. Thus, the conditions of Theorem 1.8.3 are satisfied for F_-, and hence $u_- = t_-(v_-)v_-$ is a solution of the boundary value problem (2.3.2) under the conditions on ϕ and h mentioned above. Now, let us consider the functional F_+. For this functional at the point $v_+ \in B$ we have

$$F_+(v_+) = \inf_{v \in B} F_+(v) > -\infty.$$

Then we get from representation (2.3.3), (2.3.4) for F_+ that $t_+ v_+ \neq 0$. The functional F_+ is differentiable at the point v_+, and $a(v_+) = \int_\Omega v_+^3 \neq 0$. Thus, the conditions of Theorem 1.8.3 hold for F_+, and hence, $u_+ = t_+(v_+)v_+$ is a solution of (2.3.2) under the aforementioned conditions on ϕ and h. We notice that the solutions u_- and u_+ are different. Indeed, if $u_- = u_+$ then $t_- v_- = t_+ v_+$; for v_-, v_+ it would follow that $v_- = \pm v_+$ and $|t_-| = |t_+|$. The last equalities contradict (2.3.3) under our assumptions on ϕ and h.

2.4. *A test for the absence of solutions*

We continue to demonstrate applications of the fibering method to nonlinear boundary value problems. Now we outline an application to the nonexistence problem: we first present a scheme for getting sufficient conditions for the absence of solutions. Let us consider the variational problem in the situation of Subsection 1.8, that is, we consider a Banach space X with norm differentiable away from zero and a functional $f(u) = f_0(u) + f_1(u)$ with $f_0(u) = \|u\|^p$ and $f_1 \in C^1(X)$. Then the problem

$$f_0'(u) + f_1'(u) = 0 \tag{2.4.1}$$

is equivalent to the system

$$\begin{cases} p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0, & t \neq 0, \\ |t|^{p-2}t \cdot (\|v\|^p)' + f_1'(tv) = 0, & v \in S. \end{cases}$$

From this system we obtain, for any $w \in X$, the following system of two scalar equations:

$$\begin{cases} p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0, \\ |t|^{p-2}t\langle(\|v\|^p)', w\rangle + \langle f_1'(tv), w\rangle = 0. \end{cases} \tag{2.4.2}$$

This gives us the following test for the absence of nonzero solutions for equation (2.4.1) in X.

THEOREM 2.4.1. *Let f_0 and f_1 be the functionals defined above, and suppose that there exists an element $x \in X$ such that system* (2.4.2) *is inconsistent for any value of $t \neq 0$ and $v \in S$. Then equation* (2.4.1) *does not admit nontrivial solutions in X.*

Obviously, the zero solution of (2.4.1) does not exist if

$$f_0'(0) + f_1'(0) \neq 0.$$

REMARK 2.4.2. Consider again the boundary value problem with quadratic nonlinearity (2.3.2). In this case system (2.4.2) takes the form

$$\begin{cases} t - \int_\Omega (tv+h)^2 v + \int_\Omega \phi v = 0, \\ -t\int_\Omega v\Delta\psi - \int_\Omega (tv+h)^2\psi + \int_\Omega \phi\psi = 0, \end{cases}$$

where ψ is an arbitrary function in $W_0^{1,2}(\Omega)$. Notice that the first equation in this system can be obtained from the second one by setting $\psi = v$. Therefore we now consider the second scalar equation with respect to t, namely,

$$t^2\int_\Omega \psi v^2 + t\int_\Omega(\Delta\psi + 2h\psi)v + \int_\Omega \big(h^2 - \phi\big)\psi = 0.$$

This equation clearly does not admit any real solution if there exists a function $\psi \in W_0^{1,2}(\Omega)$ such that, for all $v \in S$, one has

$$\left(\int_\Omega(\Delta\psi + 2h\psi)v\right)^2 < 4\int_\Omega \big(h^2 - \phi\big)\psi \int_\Omega \psi v^2. \tag{2.4.3}$$

On the other hand, if $\psi(x) \geqslant 0$ in Ω, then

$$\left(\int_\Omega(\Delta\psi + 2h\psi)v\right)^2 = \left(\int_\Omega \frac{\Delta\psi + 2h\psi}{\sqrt{\psi}}\sqrt{\psi}v\right)^2 \leqslant \int_\Omega \frac{(\Delta\psi + 2h\psi)^2}{\psi}\int_\Omega \psi v^2.$$

Hence, (2.4.3) holds if there exists a function $\psi \geqslant 0$ in $W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \frac{(\Delta\psi + 2h\psi)^2}{\psi} < 4\int_\Omega \left(h^2 - \phi\right)\psi,$$

or, equivalently,

$$\int_\Omega \left(\frac{(\Delta\psi)^2}{\psi} + 4h\Delta\psi + 4\phi\psi\right) < 0. \tag{2.4.4}$$

Accordingly, we get the following result:

PROPOSITION 2.4.3. *Suppose that there exists a function* $\psi \geqslant 0$ *in* $W_0^{1,2}(\Omega)$ *such that* (2.4.4) *holds. Then the boundary value problem* (2.3.1) *does not admit solutions in* $W_0^{1,2}(\Omega)$.

EXAMPLE 2.4.4. Consider problem (2.3.1) with $h_0 \equiv 0$, namely, the Ovsjannikov problem

$$\begin{cases} \Delta\Phi + \Phi^2 = \phi(x) & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then from Proposition 2.4.3 we obtain absence of solutions if we are able to find a $\psi \in W_0^{1,2}(\Omega)$ such that:

(1) $\psi > 0$ in Ω and $\psi \geqslant 0$ on $\partial\Omega$.

(2) $\int_\Omega \psi\phi < -\frac{1}{4}\int_\Omega \frac{(\Delta\psi)^2}{\psi}$.

For instance, if we take ψ such that

$$\begin{cases} \Delta\psi + \lambda_1\psi = 0, \quad \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

then we obtain that the Ovsjannikov problem does not admit any solution if

$$\int_\Omega \psi\phi < -\frac{\lambda_1^2}{4}\int_\Omega \psi.$$

In particular, we obtain absence of solutions if $\phi(x) < -\lambda_1^2/4$.

We remark that the general nonexistence test (2.4.3), in contrast with traditional tests for quasilinear elliptic equations of second order, is not a pointwise test but an integral one. We explain this feature by the following special example.

EXAMPLE 2.4.5. Consider the boundary value problem (2.3.1) where Ω is the open unit disk $D \subset \mathbb{R}^2$, $\phi \equiv 0$, and the boundary function h_0 is equal to $A\cos\theta$ in polar coordinates:

$$\begin{cases} \Delta\Phi + \Phi^2 = 0 & \text{in } D, \\ \Phi = A\cos\theta & \text{on } \partial D, \end{cases} \tag{2.4.5}$$

where A is an arbitrary real parameter. The choice of this particular example is due to two circumstances. First, this problem is given without analysis in a number of books. Second (and this is the main thing), the traditional tests for the absence of real solutions are not applicable to problem (2.4.5), since the mean of the boundary values is equal to zero:

$$\int_0^{2\pi} A\cos\theta\, d\theta = 0.$$

For problem (2.4.5) inequality (2.4.4) takes the form

$$\int_0^{2\pi} d\theta \int_0^1 dr \left(\frac{(\Delta\psi)^2}{\psi} + 4\arccos\theta \cdot \Delta\psi\right) r < 0. \tag{2.4.6}$$

We now choose ψ to be a solution of the following problem with parameter $\tau > 0$:

$$\begin{cases} \Delta\psi = -(\tau + r\cos\theta)(1 - r^2) & \text{in } D, \\ \psi = 0 & \text{on } \partial D. \end{cases}$$

This solution can be written explicitly, and $\psi \geqslant 0$ for $\tau \geqslant 1/3$. We substitute this function ψ (which depends on $\tau \geqslant 1/3$) into (2.4.6). Then we get a parametric inequality for A with $\tau \geqslant 1/3$, and it yields the following estimate for $|A|$ when $\tau = (1 + \sqrt{5/2})/3$:

$$|A| > 20.65.$$

If A satisfies this last inequality, then the boundary value problem (2.4.5) does not admit any solution in $W_0^{1,2}(D)$.

3. Application to Dirichlet problems

3.1. *An example*

In this section we apply the fibering method to the problem of existence of positive solutions for equations involving the p-Laplacian

$$\Delta_p u := \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. The particular equation we consider in this section was also studied in [21,22] for $\Omega = \mathbb{R}^N$. Essentially the same result as here was proved in [22] by using the so-called bifurcation argument [18] combined with the critical point theory. However, it appears that our approach based on the fibering method yields the existence and multiplicity of positive solutions in a more explicit and constructive way. We first discuss an example with $p = 2$ (hence $\Delta_p = \Delta$).

Consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = a(x)|u|^{q-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where $2 < q < 2^* := 2N/(N-2)$, and $a \in L^\infty(\Omega)$ satisfies the following assumptions:

(A1) a^+ (the positive part of a) is not identically zero;

(A2) $\int_\Omega a \cdot u_1^q < 0$, where u_1 is the eigenfunction associated to the first eigenvalue λ_1 of $-\Delta$, namely, $u_1 = u_1(x)$ is such that

$$\begin{cases} \Delta u_1 + \lambda_1 u_1 = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see also Lemma 3.2.3 in the next subsection).

THEOREM 3.1.1 (Alama and Tarantello, [2] 1993). *The following results hold.*

- *Let* $0 \leqslant \lambda < \lambda_1$, *and assume* (A1). *Then* (3.1.1) *has a positive solution in* $W^2_\infty(\Omega)$.
- *Let* $\lambda = \lambda_1$, *and assume* (A1) *and* (A2). *Then* (3.1.1) *has a positive solution in* $W^2_\infty(\Omega)$.
- *Let* $\lambda > \lambda_1$, *and assume* (A1) *and* (A2). *Then there exists* $\delta > 0$ *such that for* $\lambda < \lambda_1 + \delta$ (3.1.1) *has two positive solutions in* $W^2_\infty(\Omega)$.

This result is of considerable interest. Let us point out some features.

(1) Solutions of (3.1.1) are *not* small. Indeed, let $\lambda = 0$. Then we have

$$\begin{cases} -\Delta u = a(x)|u|^{q-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By multiplying by u and integrating by parts we find

$$\begin{aligned} \int_\Omega |\nabla u|^2 &= \int_\Omega a|u|^q \\ &\leqslant \|a\|_\infty \int_\Omega |u|^q \\ &\leqslant \|a\|_\infty \big(C_q \|u\|_{1,2}\big)^q, \end{aligned}$$

where C_q is (since $q < 2^*$) the Sobolev constant

$$\|u\|_q \leqslant C_q \|u\|_{1,2}.$$

Since $u = 0$ on $\partial\Omega$, we obtain

$$\|u\|_{1,2}^2 \leqslant C \|a\|_\infty \|u\|_{1,2}^q.$$

From this last inequality we obtain, since u is not identically zero and $q > 2$:

$$\|u\|_{1,2} \geqslant \left(\frac{1}{C\|a\|_\infty}\right)^{\frac{1}{q-2}} \to +\infty \quad \text{as } \|a\|_\infty \to 0.$$

Therefore, Theorem 3.1.1 does *not* follow from the classical bifurcation theory.

(2) Solutions of (3.1.1) are *positive* if λ is near to λ_1 (including $\lambda > \lambda_1$!).

(3) If $\lambda > \lambda_1$, Theorem 3.1.1 states the existence of *two* positive solutions (an *even* number of solutions) for (3.1.1).

These interesting features stimulated further investigations by S. Alama, G. Tarantello, L. Nirenberg, H. Brezis, H. Berestycki, I. Capuzzo-Dolcetta, and other mathematicians who considered elliptic problems in the form

$$\begin{cases} -Lu = f(x)g(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ \frac{\partial u}{\partial \nu} + \alpha(x)u = 0 & \text{on } \partial\Omega \end{cases}$$

with the linear principal part

$$Lu = \sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Actually, their method is based on the linear decomposition

$$W_0^{1,2}(\Omega) = \operatorname{span}\{u_1\} + W_0^{\perp},$$

associated with the linear structure of the operator L. This method does not work, if the principal part is nonlinear, in particular, for $Lu = \Delta_p u$. That is why we apply the fibering method instead.

3.2. *A problem involving the p-Laplacian*

In this subsection and in the next ones we consider an application of the fibering method to the p-Laplacian. Here we follow the paper [23], where we used some arguments from [20].

Let Ω be a bounded domain in $\mathbb{R}^N$. Assume $p, \lambda, q \in \mathbb{R}$, $1 < p < q < p^*$, where $p^* := Np/(N-p)$ for $p < N$ and $p^* := \infty$ for $p \geqslant N$. We consider the equation

$$-\Delta_p u = \lambda b(x)|u|^{p-2}u + a(x)|u|^{q-2}u \tag{3.2.1}$$

for $x \in \Omega$, under the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.2.2}$$

This problem is studied in connection with the corresponding eigenvalue problem

$$-\Delta_p u = \lambda b(x)|u|^{p-2}u. \tag{3.2.3}$$

We concentrate ourselves on the existence and multiplicity of positive solutions for (3.2.1) when $0 \leqslant \lambda < \lambda_1 + \epsilon$, where ϵ is a "small" positive number and λ_1 is the first eigenvalue of (3.2.3). In particular for $\lambda > \lambda_1$ we shall prove the existence of (at least) two solutions, similarly to the case $\Delta_p = \Delta$ that we discussed in the previous subsection.

Let us premise some definitions and notations. We work in the Sobolev space $W := W_0^{1,p}(\Omega)$ equipped with the usual norm

$$\|u\|_W = \left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

We assume $a, b \in L^\infty(\Omega)$, with $b \geqslant 0$ and b not identically zero.

DEFINITION 3.2.1. A function $u \in W$ is a weak solution for problem (3.2.1) under condition (3.2.2) iff it satisfies the integral identity

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega b \cdot |u|^{p-2}uv + \int_\Omega a \cdot |u|^{q-2}uv \tag{3.2.4}$$

for every $v \in W$.

DEFINITION 3.2.2. A real number λ is an eigenvalue for problem (3.2.3) under condition (3.2.2), and $u \in W \setminus \{0\}$ is a corresponding *eigenfunction*, iff

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega b \cdot |u|^{p-2}uv \tag{3.2.5}$$

for every $v \in W$.

The following result is now well known (see, e.g., [6,10,32]).

LEMMA 3.2.3. *There exists the first positive eigenvalue λ_1 for problem* (3.2.3) *under condition* (3.2.2). *It is characterized as the minimum of the Rayleigh quotient*:

$$\lambda_1 = \min\left\{\frac{\int_\Omega |\nabla u|^p}{\int_\Omega b|u|^p} u \in W,\ \int_\Omega b|u|^p > 0\right\}. \tag{3.2.6}$$

Moreover, λ_1 is simple (*i.e. each associated eigenfunction can be obtained from any other by multiplying by a nonzero constant*), *isolated* (*i.e. there are no eigenvalues in a suitable neighborhood of λ_1*), *and admits an eigenfunction $u_1 \in W$, which is positive in Ω.*

We denote by $\langle\ ,\ \rangle_W$ the duality between W^* and W, so that the left-hand side of (3.2.4) and (3.2.5) can be written as

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \langle -\Delta_p u, v\rangle_W.$$

Since $b \in L^\infty(\Omega)$ and $1 < q < p^*$, from the continuity of the Nemytskii operator [27] and the Sobolev Imbedding Theorem [1] follows that:

- (B0) the functional

$$B(u) := \int_\Omega b \cdot |u|^p$$

  is weakly continuous on W;
- (A0) the functional

$$A(u) := \int_\Omega a \cdot |u|^q$$

  is weakly continuous on W.

Notice that $B(u)$ is p-homogeneous and $A(u)$ is q-homogeneous.

3.3. *The application of the fibering method*

Let us consider the Euler functional

$$\begin{aligned} E_\lambda(u) &:= \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega b|u|^p - \frac{1}{q}\int_\Omega a|u|^q \\ &= \frac{1}{p}\|u\|_W^p - \frac{\lambda}{p}B(u) - \frac{1}{q}A(u) \end{aligned} \tag{3.3.1}$$

associated with (3.2.1), (3.2.2). According to (3.2.4), critical points of E_λ coincide with weak solutions of boundary value problem (3.2.1), (3.2.2). Following the fibering method, we substitute $u = tv$ (with $t \in \mathbb{R} \setminus \{0\}$ and $v \in W$) into (3.3.1) and get

$$E_\lambda(tv) = \frac{|t|^p}{p}\|v\|_W^p - \frac{\lambda|t|^p}{p}B(v) - \frac{|t|^q}{q}A(v). \tag{3.3.2}$$

We choose as the *fibering functional* H_λ the principal part of E_λ, i.e.

$$H_\lambda(v) := \int_\Omega |\nabla v|^p - \lambda\int_\Omega b|v|^p = \|v\|_W^p - \lambda B(v) \tag{3.3.3}$$

(notice that H_λ is independent from t). Then the *bifurcation equation* $\frac{\partial}{\partial t}E_\lambda(tv) = 0$ takes the form

$$|t|^{p-2}tH_\lambda(v) - |t|^{q-2}tA(v) = 0,$$

i.e., since $t \neq 0$:

$$H_\lambda(v) - |t|^{q-p} A(v) = 0.$$

From this we obtain

$$|t| = \left(\frac{H_\lambda(v)}{A(v)} \right)^{\frac{1}{q-p}} > 0 \tag{3.3.4}$$

under the *necessary* conditions

$$A(v) \neq 0, \qquad \frac{H_\lambda(v)}{A(v)} > 0. \tag{3.3.5}$$

By substituting (3.3.4) into (3.3.2) we define

$$\hat{E}_\lambda(v) := E_\lambda\big(t(v)v\big) = \left(\frac{1}{p} - \frac{1}{q} \right) \left(\frac{H_\lambda(v)}{A(v)} \right)^{\frac{q}{q-p}} A(v). \tag{3.3.6}$$

LEMMA 3.3.1. *The functional $\hat{E}_\lambda$ is* 0*-homogeneous, i.e. for every $\tau \in \mathbb{R} \setminus \{0\}$ and every $v \in W$ such that $A(v) \neq 0$ we have*

$$\hat{E}_\lambda(\tau v) = \hat{E}_\lambda(v).$$

In particular, $\hat{E}_\lambda$ is even, and its Gâteaux derivative at v in the direction v is zero:

$$\big\langle \hat{E}_\lambda(v), v \big\rangle_W = 0.$$

Moreover, if $v_c \in W$ is a critical point of $\hat{E}_\lambda$, then $|v_c|$ is one as well.

The proof of Lemma 3.3.1 is obvious. It implies that whenever we find some critical point v_c of $\hat{E}_\lambda$, we can automatically assume that v_c is *nonnegative* in Ω.

The following subsections are devoted to studying problem (3.2.1) in three distinct cases:

- $0 \leqslant \lambda < \lambda_1$;
- $\lambda = \lambda_1$;
- $\lambda_1 < \lambda < \lambda_1 + \epsilon$.

Here and in the following subsections, λ_1 is the first positive eigenvalue of (3.2.3) under condition (3.2.2). By u_1, we denote the corresponding positive eigenfunction (see Lemma 3.2.3).

3.4. *The case* $0 \leqslant \lambda < \lambda_1$

Let $0 \leqslant \lambda < \lambda_1$. We can take H_λ as the fibering functional, as defined by (3.3.3). In fact, it follows from Lemma 3.2.3 that $H_\lambda(v) \geqslant 0$ for any $v \in W$, hence the fibering constraint becomes

$$H_\lambda(v) = 1,$$

since H_λ is p-homogeneous. We still have to verify the nondegeneracy condition (cf. inequality (1.2.4)). Indeed, it follows directly from (3.3.3) that

$$\langle H_\lambda'(v), v\rangle_W = p \cdot H_\lambda(v) \neq 0$$

(we recall that in the present case the derivative of the fibering functional with respect to t is zero). Since $H_\lambda(v) \geqslant 0$, it follows from (3.3.5) that we have to consider the conditionally critical points of $\hat{E}_\lambda$ satisfying

$$A(v) = \int_\Omega a|v|^q > 0,$$

so the following hypothesis is natural (cf. hypothesis (A1) in Subsection 4.1):

(A1) a^+ is not identically zero.

By (3.3.6), the functional $\hat{E}_\lambda(v)$ under constraint $H_\lambda(v) = 1$ assumes the form

$$\hat{E}_\lambda(v) := \left(\frac{1}{p} - \frac{1}{q}\right) A^{-\frac{p}{q-p}}(v).$$

Therefore we consider the conditional variational problem

(P_λ) Find a maximizer $v_c \in W$ of the problem

$$0 < M_\lambda = \sup_{v\in W} \{A(v) \mid H_\lambda(v) = 1\}.$$

PROPOSITION 3.4.1. *Assume* (B0), (A0), (A1). *Then problem* (P_λ) *admits a nonnegative solution.*

PROOF. Let us consider the set

$$W_\lambda := \{v \in W\colon\ H_\lambda(v) = 1\}.$$

W_λ is nonempty since $H_\lambda(u_1) > 0$ and H_λ is homogeneous. Due to (3.3.3) and to the variational characterization (3.2.5) of λ_1, we get for any $v \in W_\lambda$

$$\int_\Omega |\nabla v|^p = 1 + \lambda \int_\Omega g|v|^p \leqslant 1 + \frac{\lambda}{\lambda_1} \int_\Omega |\nabla v|^p$$

and then

$$\|v\|_W^p = \int_\Omega |\nabla v|^p \leqslant \frac{\lambda_1}{\lambda_1 - \lambda}.$$

Hence, W_λ is bounded in W. Therefore, any maximizing sequence $(v_n)_{n=1}^\infty$ for problem (P_λ) is bounded in W. Consequently, we can assume

$$v_n \rightharpoonup v_c \quad \text{in } W.$$

By (A0) and (A1) one has

$$\int_\Omega a(x)|v_n|^q \to \int_\Omega a(x)|v_c|^q = M_\lambda > 0. \tag{3.4.1}$$

Moreover, we have $H_\lambda(v_n) = 1$ and due to the weak lower semicontinuity of the norm $\|\cdot\|_W$ and (B0) we get

$$\int_\Omega |\nabla v_c|^p \leqslant \lim_{n\to\infty} \inf \int_\Omega |\nabla v_n|^p,$$

$$\int_\Omega b(x)|v_c|^p = \lim_{n\to\infty} \int_\Omega b(x)|\nabla v_n|^p.$$

Hence

$$H_\lambda(v_c) := \int_\Omega |\nabla v_c|^p - \lambda \int_\Omega b(x)|v_c|^p \leqslant 1. \tag{3.4.2}$$

From (3.4.1) it follows that v_c is not identically zero, and we can assume $v_c \geqslant 0$ (cf. Lemma 3.3.1). We have only to prove that $v_c \in W_\lambda$, that is, that equality holds in (3.4.2). Assume now by contradiction that this is not the case, i.e.

$$H_\lambda(v_c) < 1.$$

Since H_λ is homogeneous, we can find $k_c > 1$ such that

$$H_\lambda(k_c v_c) = 1.$$

But then $\tilde{v}_c = k_c v_c \in W_\lambda$, and by (3.4.1)

$$\int_\Omega a(x)|\tilde{v}_c|^q = k_c^q \int_\Omega a(x)|v_c|^q = k_c^q M_\lambda > M_\lambda,$$

which contradicts the definition of M_λ. Hence $v_c \in W_\lambda$ is the desired solution of (P_λ). $\square$

Thanks to the fibering method we can state the following result.

THEOREM 3.4.2. *Let* $1 < p < q < p^*$, $0 \leqslant \lambda < \lambda_1$, $a, b \in L^\infty(\Omega)$, *and let hypothesis* (A1) *be satisfied. Then the boundary value problem* (3.2.1) *under condition* (3.2.2) *has at least one positive weak solution* $u \in W \cap L^\infty(\Omega)$. *Moreover,* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

PROOF. Recall that (B0) and (A0) hold under assumptions of the theorem. Then by Proposition 3.4.1 there is a nonnegative solution of problem (P_λ). Clearly, v_c is a conditionally critical point of $\hat{E}_\lambda(v)$ under the fibering constraint $H_\lambda(v) = 1$. Then, by means of the fibering method, we can take

$$u_c := t_c v_c \geqslant 0$$

as a critical point for E_λ (here $t_c > 0$ is defined by (3.3.5)): that is, u_c is a weak solution of (3.2.1). Following the bootstrap argument (used, e.g., in [18]) we can prove that $u \in L^\infty(\Omega)$. Then by applying the Harnack inequality due to Trudinger [56,25] we get $u > 0$ in Ω (cf. [11]). It follows from the result of Tolksdorff [55] that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ (cf. [18]). □

REMARK 3.4.3. If $a(x) > 0$ and $\lambda < \lambda_1$, then by Lyusternik–Shnirel'man theory immediately follows the existence of a countable set of nontrivial (sign-changing) solutions of (3.2.1), (3.2.2).

3.5. *The case* $\lambda = \lambda_1$

Let $\lambda = \lambda_1$. Keeping the notation of the previous subsection, we consider the conditional variational problem

(P_{λ_1}) Find a maximizer $v_c \in W$ of the problem

$$0 < M_{\lambda_1} = \sup_{v \in W} \left\{ \int_\Omega a|v|^q \;\middle|\; H_{\lambda_1}(v) = 1 \right\}.$$

We have $M_{\lambda_1} > 0$ by (A1), as in the previous subsection. In this case, however, the set

$$W_{\lambda_1} = \left\{ v \in W\colon\ H_{\lambda_1}(v) = 1 \right\}$$

is *unbounded* in W. So we are forced to require the following additional condition on f (cf. hypothesis (A2) in Subsection 3.1):

$$\text{(A2)} \qquad \int_\Omega a u_1^q < 0.$$

PROPOSITION 3.5.1. *Assume* (B0), (A0), (A1), *and* (A2). *Then problem* (P_{λ_1}) *admits a nonnegative solution.*

PROOF. Let $(v_n)_{n=1}^\infty$ be a maximizing sequence of (P_{λ_1}), i.e.

$$H_{\lambda_1}(v_n) = 1, \qquad \int_\Omega a(x)|v_n|^q \to M_{\lambda_1} > 0. \tag{3.5.1}$$

Suppose by contradiction that (v_n) is unbounded. Then we can assume $\|v_n\|_W \to \infty$. Set

$$v_n = t_n w_n \quad \text{with } |t_n| = \|v_n\|_W,\ \|w_n\|_W = 1,$$

so that, by (3.3.3),

$$H_{\lambda_1}(v_n) = |t_n|^p \left[\int_\Omega |\nabla w_n|^p - \lambda_1 \int_\Omega b(x)|w_n|^p \right] = 1.$$

Due to (3.2.6), we have

$$0 \leqslant \int_\Omega |\nabla w_n|^p - \lambda_1 \int_\Omega b(x)|w_n|^p = |t_n|^{-p} \to 0 \tag{3.5.2}$$

and then, since $\|w_n\|_W = 1$,

$$\lim_{n\to\infty} \lambda_1 \int_\Omega b(x)|w_n|^p \geqslant 1. \tag{3.5.3}$$

We can assume that $w_n \rightharpoonup \tilde{w}$ in W for some $\tilde{w} \in W$. Then (3.5.3) and (B0) imply

$$\int_\Omega b|\tilde{w}|^p = B(\tilde{w}) \geqslant \frac{1}{\lambda_1}$$

and consequently $\tilde{w} \neq 0$. Since we have also

$$\int_\Omega |\tilde{w}|^p \leqslant \lim_{n\to\infty} \inf \int_\Omega |\nabla w_n|^p = 1,$$

it follows from (3.5.2) that

$$0 \leqslant \int_\Omega |\nabla \tilde{w}|^p - \lambda_1 \int_\Omega b(x)|\tilde{w}|^p \leqslant 0,$$

that is, $H_{\lambda_1}(\tilde{w}) = 0$. By Lemma 3.2.3, $\tilde{w}$ is a multiple of the first eigenfunction u_1, i.e. $\tilde{w} = ku_1$ for a suitable $k \neq 0$. On the other hand, thanks to (3.5.1) we get

$$\int_\Omega a(x)|v_n|^q = |t_n|^q \int_\Omega a(x)|v_n|^q = m_n \to M_{\lambda_1} > 0,$$

i.e.

$$\int_\Omega a(x)|w_n|^q = \frac{m_n}{|t_n|^q} \to 0.$$

Then, by (A0), we have

$$\int_\Omega a(x)|\tilde{w}|^q \geqslant 0$$

and consequently

$$\int_\Omega a(x)|u_1|^q \geqslant 0,$$

which contradicts (A2). Hence the *maximizing sequence is bounded*, and we can assume

$$v_n \rightharpoonup v_c \quad \text{in } W$$

for some $v_c \in W$. By (A0), we have

$$\int_\Omega a(x)|v_n|^q \to \int_\Omega a(x)|v_c|^q = M_{\lambda_1} > 0 \tag{3.5.4}$$

and hence $v_c \neq 0$. From (3.5.1), (3.2.6), and (B0) it follows that

$$0 \leqslant \int_\Omega |\nabla v_c|^p - \lambda_1 \int_\Omega b(x)|v_c|^p \leqslant 1.$$

First we prove that

$$\int_\Omega |\nabla v_c|^p - \lambda_1 \int_\Omega b(x)|v_c|^p > 0.$$

Indeed, otherwise the equality together with Lemma 3.2.3 would yield the existence of $k \neq 0$ such that $v_c(x) = ku_1(x)$. If we substitute it into (3.5.4), then

$$|k|^q \int_\Omega a(x)|u_1|^q = \int_\Omega a(x)|v_c|^q = M_{\lambda_1} > 0,$$

in contradiction with (A2). Now we prove that

$$\int_\Omega |\nabla v_c|^p - \lambda_1 \int_\Omega b(x)|v_c|^p = 1. \tag{3.5.5}$$

Let us assume

$$0 < \int_\Omega |\nabla v_c|^p - \lambda_1 \int_\Omega b(x)|v_c|^p \leqslant 1.$$

Then there exists $k_c > 1$ such that the function $\tilde{v}_c = t_c v_c$ satisfies $H_{\lambda_1}(\tilde{v}_c) = 1$ and simultaneously

$$\int_\Omega a(x)|\tilde{v}_c|^q = k_c^q \int_\Omega a(x)|v_c|^q = k_c^q \cdot M_{\lambda_1}$$
$$> M_{\lambda_1} = \sup\left\{\int_\Omega a(x)|v|^q\colon\ H_{\lambda_1}(v) = 1\right\},$$

a contradiction. This proves (3.5.5), and hence v_c is a maximizer to problem (P_{λ_1}). We can assume $v_c \geqslant 0$ in Ω due to Lemma 3.3.1. □

THEOREM 3.5.2. *Let* $1 < p < q < p^*$, *let* $a, b \in L^\infty(\Omega)$, *and* (A1), (A2) *be satisfied. Then problem* (3.2.1) *with* $\lambda = \lambda_1$ *under Dirichlet condition* (3.2.2) *admits at least one positive weak solution* $u \in W \cap L^\infty(\Omega)$. *Moreover,* $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

PROOF. The proof is based on Proposition 3.5.1 and follows the same ideas as the proof of Theorem 3.4.2. □

3.6. *The case $\lambda > \lambda_1$*

We consider again problem (3.2.1) under condition (3.2.2), with $\lambda > \lambda_1$ but close enough to λ_1. The main result of this subsection is formulated in the next theorem.

THEOREM 3.6.1. *Let $1 < p < q < p^*$; let $a, b \in L^\infty(\Omega)$ and* (A1), (A2) *be satisfied. Then there exists $\epsilon > 0$ such that for $\lambda_1 < \lambda < \lambda_1 + \epsilon$ problem* (3.2.1) *under condition* (3.2.2) *admits* two *positive weak solutions $u_1, u_2 \in W \cap L^\infty(\Omega)$, and both solutions belong to $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

To prove this multiplicity result, we will consider two variational problems:
(P^1_λ) Find a maximizer $v_1 \in W$ of the problem

$$M_\lambda = \sup_{v \in W} \left\{ \int_\Omega a(x)|v|^q \;\Big|\; H_\lambda(v) = +1 \right\}.$$

(P^2_λ) Find a minimizer $v_2 \in W$ of the problem

$$m_\lambda = \inf_{v \in W} \left\{ H_\lambda(v) \;\Big|\; \int_\Omega a(x)|v|^q = -1 \right\}.$$

Notice that in problem (P^2_λ) we no longer consider H_λ, but rather $A(v)$ as the fibering functional.

In the next two sub-subsections we consider separately problem (P^1_λ) and (P^2_λ).

3.6.1. *Problem (P^1_λ).* First, we state an equivalence. Consider the problem
$(\tilde{P}^1_\lambda)$ Find a maximizer $\tilde{v}_1 \in W$ of the problem

$$\tilde{M}_\lambda = \sup_{v \in W} \left\{ \int_\Omega a(x)|v|^q \;\Big|\; H_\lambda(v) \leqslant 1 \right\}.$$

Note that $\tilde{M}_\lambda > 0$ if we assume (A1). Then the following statement holds.

LEMMA 3.6.2. *Assume* (A1). *Then problem (P^1_λ) is equivalent to $(\tilde{P}^1_\lambda)$.*

PROOF. Let $\tilde{v}_1 \in W$ be a maximizer of $(\tilde{P}^1_\lambda)$, and suppose by contradiction that

$$H_\lambda(\tilde{v}_1) < 1. \tag{3.6.1}$$

Then for a sufficiently small $k > 1$ one has

$$H_\lambda(k\tilde{v}_1) \leqslant 1,$$

and

$$\int_\Omega a(x)|k\tilde{v}_1|^q = k^q M_\lambda > M_\lambda$$

since $M_\lambda > 0$, by (A1). But this contradicts the fact that $\tilde{v}_1$ is a maximizer of $(\tilde{P}^1_\lambda)$. □

REMARK 3.6.3. Alternatively, one can prove this lemma by deducing from assumption (3.6.1) that for arbitrary $w \in W$ there is sufficiently small $\eta > 0$ such that for $\tilde{v}_\eta = \tilde{v}_1 + \eta w$ one has

$$H_\lambda(\tilde{v}_\eta) \leqslant 1. \tag{3.6.2}$$

Then

$$\begin{aligned}\int_\Omega a(x)|\tilde{v}_1 + \eta w|^q &= \int_\Omega a(x)|\tilde{v}_1|^q + \eta \int_\Omega a(x)|\tilde{v}_1|^{q-2}\tilde{v}_1 w + o(\eta) \\ &> \int_\Omega a(x)|\tilde{v}_1|^q \end{aligned} \tag{3.6.3}$$

if η is small enough and

$$\int_\Omega a(x)|\tilde{v}_1|^{q-2}\tilde{v}_1 w > 0.$$

Note that it is valid if, e.g., $w = \tilde{v}_1$, because we have

$$\int_\Omega a(x)|\tilde{v}_1|^{q-2}\tilde{v}_1 \cdot \tilde{v}_1 = \int_\Omega a(x)|\tilde{v}_1|^q = \tilde{M}_\lambda > 0.$$

Hence (3.6.13) and (3.6.14) are also incompatible with the fact that $\tilde{v}_1$ is a maximizer of $(\tilde{P}^1_\lambda)$.

PROPOSITION 3.6.4. *Assume* (B0), (A0), (A1), *and* (A2). *Then there exists* $\epsilon_1 > 0$ *such that for* $\lambda_1 < \lambda < \lambda_1 + \epsilon_1$ *problem* (P^1_λ) *has a nonnegative solution.*

PROOF. Due to Lemma 3.6.2, it suffices to show that there exists $\epsilon_1 > 0$ such that problem $(\tilde{P}^1_\lambda)$ admits a nonnegative solution for any λ, $\lambda_1 < \lambda < \lambda_1 + \epsilon_1$. Assume by contradiction that there is a sequence $\epsilon_k \to 0^+$ such that for any $\lambda^k := \lambda_1 + \epsilon_k$ problem $(\tilde{P}^1_{\lambda^k})$ has no (nonnegative) solution. Let us fix $k \in \mathbb{N}$ and consider the problem $(\tilde{P}^1_{\lambda^k})$. Let $(v^k_n)^\infty_{n=1}$ be a maximizing sequence of $(\tilde{P}^1_{\lambda^k})$, i.e.

$$\int_\Omega \left|\nabla v^k_n\right|^p - \lambda^k \int_\Omega b(x)\left|v^k_n\right|^p \leqslant 1$$

and

$$\int_\Omega a(x)\left|v_n^k\right|^q \to M_{\lambda^k} > 0$$

as $n \to \infty$. If the sequence $(v_n^k)_{n=1}^\infty$ is bounded, the variational problem $(\tilde{P}^1_{\lambda^k})$ has a (non-negative) solution, and this would be a contradiction. Indeed, if $(v_n^k)_{n=1}^\infty$ is bounded, we can assume

$$v_n^k \rightharpoonup v_0^k \quad \text{in } W$$

as $n \to \infty$. By using (B0), (A0), (A1), and repeating arguments from the proof of Proposition 3.5.1, we obtain

$$\int_\Omega a(x)\left|v_0^k\right|^q = M_{\lambda^k} > 0$$

and

$$\int_\Omega \left|\nabla v_0^k\right|^p - \lambda^k \int_\Omega b(x)\left|v_0^k\right|^p \leqslant 1.$$

Hence, as a contradiction we get that v_0^k is a solution of $(\tilde{P}^1_{\lambda^k})$.

Thus, for any k the sequence $(v_n^k)_{n=1}^\infty$ must be *unbounded*. Then we may assume

$$\left\|v_n^k\right\|_W \to \infty \quad \text{as } n \to \infty.$$

Set $v_n^k = t_n^k w_n^k$ with $t_n^k = \|v_n^k\|_W$ and $\|w_n^k\|_W = 1$. We can assume again

$$w_n^k \rightharpoonup w_0^k \quad \text{in } W \text{ as } n \to \infty.$$

We have

$$\int_\Omega a(x)\left|v_n^k\right|^q = \left(t_n^k\right)^q \int_\Omega a(x)\left|w_n^k\right|^q \to M_{\lambda^k} > 0$$

and therefore

$$\int_\Omega a(x)\left|w_0^k\right|^q \geqslant 0. \tag{3.6.4}$$

Furthermore

$$\left(t_n^k\right)^p \left[\int_\Omega \left|\nabla w_n^k\right|^p - \lambda^k \int_\Omega b(x)\left|w_n^k\right|^p\right] \leqslant 1$$

and so

$$\int_\Omega \left|\nabla w_n^k\right|^p - \lambda^k \int_\Omega b(x)\left|w_n^k\right|^p \leqslant \frac{1}{(t_n^k)^p}. \tag{3.6.5}$$

This together with (B0) and the weak lower semicontinuity of the norm in W imply

$$\int_\Omega \left|\nabla w_0^k\right|^p - \lambda^k \int_\Omega b(x)\left|w_0^k\right|^p \leqslant 0. \tag{3.6.6}$$

It follows from (3.6.5) that

$$\lambda^k \int_\Omega b(x)\left|w_n^k\right|^p \geqslant \int_\Omega \left|\nabla w_n^k\right|^p - \frac{1}{(t_n^k)^p}$$

and letting $n \to \infty$, we get from here (using (B0) again):

$$\lambda^k \int_\Omega b(x)\left|w_0^k\right|^p \geqslant 1. \tag{3.6.7}$$

Obviously, we have

$$\int_\Omega \left|\nabla w_0^k\right|^p \leqslant 1. \tag{3.6.8}$$

Now we pass to the limit as $k \to \infty$. Then $\lambda^k \to \lambda_1$ and due to (3.6.8) we may assume that

$$w_0^k \rightharpoonup \overline{w}_0 \quad \text{in } W \text{ as } k \to \infty.$$

It follows from (3.6.7) and (B0) that

$$\lambda_1 \int_\Omega b(x)|w_0|^p \geqslant 1, \tag{3.6.9}$$

and (3.6.6) yields

$$\int_\Omega |\nabla w_0|^p - \lambda_1 \int_\Omega b(x)|w_0|^p \leqslant 0. \tag{3.6.10}$$

Inequality (3.6.4) and (A0) imply

$$\int_\Omega a(x)|w_0|^q \geqslant 0. \tag{3.6.11}$$

It follows from (3.6.9) that $w_0 \not\equiv 0$, and then (3.6.10) together with Lemma 3.2.3 imply for a suitable $k_1 \neq 0$

$$w_0(x) = k_1 \cdot u_1(x), \tag{3.6.12}$$

where $u_1 > 0$ is the eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta_p$. If we substitute this into (3.6.11), we get

$$|k_1|^q \int_\Omega a(x)|u_1|^q \geqslant 0,$$

which contradicts (A2).

Hence for some $\epsilon_1 > 0$ problem $(\tilde{P}^1_\lambda)$ (and hence (P^1_λ), due to Lemma 3.6.1) admits at least one (nonnegative) solution for $\lambda_1 < \lambda < \lambda_1 + \epsilon_1$. □

3.6.2. *Problem* (P^2_λ). Now we choose $A(v)$ as the fibering functional, and

$$A(v) = -1$$

as the fibering constraint. The nondegeneracy condition

$$\langle A'(v), v\rangle \neq 0 \quad \text{as } A(v) = -1$$

can be proved the same way as for H_λ in the previous cases. By (3.3.6), the functional $\hat{E}_\lambda$ takes the form

$$\hat{E}_\lambda(v) = -\left(\frac{1}{p} - \frac{1}{q}\right)\left(-H_\lambda(v)\right)^{\frac{q}{q-p}}.$$

We have to search a conditionally critical point of $\hat{E}_\lambda$ satisfying (3.3.5), i.e.

$$H_\lambda(v) < 0.$$

Thus, solving problem (P^2_λ) is a natural way to find such a critical point.

First, we prove that problem (P^2_λ) makes sense under hypothesis (A2).

LEMMA 3.6.5. *Assume* (A2). *Then the set*

$$W^- := \left\{v \in W \;\middle|\; A(v) = \int_\Omega a(x)|v|^q = -1\right\}$$

is nonempty, and $m_\lambda < 0$ *for any* $\lambda > \lambda_1$.

PROOF. For $v = tu_1$ (u_1 being the first eigenfunction of $-\Delta_p$) we get

$$\int_\Omega a(x)|v|^q = |t|^q \int_\Omega a(x)|u_1|^q.$$

Due to (A2) we have $\int_\Omega a(x)|u_1|^q < 0$ and so we can find t_1 such that for $v_1 = t_1 u_1$ one has

$$\int_\Omega a(x)|v_1|^q = |t_1|^q \int_\Omega a(x)|u_1|^q = -1.$$

We have also

$$H_\lambda(v_1) = |t_1|^p\left[\int_\Omega |\nabla u_1|^p - \lambda^k \int_\Omega b(x)|u_1|^p\right]$$
$$= |t_1|^p(\lambda_1 - \lambda)\int_\Omega b(x)|u_1|^p < 0$$

for $\lambda > \lambda_1$ (see Lemma 3.3.1). Thus the infimum of H_λ in W^- is negative. □

PROPOSITION 3.6.6. *Assume* (B0), (A0), (A1) *and* (A2). *Then there exists* $\epsilon_2 > 0$ *such that for* $\lambda_1 < \lambda < \lambda_1 + \epsilon_2$ *problem* (P^2_λ) *admits a nonnegative solution* $v_2 \in W$.

PROOF. Assume that the statement of Proposition 3.6.6 is not true. Then there exists a sequence $\epsilon_k \to 0^+$ such that for $\lambda^k := \lambda_1 + \epsilon_k$ problem $(P^2_{\lambda^k})$ has no (nonnegative) solution. Let us consider a fixed $k \in \mathbb{N}$ and a minimizing sequence $(v_n^k)_{n=1}^\infty$ for $(P^2_{\lambda^k})$, i.e.

$$\int_\Omega a(x)\left|v_n^k\right|^q = -1,$$

$$\int_\Omega \left|\nabla v_n^k\right|^p - \lambda^k \int_\Omega b(x)\left|v_n^k\right|^p \to m_{\lambda^k} < 0 \quad \text{as } n \to \infty.$$

If $(v_n^k)_{n=1}^\infty$ is bounded, then $m_{\lambda^k} > -\infty$ and similarly to the proof of Proposition 3.6.4 we arrive at $v_n^k \rightharpoonup v_0^k$ as $n \to \infty$ and

$$\int_\Omega a(x)\left|v_0^k\right|^q = -1, \qquad \int_\Omega \left|\nabla v_0^k\right|^p - \lambda^k \int_\Omega b(x)\left|v_0^k\right|^p = m_{\lambda^k}.$$

Hence v_0^k is a solution of problem $(P^2_{\lambda^k})$, a contradiction.

Let us assume now that for any k the sequence $(v_n^k)_{n=1}^\infty$ is unbounded. Similarly to the proof of Proposition 3.6.4 we arrive at $v_n^k = t_n^k w_n^k$, $t_n^k = \|v_n^k\|_W \to \infty$, $\|w_n^k\|_W = 1$, $w_n^k \rightharpoonup w_0^k$ in W as $n \to \infty$. In this case we have

$$-1 = \int_\Omega a(x)\left|v_n^k\right|^q = \left(t_n^k\right)^q \int_\Omega a(x)\left|w_n^k\right|^q,$$

i.e.

$$\int_\Omega a(x)\left|w_n^k\right|^q = -\frac{1}{(t_n^k)^q} \to 0$$

as $n \to \infty$. Due to (A0), we have

$$\int_\Omega a(x)\left|w_0^k\right|^q = 0. \tag{3.6.13}$$

On the other hand,

$$H_{\lambda^k}\big(v_n^k\big) = \big(t_n^k\big)^p \left[\int_\Omega \big|\nabla w_n^k\big|^p - \lambda^k \int_\Omega b(x)\big|w_n^k\big|^p\right] \leqslant 0$$

and consequently

$$\int_\Omega \big|\nabla w_n^k\big|^p - \lambda^k \int_\Omega b(x)\big|w_n^k\big|^p \leqslant 0. \tag{3.6.14}$$

Similarly as in the proof of Proposition 3.6.4, we get from (3.6.14) that

$$\int_\Omega \big|\nabla w_0^k\big|^p - \lambda^k \int_\Omega b(x)\big|w_0^k\big|^p \leqslant 0 \tag{3.6.15}$$

and

$$\lambda^k \int_\Omega b(x)\big|w_0^k\big|^p \geqslant 1. \tag{3.6.16}$$

Now, we shall pass to the limit for $k \to \infty$, i.e. we have $\lambda^k \to \lambda_1$. We may assume again that

$$w_0^k \rightharpoonup w_0 \quad \text{in } W \text{ as } k \to \infty.$$

Using (A0) and (B0), we get from (3.6.14)–(3.6.16) that

$$\lambda_1 \int_\Omega b(x)|w_0|^p \geqslant 1 \quad (\text{i.e. } w_0 \not\equiv 0), \tag{3.6.17}$$

$$\int_\Omega |\nabla w_0|^p - \lambda_1 \int_\Omega b|w_0|^p \leqslant \liminf_{k\to\infty}\left[\int_\Omega \big|\nabla w_0^k\big|^p - \lambda^k \int_\Omega b\big|w_0^k\big|^p\right] \leqslant 0, \tag{3.6.18}$$

and

$$\int_\Omega a(x)|w_0|^q = \lim_{k\to\infty} \int_\Omega a(x)\big|w_0^k\big|^q = 0. \tag{3.6.19}$$

Now, (3.6.17) and (3.6.18) imply $w_0 = ku_1$, with $k \neq 0$. If we substitute w_0 into (3.6.19), we get

$$\int_\Omega a(x)|u_1|^q = 0,$$

a contradiction with (A2).

It follows from the above considerations that there exists $\epsilon_2 > 0$ such that problem (P_λ^2) admits at least one (nonnegative) solution $v_2 \in W$ for $\lambda_1 < \lambda < \lambda_1 + \epsilon_2$. □

3.6.3. *Proof of Theorem 3.6.1.* Set $\epsilon = \min(\epsilon_1, \epsilon_2)$ and consider $\lambda_1 < \lambda < \lambda_1 + \epsilon$. By applying the fibering method, we obtain weak solutions of (3.2.1) under condition (3.2.2), as in the previous subsections. Namely, let $t_1, t_2 > 0$ be determined by (3.3.4) for $v = v_1, v_2$. Then

$$u_1 := t_1 \cdot v_1, \qquad u_2 := t_2 \cdot v_2$$

are nonnegative weak solutions for the problem under consideration. The first weak solution u_1 belongs to the set

$$W_1 = \left\{u \in W\colon \int_\Omega |\nabla u|^p - \lambda \int_\Omega b(x)|u|^p > 0\right\},$$

because

$$\int_\Omega |\nabla u_1|^p - \lambda \int_\Omega b(x)|u_1|^p = (t_1)^p H_\lambda(v_1) = (t_1)^p > 0.$$

The second weak solution u_2 belongs to the set

$$W_2 = \left\{u \in W\colon \int_\Omega |\nabla u|^p - \lambda \int_\Omega b(x)|u|^p \leqslant 0\right\},$$

because

$$\int_\Omega |\nabla u_2|^p - \lambda \int_\Omega b(x)|u_2|^p = (t_2)^p H_\lambda(v_2) \leqslant 0.$$

Since

$$W_1 \cap W_2 = \emptyset,$$

we have

$$u_1 \not\equiv u_2.$$

Other properties of u_1 and u_2, such as positivity, L^∞-boundedness and $C^{1,\alpha}_{\text{loc}}$-regularity, can be derived the same way as in the proofs of the previous theorems.

3.7. *Problems on* $\mathbb{R}^N$

In this subsection we deal with equation (3.7.1) on the whole of $\mathbb{R}^N$. We study the problem

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u + a(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, \quad x \in \mathbb{R}^N, \end{cases} \tag{3.7.1}$$

where $0 \leqslant \lambda < \lambda_1$, $\lambda = \lambda_1$ and $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$ with some $\epsilon > 0$. Let us point out that results in this subsection are closely related to [22]. The method applied here is, however, completely different. The basic idea is to use the fibering method as in Subsections 3.4–3.6 but we point out some differences in proofs, which arise due to the character of the function space. More specially, we consider $u \in V$, where V is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_V := \left(\int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} w(x)|u|^p\right)^{1/p}$$

with

$$w(x) := \max\left\{b^-(x), \frac{1}{(1+|x|)^p}\right\}, \quad x \in \mathbb{R}^N.$$

V is a uniformly convex Banach space, and the notions of the weak solution to (3.7.1) and the eigenvalue (eigenfunction) of

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{3.7.2}$$

can be defined as in Subsection 3.1, where $u, v \in V$ and all integrals are taken over $\mathbb{R}^N$.

The following assertion is an analogy to Lemma 3.1.1. The proofs can be found in [3,22].

LEMMA 3.7.1. *Let $b^+ \not\equiv 0$ be as above. Then* (3.7.2) *has the first positive eigenvalue λ_1 characterized as the minimum of the Rayleigh quotient. Moreover, λ_1 is simple, isolated, and there exists the first eigenfunction u_1 positive in $\mathbb{R}^N$.*

In [22] it is also proved that $b^+ \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ implies the following:
(B1) the functional

$$u \mapsto \int_{\mathbb{R}^N} b^+(x)|u|^p$$

is weakly continuous on V.

If $q < p^*$, $q_1 = \frac{p^*}{p^*-q}$, $a \in L^\infty(\mathbb{R}^N) \cap L^{q_1}(\mathbb{R}^N)$, then
(A0') the functional

$$u \mapsto \int_{\mathbb{R}^N} a(x)|u|^q$$

is weakly continuous on V.

REMARK 3.7.2. In the case $p \geqslant N$, we replace p^* by $\tilde{p} = \frac{N_0 p}{N_0 - p}$, where $N_0 > p$ is an integer. If we assume $b \geqslant \eta$ with some $\eta > 0$ and $b \in L^\infty(\mathbb{R}^N) \cap L^{N_0/p}(\mathbb{R}^N)$, then Lemma 3.7.1 and (B1), (A0') remain true (cf. [21,22]).

REMARK 3.7.3. Note that the function $\omega(x)$ is exactly the weight function in the following Hardy's inequality:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{(1+|x|)^p} \leqslant \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p. \tag{3.7.3}$$

In particular, if $\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \leqslant c_1$, then we have

$$\int_{\mathbb{R}^N} \frac{|u|^p}{(1+|x|)^p} \leqslant c_1 \left(\frac{p}{N-p}\right)^p.$$

If, moreover, also $\int_{\mathbb{R}^N} b^-(x)|u|^p \leqslant c_2$, then we have

$$\|u\|^p_V \leqslant c,$$

where

$$c = c_1 + \max\left\{c_1\left(\frac{p}{N-p}\right)^p, c_2\right\}.$$

REMARK 3.7.4. Let $1 < p < N$. Assume that $\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \leqslant c$. Then it follows from the Hölder and the Sobolev inequalities that

$$\begin{aligned}\int_{\mathbb{R}^N} b^+(x)|u|^p &\leqslant \left(\int_{\mathbb{R}^N} \left(b^+(x)\right)^{N/p}\right)^{p/N} \cdot \left(\int_{\mathbb{R}^N} |u|^{p^*}\right)^{p/p^*} \\ &\leqslant c_1\left(\int_{\mathbb{R}^N} \left(b^+(x)\right)^{N/p}\right)^{p/N} \cdot \left(\int_{\mathbb{R}^N} |\nabla u|^p\right) \\ &\leqslant c \cdot c_1 \left\|b^+\right\|_{L^{N/p}(\mathbb{R}^N)}.\end{aligned} \tag{3.7.4}$$

In particular, it means that the integral on the left-hand side of (3.7.4) is uniformly bounded independently of u. It follows also from (3.7.4) that

$$\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \to 0$$

implies $\int_{\mathbb{R}^N} b^+(x)|u|^p \to 0$.

We formulate the main assertions only in the case $p < N$. For $p \geqslant N$ all assumptions can be modified in the sense of Remark 3.7.2.

THEOREM 3.7.5. *Let* $1 < p < q < p^*$, $p < N$, $0 \leqslant \lambda < \lambda_1$, $q_1 = \frac{p^*}{p^*-q}$, $b^+ \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, $a \in L^\infty(\mathbb{R}^N) \cap L^{q_1}(\mathbb{R}^N)$ *and* (A1) *be satisfied. Then problem* (3.7.1) *has at least one positive weak solution* $u \in V \cap L^\infty(\mathbb{R}^N)$. *Moreover,* $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

SKETCH OF THE PROOF. We can follow calculations in Subsection 3.4 where all integrals are now taken over $\mathbb{R}^N$ and the function space we work in is V. The main difference here consists in the fact that we assume (B1) instead of (B0) in Proposition 3.4.1 and $\int_{\mathbb{R}^N} |\nabla v|^p$ is no longer a norm on V. However, the proof of Proposition 3.4.1 can be performed in a similar way. Indeed, consider the set

$$V_\lambda := \left\{v \in V\colon\ H_\lambda(v) = 1\right\},$$

where

$$H_\lambda = \int_{\mathbb{R}^N} |\nabla v|^p - \lambda \int_{\mathbb{R}^N} b(x)|v|^p.$$

To prove that V_λ is bounded we have to proceed as follows. Due to the variational characterization of λ_1 we get (as in the proof of Proposition 3.4.1) for $0 \leqslant \lambda < \lambda_1$ that

$$\int_{\mathbb{R}^N} |\nabla v|^p \leqslant \frac{\lambda_1}{\lambda_1 - \lambda}.$$

Due to Remarks 3.7.3 and 3.7.4 also $\int_{\mathbb{R}^N} \frac{|v|^p}{(1+|x|)^p}$ and $\int_{\mathbb{R}^N} b^+(x)|v|^p$ are uniformly bounded independently of v. But then it follows from the condition $H_\lambda(v) = 1$ that also $\int_{\mathbb{R}^N} b^-(x)|v|^p$ is uniformly bounded independently of v. Hence there exists $c > 0$ such that for any $v \in V_\lambda$ we have

$$\|v\|_V \leqslant c$$

(see Remark 3.7.3). Consequently we may assume again that $v_n \rightharpoonup v_c$ in V. The rest of the proof runs along the same lines as the proof of Proposition 3.4.1 and Theorem 3.4.2 using (B1), (A0'), and the weak lower semicontinuity of $\int_{\mathbb{R}^N} |\nabla v|^p$ and $\int_{\mathbb{R}^N} b^-(x)|v|^p$. The *decay* of solution $u(x)$ as $|x| \to \infty$ follows from the estimate of Serrin [53] as in [19]. □

THEOREM 3.7.6. *Let* $1 < p < s < p^*$, $p < N$, $s_1 = \frac{p^*}{p^*-s}$, $b^+ \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, $a \in L^\infty(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N)$, (A1) *and* (A2) *be satisfied*. *Then problem* (3.7.1) *with* $\lambda = \lambda_1$ *has at least one positive weak solution* $u \in V \cap L^\infty(\mathbb{R}^N)$. *Moreover,* $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

SKETCH OF THE PROOF. The idea of the proof is based on an analogy with Proposition 3.5.1 when working with the space V instead of W. So we point out only the differences which occur in the proof due to the different norm in the space V. Let the maximizing sequence of (P_{λ_1}) be unbounded. Coming to (3.5.2), we have

$$0 \leqslant H_{\lambda_1}(w_n) = \frac{1}{t_n^p} \to 0. \tag{3.7.5}$$

This, together with Remarks 3.7.3 and 3.7.4 and with the fact that $\|w_n\|_V = 1$ imply

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^p \geqslant c > 0.$$

Indeed, if $\int_{\mathbb{R}^N} |\nabla w_n|^p \to 0$ then $\int_{\mathbb{R}^N} b^+(x)|w_n|^p \to 0$ and hence also

$$\int_{\mathbb{R}^N} b^-(x)|w_n|^p \to 0$$

due to (3.7.5). This implies $\|w_n\|_V \to 0$, a contradiction. Hence, instead of (3.5.3), we get

$$\liminf_{n\to\infty} \lambda_1 \int_{\mathbb{R}^N} b^+(x)|w_n|^p \geqslant c > 0, \tag{3.7.6}$$

and for the weak limit w_0 of $\{w_n\}_{n=1}^{\infty}$ we have due to (B1):

$$\int_{\mathbb{R}^N} b^+(x)|w_0|^p \geqslant \frac{c}{\lambda_1} > 0,$$

i.e. $w_0 \not\equiv 0$. Using (A0), the weak lower semicontinuity of $\int_{\mathbb{R}^N} |\nabla w|^p$ and $\int_{\mathbb{R}^N} b^-(x)|w|^p$ and Lemma 3.2.3, we can proceed as in the proof of Proposition 3.5.1 and derive a contradiction with (A2).

Similar arguments yield a contradiction also in the case of bounded maximizing sequence of (P_{λ_1}). The rest of the proof is identical to that of Theorem 3.5.2. □

THEOREM 3.7.7. *Let* $1 < p < s < p^*$, $p < N$, $s_1 = \frac{p^*}{p^*-s}$, $b \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, $a \in L^\infty(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N)$, (A1) *and* (A2) *be satisfied. Then there exists* $\epsilon > 0$ *such that for any* $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$ *problem* (3.7.1) *has two distinct positive solutions* $u_1, u_2 \in V \cap L^\infty(\mathbb{R}^N)$, *and both belong to* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

The proof follows the lines of the proof of Theorem 3.6.1. The only differences which occur are caused again by the norm in V and so, in order to prove analogies of Propositions 3.6.4 and 3.6.6, we have to modify their proofs in the sense mentioned above (cf. the sketches of the proofs of Theorems 3.7.5 and 3.7.6).

3.8. *Nonexistence results*

Here we use a variational identity to show that our assumptions are essential for existence of solutions.

Let us assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which has the following property: there exists a unit normal vector $\nu(x) = (\nu_1(x), \dots, \nu_N(x))$ at every point $x \in \partial\Omega$ and

$$\sum_{i=1}^{N} x_i \nu_i(x) \geqslant 0 \tag{3.8.1}$$

for any $x = (x_1, \dots, x_N) \in \partial\Omega$. We assume that $b, c \in C^1(\overline{\Omega})$ and denote for $x \in \Omega$

$$\langle b', x\rangle = \sum_{i=1}^{N} x_i \frac{\partial b_i}{\partial x_i}, \qquad \langle c', x\rangle = \sum_{i=1}^{N} x_i \frac{\partial c_i}{\partial x_i}.$$

Let us consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u + c(x)|u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8.2}$$

and assume that $u_0 \in W^{2,p_1}(\Omega),\ p_1 > N$ is a nontrivial solution to (3.8.2). Following [41] and taking (3.8.1) into account, we obtain from here that necessarily

$$\begin{aligned} &\frac{N-p}{p}\int_\Omega |\nabla u_0|^p\,dx + \int_\Omega \left[-\frac{\lambda N}{p} b(x) - \frac{\lambda}{p}\langle b', x\rangle\right]|u_0|^p\,dx \\ &\quad + \int_\Omega \left[-\frac{N}{s} c(x) - \frac{1}{s}\langle c', x\rangle\right]|u_0|^s\,dx = -\left(1 - \frac{1}{p}\right)\int_{\partial\Omega} |\nabla u_0|^p \sum_{i=1}^{N} x_i \nu_i\,dS. \end{aligned}$$

Since also (by using u_0 as a test function)

$$\int_\Omega |\nabla u_0|^p\,dx - \lambda\int_\Omega b(x)|u_0|^p\,dx - \int_\Omega c(x)|u_0|^s\,dx = 0,$$

we obtain that the following identity holds for any $\alpha \in \mathbb{R}$:

$$\begin{aligned} &\left(\frac{N-p}{p} + \alpha\right)\int_\Omega |\nabla u_0|^p\,dx + \int_\Omega \left[\left(-\frac{\lambda N}{p} - \alpha\lambda\right) b(x) - \frac{\lambda}{p}\langle b', x\rangle\right]|u_0|^p\,dx \\ &\quad + \int_\Omega \left[\left(-\frac{N}{s} - \alpha\right) c(x) - \frac{1}{s}\langle c', x\rangle\right]|u_0|^s\,dx \\ &\qquad = -\left(1 - \frac{1}{p}\right)\int_{\partial\Omega} |\nabla u_0|^p \sum_{i=1}^{N} x_i \nu_i\,dS. \end{aligned} \tag{3.8.3}$$

Now it follows from (3.8.3) that the following inequalities

$$\begin{aligned} &\frac{N-p}{p} + \alpha \geqslant 0, \\ &\left(-\frac{\lambda N}{p} - \alpha\lambda\right) b(x) - \frac{\lambda}{p}\langle b', x\rangle \geqslant 0, \\ &\left(-\frac{N}{s} - \alpha\right) c(x) - \frac{1}{s}\langle c', x\rangle \geqslant 0 \end{aligned} \tag{3.8.4}$$

cannot hold simultaneously with at least one strict inequality sign. So we have the following *nonexistence result*.

THEOREM 3.8.1. *Let* $b, c \in C^1(\overline{\Omega})$, *and let* $\partial\Omega$ *have the property stated above. Let* (3.8.4) *hold with at least one strict inequality sign. Then the boundary value problem* (3.8.2) *has no nontrivial solution* $u \in W^{2,p_1}(\Omega)$ *with* $p_1 > N$.

Let us consider a special case of the boundary value problem (3.8.2) with $p < N$:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8.5}$$

i.e. $b(x) \equiv c(x) \equiv 1$. Clearly, we have $\langle b', x\rangle = \langle c', x\rangle = 0$, $x \in \Omega$ in this case. Elementary calculation yields that for $\alpha = \frac{p-N}{p}$, $\lambda \leqslant 0$ and $s > \frac{Np}{N-p}$ inequalities (3.8.4) hold with strict sign in the third one. So we have the following assertion.

COROLLARY 3.8.2. *Let* $\partial\Omega$ *have the property stated above,* $\lambda \leqslant 0$ *and* $p < N$, $s > p^*$. *Then the boundary value problem* (3.8.5) *has no nontrivial solution* $u \in W^{2,p_1}(\Omega)$ *with* $p_1 > N$.

Let us assume now that $\Omega = \mathbb{R}^N$, $b^+(x) \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, and $b^-(x) \in L^\infty(\mathbb{R}^N)$. We consider $u_0 \in V$, where V is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_V := \left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{\mathbb{R}^N} w(x)|u|^p\,dx\right)^{1/p}$$

with

$$w(x) := \max\left\{b^-(x), \frac{1}{(1+|x|)^p}\right\}, \quad x \in \mathbb{R}^N.$$

Then there exists a sequence of balls $B_{R_k} := \{x \in \mathbb{R}^N\colon\ |x| \leqslant R_k\}$ with $R_k \to +\infty$ $(k \to +\infty)$ such that

$$\int_{\partial B_{R_k}} |\nabla u_0|^p\,dS \to 0. \tag{3.8.6}$$

Indeed, we have

$$\int_{B_{R_k}} |\nabla u_0|^p\,dx = \int_0^{R_k}\left(\int_{\partial B_r} |\nabla u_0|^p\,dS\right)dr. \tag{3.8.7}$$

If we assume the contrary to (3.8.6), i.e.

$$\liminf_{R\to\infty} \int_{\partial B_R} |\nabla u_0|^p\,dS = \epsilon > 0$$

with some $\epsilon > 0$, due to (3.8.7) we get that

$$\int_{B_{R_k}} |\nabla u_0|^p \, dx \to +\infty$$

as $k \to \infty$, a contradiction with $u_0 \in V$. It follows from (3.8.6) that considering $\Omega_k = B_{R_k}$ we arrive at (3.8.3) where $\Omega = \mathbb{R}^N$ and the right-hand side is identically equal to zero:

$$\left(\frac{N-p}{p} + \alpha\right) \int_{\mathbb{R}^N} |\nabla u_0|^p \, dx + \int_{\mathbb{R}^N} \left[\left(-\frac{\lambda N}{p} - \alpha\lambda\right) b(x) - \frac{\lambda}{p}\langle b', x\rangle\right] \cdot |u_0|^p \, dx + \int_{\mathbb{R}^N} \left[\left(-\frac{N}{s} - \alpha\right) c(x) - \frac{1}{s}\langle c', x\rangle\right] |u_0|^s \, dx = 0.$$

Then (3.8.4) follows but we have now that also opposite inequalities:

$$\frac{N-p}{p} + \alpha \leqslant 0,$$
$$\left(-\frac{\lambda N}{p} - \alpha\lambda\right) b(x) - \frac{\lambda}{p}\langle b', x\rangle \leqslant 0,$$
$$\left(-\frac{N}{s} - \alpha\right) c(x) - \frac{1}{s}\langle c', x\rangle \leqslant 0 \tag{3.8.8}$$

cannot hold simultaneously with at least one strict inequality sign. So we have

THEOREM 3.8.3. *Assume $b, c \in C^1(\mathbb{R}^N)$. Let either* (3.8.4) *or* (3.8.8) *hold with at least one strict inequality sign. Then the boundary value problem*

$$-\Delta_p u = \lambda b(x)|u|^{p-2}u + c(x)|u|^{s-2}u \quad \text{in } \mathbb{R}^N$$

has no nontrivial solution $u \in W^{2,p_1}_{\mathrm{loc}}(\mathbb{R}^N) \cap V$ with $p_1 > N$.

In the special case

$$-\Delta_p u = \lambda |u|^{p-2}u + |u|^{s-2}u \quad \text{in } \mathbb{R}^N \tag{3.8.9}$$

we get

COROLLARY 3.8.4. *Let either $\lambda \leqslant 0$, $p < N$, $s > p^*$ or $\lambda > 0$, $p < N$, $s < p^*$. Then the boundary value problem* (3.8.9) *has no nontrivial solution $u \in W^{2,p_1}_{\mathrm{loc}}(\mathbb{R}^N) \cap V$ with $p_1 > N$.*

4. Positive solutions to Neumann problems

Assume Ω is a connected and bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The problem we are interested in deals with the construction of multiple positive solutions of equations of the following type:

$$\begin{cases} -\Delta_p u + \langle \nabla\psi, \nabla u\rangle |\nabla u|^{p-2} = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = g(x,u) & \text{on } \partial\Omega \end{cases} \tag{4.0.1}$$

($p > 1$) for some specific f and g the assumptions on which will be stated below. When $p = 2$, this type of problems can be studied via classical bifurcation techniques, but this approach is uneasy when $p \neq 2$, although some results in this direction have been obtained by Véron in [57,58] in the framework of nonmonotone perturbations of monotone operators of p-Laplacian type. The problems studied here are of nonmonotone type and we shall work in the framework of the fibering method.

We shall mostly assume that f and g have the following form:

$$f(x,u) = \lambda b(x)|u|^{p-2}u + c(x)|u|^{s-2}u + a(x)|u|^{q-2}u, \tag{4.0.2}$$

$$g(x,u) = k(x)|u|^{r-2}u, \tag{4.0.3}$$

where a, b, c and k are bounded and measurable functions defined on Ω and $\partial\Omega$. In order to avoid noncompactness we shall always assume that the problem is subcritical, in the sense that we define two critical exponents, one for Ω:

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N, \\ \infty & \text{if } p \geqslant N, \end{cases} \tag{4.0.4}$$

and the other for $\partial\Omega$:

$$\tilde{p} = \begin{cases} \frac{N(p-1)}{N-p} & \text{if } 1 < p < N, \\ \infty & \text{if } p \geqslant N, \end{cases} \tag{4.0.5}$$

and our assumption is that the exponents q and s (respectively r) are smaller than p^* (respectively $\tilde{p}$). (Note that $p < p^*$ automatically.) Concerning λ, we shall always assume its positivity. By a solution of (4.0.1), we mean a $W^{1,p}(\Omega)$-function which is a critical point of the functional

$$E(v) = \int_\Omega \left(-\frac{1}{p}|\nabla v|^p + \frac{\lambda}{p} b|v|^p + \frac{1}{s} c|v|^s + \frac{1}{q} a|v|^q \right) \rho\, dx + \frac{1}{r}\int_{\partial\Omega} k|v|^r \rho\, ds,$$
$$\text{with } \rho = e^{\psi} \tag{4.0.6}$$

and therefore satisfies

$$E(v) = \int_\Omega \big(|\nabla v|^{p-2}(\nabla v, \nabla\eta) - f(x,u)\eta\big)\rho\,dx - \int_{\partial\Omega} g(x,u)\eta\rho\,ds = 0$$

for every $\eta \in W^{1,p}(\Omega)$. In this formula we identify the trace of v on $\partial\Omega$ with $\gamma_0(v)$, where γ_0 is the trace operator from $W^{1,p}(\Omega)$ into $W^{1-1/p,p}(\partial\Omega)$.

This section is organized as follows:

– Subsection 4.1: the case $p = 2$, $\psi \equiv 0$, $b \equiv 0 \not\equiv c$, $k \not\equiv 0$.
– Subsection 4.2: the case $p = 2$, $\psi \not\equiv 0$, $b \equiv c \equiv 0$, $k \equiv 0$.
– Subsection 4.3: the case $p > 1$, $\psi \not\equiv 0$, $b \equiv c \equiv 0$, $k \equiv 0$.
– Subsection 4.4: the case $p > 1$, $\psi \not\equiv 0$, $b \not\equiv c \equiv 0$, $k \equiv 0$.
– Subsection 4.5: the case $p > 1$, $\psi \equiv 0$, $k \equiv 0 \equiv c$.
– Subsection 4.6: the case $p > 1$, $\psi \equiv 0$, $a \equiv 0 \equiv c$.
– Subsection 4.7: the case $p > 1$, $\psi \equiv 0$, $r < q$.
– Subsection 4.8: the case $p > 1$, $\psi \equiv 0$, $r = q$.
– Subsection 4.9: the case $p > 1$, with more general nonlinear boundary conditions.

Most of these results were obtained jointly with A. Tesei in [49,50] (Subsections 4.1–4.4) and with L. Véron in [51] (Subsections 4.5–4.8).

4.1. *The homogeneous semilinear case*

In this subsection we deal with positive solutions of the following class of nonlinear Neumann problems:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \subset \mathbb{R}^N, \\ \frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and ν is the external normal to $\partial\Omega$.

Here $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is a function from the class $C^\alpha(\overline{\Omega} \times \mathbb{R}^+)$, $\alpha \in (0,1)$, satisfying the inequality

$$\big|f(x,u)\big| \leqslant c\big(1 + u^{s-1}\big) \tag{4.1.2}$$

for $u \in \mathbb{R}^+ = [0, +\infty)$ with a certain constant $c > 0$ and

$$2 < s < 2^*,$$

where

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N = 1, 2. \end{cases}$$

Throughout this subsection we assume that the function $g : \partial\Omega \times \mathbb{R}^+ \to \mathbb{R}$, $g \in C^\alpha(\overline{\Omega} \times \mathbb{R}^+)$, satisfies the inequality

$$\left|g(x,u)\right| \leqslant c_1\left(1 + u^{r-1}\right) \tag{4.1.3}$$

for $u \in \mathbb{R}^+$ with a certain constant $c_1 > 0$ and

$$2 < r < \tilde{2} = \begin{cases} \frac{2(N-1)}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N = 1, 2. \end{cases}$$

It stands to reason that the functions f and g satisfy additional conditions which imply the existence of positive solutions to (4.1.1).

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega \in C^{2,\alpha}$. We shall work in the Sobolev space $W^{1,2}(\Omega)$.

We shall define some necessary conditions for the existence of a positive solution to (4.1.1). Multiplying the equation from (4.1.1) by u and integrating by parts, with due account of Neumann boundary conditions, we obtain

$$\int_\Omega f(x,u)\,dx + \int_{\partial\Omega} g(x,u)\,d\sigma = 0. \tag{4.1.4}$$

We consider the function

$$v(x) = G\big(x, u(x)\big),$$

where $G \in C^2(\overline{\Omega}, \mathbb{R})$. Then, if $u(x)$ is a solution to (4.1.1), we have

$$\Delta v = \Delta_x G + 2\langle \nabla_x G_u, \nabla u\rangle + G_{uu}|\nabla u|^2 - G_u f(x,u) \tag{4.1.5}$$

and

$$\frac{\partial u}{\partial \nu} = G_\nu(x,u) + G_u g(x,u). \tag{4.1.6}$$

Here G_ν is a projection of $\nabla_x G$ onto the direction of the external normal ν to $\partial\Omega$.

Next, integrating (4.1.5) by parts and considering (4.1.6), we get

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma &= \int_\Omega \Delta_x G\,dx + 2\int_\Omega \langle \nabla_x G_u, \nabla u\rangle\,dx \\ &\quad + \int_\Omega G_{uu}|\nabla u|^2\,dx - \int_\Omega G_u f(x,u)\,dx. \end{aligned} \tag{4.1.7}$$

However,

$$\Delta_x G = \sum \frac{\partial^2 G}{\partial x_i^2} = \sum \frac{\partial}{\partial x_i} G_{x_i}(x,u) = \sum \frac{d}{dx_i} G_{x_i}(x,u) - \sum G_{x_i u}\frac{\partial u}{\partial x_i}.$$

Hence,

$$\int_{\Omega} \Delta_x G\, dx = \int_{\partial\Omega} G_\nu(x,u)\, d\sigma - \int_{\Omega} \langle \Delta_x G_u, \nabla u\rangle\, dx.$$

Then (4.1.7) implies

$$\begin{aligned}
&\int_{\partial\Omega} G_\nu(x,u)\, d\sigma + \int_{\partial\Omega} G_u g(x,u)\, d\sigma \\
&\quad = \int_{\partial\Omega} G_\nu(x,u)\, d\sigma + \int_{\Omega} \langle \Delta_x G_u, \nabla u\rangle\, dx + \int_{\Omega} G_{uu} |\nabla u|^2\, dx \\
&\qquad - \int_{\Omega} G_u f(x,u)\, dx.
\end{aligned}$$

Thus, we have

$$\int_{\Omega} \langle \Delta_x G_u, \nabla u\rangle\, dx + \int_{\Omega} G_{uu} |\nabla u|^2\, dx - \int_{\Omega} G_u f(x,u)\, dx = \int_{\partial\Omega} G_u g(x,u)\, d\sigma. \tag{4.1.8}$$

We now take

$$G = G(u)$$

and study the case

$$\begin{aligned}
g(x,u) &= a(x) g_0(u), \\
f(x,u) &= b(x) f_0(u).
\end{aligned}$$

Then (4.1.8) gives

$$\int_{\Omega} G'(u) b(x) f_0(u)\, dx + \int_{\partial\Omega} G'(u) a(x) g_0(u)\, d\sigma = \int_{\Omega} G_{uu} |\nabla u|^2\, dx. \tag{4.1.9}$$

This implies

COROLLARY 4.1.1. *Let $b(x) \equiv 0$. Then*

$$\int_{\partial\Omega} G'(u) a(x) g_0(u)\, d\sigma = \int_{\Omega} G_{uu} |\nabla u|^2\, dx. \tag{4.1.10}$$

Now, if we additionally assume that the function $g_0(u)$ is such that there exists

$G : \mathbb{R}^+ \to \mathbb{R}$ *from the class C^2*, *for which*

$$G'(u)g_0(u) = 1,$$

then we obtain from (4.1.10)

$$\int_{\partial\Omega} a(x)\,d\sigma = -\int_{\Omega} \frac{g_0'(u)}{g_0^2(u)} |\nabla u|^2\,dx. \tag{4.1.11}$$

Finally, if we assume that

$$G_0'(u) > 0 \quad \textit{for } u > 0,$$

then we get

$$\int_{\partial\Omega} a(x)\,d\sigma < 0. \tag{4.1.12}$$

This is a necessary condition for the existence of a positive solution to the problem

$$\begin{cases} -\Delta u = 0 & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ \frac{\partial u}{\partial \nu} = a(x)g_0(u) & \textit{on } \partial\Omega \end{cases} \tag{4.1.13}$$

with

$$\begin{cases} g_0(u) \neq 0 & \textit{for } u > 0, \\ g_0'(u) > 0 & \textit{for } u > 0. \end{cases}$$

EXAMPLE 4.1.2. Consider $g_0(u) = u^{r-1}$ with $r > 1$. Then (4.1.10) takes the form

$$\int_{\partial\Omega} G'(u)a(x)u^{r-1}\,d\sigma = \int_{\Omega} G''(u)|\nabla u|^2\,dx. \tag{4.1.14}$$

We choose $G : \mathbb{R}^+ \to \mathbb{R}$ according to the formula

$$G'(u)u^{r-1} = 1,$$

i.e.

$$G'(u) = \frac{1}{u^{r-1}},$$

$$G(u) = \frac{1}{(2-r)u^{r-2}},$$

$$G''(u) = \frac{1-r}{u^r}.$$

We use a regularization procedure. Instead of G we introduce

$$G_\varepsilon(u) = \frac{1}{(2-r)(u+\varepsilon)^{r-2}}$$

with $\varepsilon > 0$.

Then (4.1.14) takes the form

$$\int_{\partial\Omega} a(x)\frac{u^{r-1}}{(u+\varepsilon)^{r-1}}\,d\sigma = -(r-1)\int_\Omega \frac{1}{(u+\varepsilon)^r}|\nabla u|^2\,dx.$$

Setting $\varepsilon \to 0$ and repeating the previous reasoning, we obtain

$$\int_{\partial\Omega} a(x)\,d\sigma < 0. \tag{4.1.15}$$

Now let us consider the variational boundary value problem

$$\begin{cases} -\Delta u = b(x)u^{s-1} & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = a(x)u^{r-1} & \text{on } \partial\Omega. \end{cases} \tag{4.1.16}$$

Here, the functions a and b are assumed to be continuous in $\partial\Omega$ and $\overline{\Omega}$, respectively, and at least one of them is not identically zero. The exponents s and r are assumed to satisfy the condition

(H0) $1 < s < 2^*$, $1 < q < \tilde{2}$,

where 2^* and $\tilde{2}$ are the constants defined above.

Problem (4.1.16) occurs in some mathematical models in applied sciences (see, e.g., [4,36,47]). It is also related to the class of boundary value problems arising in the theory of conformal transformations of Riemannian metrics (see [44], as well as [38] and the references therein). The case $a \equiv 0$ was examined for $1 < s < 2$ in [9] and for $2 < s < 2^*$ in [12]. The case $a \not\equiv 0$, $q = 1$ was analyzed in [8].

We consider the existence of positive solutions to problem (4.1.16) in the general case. Integration by parts yields the equality

$$\int_\Omega b(x)u^{s-1}\,dx + \int_{\partial\Omega} a(x)u^{r-1}\,d\sigma = 0. \tag{4.1.17}$$

Hence, any nonnegative solution of (4.1.16) is trivial if a and b have the same constant sign.

In what follows, we consider only the case

(H1) $b \leqslant 0$ in Ω.

From (4.1.17) there follows a necessary condition for the existence of nontrivial nonnegative solutions to problem (4.1.16):

(H2) $a_+ := \max\{a, 0\} \not\equiv 0$.

If $b \equiv 0$, a further necessary condition for the existence of such solutions is

(H3) $\int_{\partial\Omega} a\, d\sigma < 0$

(see Proposition 4.1.6). In view of (H2) and (H3), the coefficient a must be alternating on $\partial\Omega$ if $b \equiv 0$. Note that conditions similar to (H2) and (H3) are imposed on the coefficient b if $a \equiv 0$ [9,12].

In what follows, the conditions (H0)–(H3) are assumed to hold. When $b \equiv 0$, it is possible to prove the following result.

THEOREM 4.1.3. *Let assumptions* (H0), (H2), *and* (H3) *be satisfied, and let* $b \equiv 0$. *Then there exists a nontrivial nonnegative solution* $u \in H^1(\Omega) \cap L^\infty(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *to problem* (4.1.16).

In the case (H1), the relation between s and r is similar. Namely, if

(A) $r > \max\{2, s\}$,

(B) $r < \min\{2, s\}$,

it is possible to prove the following theorem.

THEOREM 4.1.4. *Let conditions* (A) *or* (B) *be fulfilled and assumptions* (H0)–(H3) *hold. Then there exists a nontrivial nonnegative solution* $u \in H^1(\Omega) \cap L^\infty(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *to problem* (4.1.16).

Note that similar results are valid for the more general problem

$$\begin{cases} -\Delta_p u = b(x)u^{s-1} & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = a(x)u^{r-1} & \text{on } \partial\Omega, \end{cases} \tag{4.1.18}$$

where Δ_p denotes the p-Laplacian, $p > 1$ (see next subsections).

We work in the Sobolev space $X = H^1(\Omega)$ with an ordinary norm. As usual, a weak solution to problem (4.1.16) is defined as a critical point of the functional

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{r}\int_{\partial\Omega} a|u|^r\, d\sigma + \frac{1}{s}\int_\Omega b|u|^s\, dx.$$

By assumption (H0), we have $H^1(\Omega) \hookrightarrow L^s(\Omega) \cap L^r(\partial\Omega)$; therefore, f is defined on X.

To examine the critical points of f, we relate it to the functional $F : \mathbb{R} \times X \to \mathbb{R}$ assuming that, for every $t \in \mathbb{R}$ and $v \in X$,

$$F(t, v) := f(tv) = -\frac{|t|^2}{2}\int_\Omega |\nabla v|^2\, dx + \frac{|t|^r}{r}A(v) - \frac{|t|^s}{s}B(v) \tag{4.1.19}$$

where

$$A(v) := \int_{\partial\Omega} a|v|^r\, d\sigma,$$

$$B(v) := \int_\Omega b|v|^s\, dx$$

(note that here we used assumption (H1)).

If $u = tv$ is a critical point of f, then the bifurcation equation

$$f_t(t, v) = A(v)|t|^{r-2}t - B(v)|t|^{s-2}t - \int_\Omega |\nabla v|^2\, dx \cdot t = 0 \tag{4.1.20}$$

holds. Assume that, for any v in an open subset $E \subset X \setminus \{0\}$, there exists a root $t = t(v) \neq 0$ of (4.1.20). Let $t(\cdot) \in C^1(E)$. Then the reduced functional

$$\tilde{f}(v) := F\big(t(v), v\big)$$

is defined and continuously differentiable in E. It is possible to prove the following result (see Theorems 2.2.1, 2.2.2).

THEOREM 4.1.5. *Let $v \in E$ be a constrained extremum point of the functional $\tilde{f}(v)$ with $t(v) \neq 0$ under the condition*

$$\int_\Omega |\nabla v|^2\, dx = 1. \tag{4.1.21}$$

Then $u := t(v)v$ is a nonzero critical point of f.

The previous remark suggests the following approach to the investigation of critical points of $\tilde{f}$. First, the bifurcation equation (4.1.19) is analyzed, and the reduced functional (4.1.20) is determined. Next, following Theorem 4.1.4, $\tilde{f}$ is maximized (or minimized) under constraint (4.1.21).

Note that, if $t \neq 0$, the bifurcation equation (4.1.19) is equivalent to

$$\phi(t, v) = \int_\Omega |\nabla v|^2\, dx,$$

where

$$\phi(t, v) := A(v)|t|^{r-2} - B(v)|t|^{s-2}. \tag{4.1.22}$$

Put

$$E := \big\{v \in X \;\big|\; A(v) > 0\big\}; \tag{4.1.23}$$

by assumption (H2), the set E is nonempty. An elementary inspection of ϕ shows that, for any $v \in E$, the bifurcation equation has a unique positive root $t = t(v)$ if either $b \equiv 0$, (A), or (B) holds. Moreover, for any $v \in E$, we have

$$F_{tt}(t(v), v) = (r-2)\int_\Omega |\nabla v|^2\, dx + (r-s)B(v)\big|t(v)\big|^{s-2} \neq 0;$$

hence, $t \in C^1(E)$ (note that, by virtue of (H3), E does not contain constant functions). Consider the reduced functional

$$\begin{aligned}\tilde{f}(v) &= \left(\frac{1}{r} - \frac{1}{2}\right) \int_\Omega |\nabla v|^2\,dx \big|t(v)\big|^2 + \left(\frac{1}{r} - \frac{1}{s}\right) B(v) \big|t(v)\big|^s \\ &= \left(\frac{1}{r} - \frac{1}{2}\right) A(v) \big|t(v)\big|^r + \left(\frac{1}{r} - \frac{1}{s}\right) B(v) \big|t(v)\big|^s. \end{aligned} \tag{4.1.24}$$

Let us prove the following result.

PROPOSITION 4.1.6. *Let assumptions* (H0)–(H3) *be fulfilled. Then in both cases*

(i) $b \equiv 0$ *in* Ω,

(ii) (A) *or* (B) *holds*,

a maximum

$$\max_{x \in E} \tilde{f}(v) \text{ under the constraint } \int_\Omega |\nabla v|^2\,dx = 1 \tag{4.1.25}$$

is reached on a function $\bar{v} \geqslant 0$, $\bar{v} \not\equiv 0$ *in* Ω.

Theorems 4.1.3 and 4.1.4 follow immediately from Proposition 4.1.6.

Let us prove Proposition 4.1.6. The proof follows that used in [49] in an analogous situation and is given here for the convenience of the reader.

PROOF OF PROPOSITION 4.1.6. (a) Put

$$S := \left\{ v \in X \;\Big|\; \int_\Omega |\nabla v|^2 dx = 1 \right\}.$$

Let us prove that the set $E \cap S$ is bounded in X by contradiction. Suppose that there exists $\{v_n\} \subset E \cap S$ such that

$$\int_\Omega |v_n|^2\,dx + \int_\Omega |\nabla v_n|^2\,dx \to \infty$$

as $n \to \infty$. Let

$$v_n = t_n + w_n,$$

where

$$t_n := \frac{1}{|\Omega|} \int_\Omega v_n\,dx, \qquad w_n := v_n - t_n.$$

Since

$$\int_\Omega |\nabla w_n|^2\,dx = \int_\Omega |\nabla v_n|^2\,dx = 1,$$

$$\int_\Omega w_n\,dx = 0,$$

by the Poincaré embedding theorem, there exists $C > 0$ such that

$$\|w_n\|_X \leqslant C \quad \text{for any } n \in \mathbb{N}.$$

Then the above assumption implies that $|t_n| \to \infty$. Moreover, by the second inequality in (H0), the space X is compactly embedded into $L^q(\partial\Omega)$; therefore, it is possible to assume that the sequence of traces of $\{w_n\}$ converges in this space. Then, by assumption (H3), we have

$$\int_{\partial\Omega} a|v_n|^r\,d\sigma = |t_n|^r \int_{\partial\Omega} a\left|1 + \frac{w_n}{t_n}\right|^r d\sigma \to -\infty.$$

We obtained a contradiction to the definition of E, which completes the proof.

(b) Consider only case (ii), because case $b = 0$ is simpler by virtue of the homogeneity of the reduced functional.

Put

$$M := \sup\bigl\{\tilde{f}(v) \bigm| v \in E \cap S\bigr\}, \tag{4.1.26}$$

where $\tilde{f}$ is the reduced functional (4.1.25). It is easy to see that $M \in (-\infty, 0)$ if (A) holds or $M \in (0, \infty)$ if (B) is satisfied. Let $\{v_n\} \subset E \cap S$ be a maximizing sequence. By virtue of the results of (a), we can assume that $\{v_n\}$ weakly converges in X to some $\bar{v}$. By assumption (H0), it follows that $v_n \to \bar{v}$ in both $L^s(\Omega)$ and $L^r(\partial\Omega)$ (more precisely, the traces of v_n strongly converge to the trace of $\bar{v}$ in $L^r(\partial\Omega)$). Let us prove that $\bar{v} \in E \cap S$.

Because $\{v_n\} \subset E \cap S$, the bifurcation equation yields

$$A(v_n)\bigl|t(v_n)\bigr|^{r-2} \geqslant 1 \quad \text{for any } n \in \mathbb{N}. \tag{4.1.27}$$

On the other hand, since $v_n \to \bar{v}$ in $L^r(\partial\Omega)$, we have

$$A(v_n) \to A(\bar{v}) \quad \text{as } n \to \infty.$$

Assume the converse. Let $A(\bar{v}) = 0$. If (A) is fulfilled, then rewriting (4.1.27) as

$$\bigl|t(v_n)\bigr| \geqslant \bigl[A(v_n)\bigr]^{-1/(r-2)},$$

we conclude that $|t(v_n)| \to \infty$. By virtue of (4.1.25),

$$\tilde{f}(v_n) \leqslant \left(\frac{1}{r} - \frac{1}{2}\right)\bigl|t(v_n)\bigr|^2$$

for any $n \in \mathbb{N}$; i.e., $\tilde{f}(v_n) \to -\infty$, which is impossible. If (B) is fulfilled, then writing (4.1.27) as

$$A(v_n) \geqslant \left|t(v_n)\right|^{2-r}, \tag{4.1.28}$$

we see that $|t(v_n)| \to 0$. Since $B(v_n) \to B(\bar{v}) < \infty$, $\tilde{f}(v_n) \to 0$, which contradicts the inequality $M > 0$. Therefore, $\bar{v} \in E$.

Let us show that $\bar{v} \in S$. The weak convergence of $\{v_n\}$ in X implies that

$$\int_\Omega |\nabla \bar{v}|^2 \, dx \leqslant 1.$$

It follows from $A(\bar{v}) > 0$ and (H3) that

$$\int_\Omega |\nabla \bar{v}|^2 \, dx > 0.$$

If the first inequality is nonstrict, it is possible to find a $k > 1$ such that

$$\int_\Omega \left|\nabla (k\bar{v})\right|^2 dx = 1;$$

therefore, $k\bar{v} \in E \cap S$. The root $t = t(k\bar{v})$ of the bifurcation equation satisfies the equation

$$A(k\bar{v})\left|t(k\bar{v})\right|^{r-2} - B(k\bar{v})\left|t(k\bar{v})\right|^{s-2} = 1. \tag{4.1.29}$$

Since

$$A(k\bar{v}) = k^r A(\bar{v}),$$
$$B(k\bar{v}) = k^s B(\bar{v}),$$

this gives

$$A(\bar{v})\left|kt(k\bar{v})\right|^{r-2} - B(\bar{v})\left|kt(k\bar{v})\right|^{s-2} = k^{-2} < 1. \tag{4.1.30}$$

On the other hand, it is easy to see that the sequence $\{t(v_n)\}$ is bounded. Indeed, in case (B), this follows from inequality (4.1.28). In case (A), we rewrite the bifurcation equation for $v = v_n$ as

$$\left|t(v_n)\right|^{r-2}\left\{A(v_n) - B(v_n)\left|t(v_n)\right|^{s-r}\right\} = 1.$$

Because $A(v_n) \to A(\bar{v}) > 0$ and $\{B(v_n)\}$ converges, it follows that, for any divergent subsequence $\{t(v_n)\}$, the left-hand side of the above equality is divergent, which is impossible.

Since $\{t(v_n)\}$ is bounded, some of its subsequences is convergent. Therefore, its limit $\bar{t}$ satisfies the relation

$$A(\bar{v})|\bar{t}|^{r-2} - B(\bar{v})|\bar{t}|^{s-2} = 1. \tag{4.1.31}$$

Comparing (4.1.30) with (4.1.31) immediately yields

$$kt(k\bar{v}) < \bar{t},$$

if (A) holds, and

$$kt(k\bar{v}) > \bar{t},$$

if (B) is fulfilled. Furthermore, an elementary inspection of the function

$$\psi(\xi) := \left(\frac{1}{r} - \frac{1}{2}\right)A(\bar{v})\xi^r - \left(\frac{1}{s} - \frac{1}{2}\right)B(\bar{v})\xi^s, \tag{4.1.32}$$

$$\xi > 0$$

shows that

$$\tilde{f}(k\bar{v}) = \psi\big(k\big|t(k\bar{v})\big|\big) > \psi(\bar{t}) = M \tag{4.1.33}$$

in both cases. Thus, we arrive at a contradiction. This means that $\bar{v} \in S$, which was to be proved.

Because (4.1.33) holds with $k = 1$, we take $t(\bar{v}) = \bar{t}$ (see (4.1.31)); thus, $M = \tilde{f}(\bar{v})$, from which the required conclusion follows. □

Finally, let us prove the following result.

PROPOSITION 4.1.7. *Let $b \equiv 0$ and $a \not\equiv 0$. Assume that there exists a nontrivial nonnegative solution to problem* (4.1.16). *Then condition* (H3) *is satisfied*.

PROOF. Following [9], we define $h_\varepsilon : \overline{\Omega} \to \mathbb{R}$, $\varepsilon > 0$, assuming

$$h_\varepsilon := -\frac{(u+\varepsilon)^{-(r-2)}}{r-2}.$$

It is easy to see that h_ε satisfies the problem

$$\Delta h_\varepsilon = -(r-1)(u+\varepsilon)^{-r}|\nabla u|^2 \quad \text{in } \Omega,$$

$$\frac{\partial h_\varepsilon}{\partial \nu} = a(x)\left(\frac{u}{u+\varepsilon}\right)^{r-1} \quad \text{on } \partial\Omega.$$

Passing to the limits as $\varepsilon \to 0$, we have

$$\int_{\partial\Omega} a\, d\sigma = -(r-1)\int_{\Omega} u^{-r}|\nabla u|^2 dx \leqslant 0.$$

In the case of an equality, we have $u \equiv \text{const}$ in Ω, from which

$$0 = ac^{r-1} \quad \text{on } \partial\Omega.$$

This contradicts the assumption $a \not\equiv 0$, which completes the proof. □

REMARK 4.1.8. The obtained conditions for solvability are not improvable. This follows from the example

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^N,\ N > 2, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = a(x)u^p & \text{on } \partial\Omega \end{cases}$$

with $1 < p < \frac{N}{N-2}$ and $a \in L^\infty(\Omega)$ satisfying (H2) and (H3).

4.2. *The inhomogeneous semilinear case*

In this section we consider the application of the fibering method to a problem with Neumann boundary conditions. Here we follow the paper by A. Tesei and S. Pohozaev [38], where this problem is considered in a more general setting. Nevertheless, results stated in this section generalize some results from [9]. We begin with the case of a linear differential operator.

Let Ω be a bounded domain in $\mathbb{R}^N$. We consider the boundary value problem

$$\begin{cases} \Delta u + \langle \nabla\psi, \nabla u\rangle + a(x)|u|^{q-2}u = 0, \quad u \geqslant 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2.1}$$

with $2 < q < 2^*$ and $\psi \in C^1(\overline{\Omega})$, under the following assumptions:

- (A1) $a \in L^\infty(\Omega)$;
- (A2) $a^+(x) := \max\{a(x), 0\}$ is not identically zero;
- (A3) $\int_\Omega a(x)\rho(x)\, dx < 0$, with $\rho(x) := e^{\psi(x)}$.

THEOREM 4.2.1. *Assume* (A1)–(A3) *and* $2 < q < 2^*$. *Then there exists a nonnegative* (*nontrivial*) *solution of* (4.2.1).

The proof is divided into several steps.

4.2.1. *Step 1.* We consider the functional E defined by

$$E(u) := -\frac{1}{2}\int_{\Omega} \left|\nabla u(x)\right|^2 \rho(x)\,dx + \frac{1}{q}\int_{\Omega} a(x)\left|u(x)\right|^q \rho(x)\,dx.$$

Thanks to (A1) and $2 < q < 2^*$, this functional is well defined on the Sobolev space $W := W^{1,2}_{\rho}(\Omega)$, defined by

$$W = W^1_2(\Omega), \quad \|u\|_W = \left(\int_{\Omega} \left|\nabla u(x)\right|^2 \rho(x)\,dx\right)^{1/2}.$$

Following the fibering method, we set

$$u(x) = t v(x), \quad t \in \mathbb{R} \setminus \{0\},\ v \in W$$

and take the norm-type fibering functional

$$H(u) := \|v\|^2_W = \int_{\Omega} |\nabla v|^2 \rho\,dx.$$

Accordingly, under the fibering constraint

$$H(v) = 1$$

the Euler functional $E(u)$ reduces to

$$\tilde{E}(t, v) = -\frac{1}{2}t^2 + \frac{1}{q}|t|^q E_1(v) \tag{4.2.2}$$

with

$$E_1(v) := \int_{\Omega} a(x)\left|v(x)\right|^q \rho(x)\,dx. \tag{4.2.3}$$

From the bifurcation equation $\tilde{E}'_t = 0$, i.e.

$$-t + |t|^{q-2} t E_1(v) = 0,$$

we obtain for $t \neq 0$:

$$\left|t(v)\right| = \left[E_1(v)\right]^{-\frac{1}{q-2}}. \tag{4.2.4}$$

Thus we define the functional $\hat{E}$ as

$$\hat{E}(v) := \tilde{E}\big(t(v), v\big) = \left(\frac{1}{q} - \frac{1}{2}\right)\left[E_1(v)\right]^{-\frac{2}{q-2}}. \tag{4.2.5}$$

4.2.2. *Step 2.* Now we search an extremal point of E_1, i.e. of $\hat{E}$ under the fibering constraint $H(v) = 1$.

LEMMA 4.2.2. *Let*

$$M_0 := \sup_{v \in W} \{ E_1(v) \mid v \in W, \ H(v) = 1 \}. \tag{4.2.6}$$

Then $0 < M_0 < \infty$, *and any maximizing sequence of* (4.2.6) *is bounded in* W.

PROOF. From (A2) and (4.2.3) it follows immediately that $M_0 > 0$. Thus, we have only to prove the boundedness of an arbitrary maximizing sequence (v_n), because from this we obtain also $M_0 < \infty$. Let

$$H(v_n) = 1, \quad E_1(v_n) \to M_0,$$

and put

$$v_n(x) =: \alpha_n + \bar{v}_n(x) \quad \text{with} \int_\Omega \bar{v}_n \rho(x)\, dx = 0, \tag{4.2.7}$$

where α_n are constants. Then $\nabla v_n = \nabla \bar{v}_n$, and by virtue of the Sobolev imbedding theorem (the Poincaré inequality) we have

$$\left| E_1(v_n) \right| \leqslant \tilde{M} < \infty \tag{4.2.8}$$

thanks to (4.2.7). Here $\tilde{M}$ does not depend on $\bar{v}_n$. Suppose by contradiction that $\|v_n\|_W \to \infty$. By (4.2.7) and (4.2.8) this implies

$$\alpha_n \to \infty. \tag{4.2.9}$$

Further, we have

$$E_1(v_n) = E_1(\alpha_n + \bar{v}_n) = |\alpha_n|^q \int_\Omega a(x) \left| 1 + \frac{\bar{v}_n(x)}{\alpha_n} \right|^q \rho(x)\, dx.$$

Then by (4.2.8), (4.2.9), (4.2.4), and (A3) we get

$$|\alpha_n|^{-q} E_1(v_n) = \int_\Omega a(x) \left| 1 + \frac{\bar{v}_n(x)}{\alpha_n} \right|^q \rho(x)\, dx \to \int_\Omega a(x)\rho(x)\, dx < 0,$$

which contradicts the fact that $M_0 > 0$. Hence, (v_n) is bounded in W. □

LEMMA 4.2.3. *There exists a maximizer* $\bar{v} \geqslant 0$ *of* (4.2.6) *in* W.

PROOF. By $2 < q < 2^*$ and (A1), we can use the Kondrashov imbedding theorem

$$W = W_\rho^{1,2}(\Omega) \Subset L_\rho^q(\Omega),$$

where

$$\|v\|_{L_\rho^q}^q = \int_\Omega |v|^q \rho\, dx.$$

Thus, by Lemma 4.2.2, we can take a maximizing sequence that weakly converges to a v_0 in W, so that

$$E_1(v_0) = M_0, \qquad H(v_0) = \|v_0\|_W^2 \leqslant 1.$$

We need to prove that actually $H(v_0) = 1$. First, notice that $H(v_0) = \|v_0\|_W^2 \neq 0$, because otherwise one would have $v_0(x) \equiv \overline{C}$, and then by (A3)

$$0 < M_0 = E_1(v_0) = |\overline{C}|^q \int_\Omega a(x)\rho(x)\, dx \leqslant 0.$$

Second, suppose that $0 < H(v_0) < 1$. Hence, for a suitable $k > 1$,

$$H(kv_0) = |k|^2 H(v_0) = 1$$

and

$$E_1(kv_0) = |k|^q E_1(v_0) = |k|^q M_0 > M_0,$$

which contradicts (4.2.6). Finally, from the general properties of the Sobolev space W, we have $|v_0| \in W$. We have also

$$H\big(|v_0|\big) = H(v_0) = 1 \quad \text{and} \quad E_1\big(|v_0|\big) = E_1(v_0) = M_0,$$

therefore it is not restrictive to consider $v_0 \geqslant 0$. □

4.2.3. *Step 3.* Let $v_0 \geqslant 0$ be the maximizer of (4.2.6) found in Lemma 4.2.3, and set

$$u_0(x) = t_0 \cdot v_0(x)$$

with $t_0 = \bar{t}(v_0) > 0$ defined by (4.2.4), i.e.

$$t_0 = \big[E_1(v_0)\big]^{-\frac{1}{q-2}}.$$

By applying the fibering method it follows that u_0 is a nonnegative solution of the boundary value problem (4.2.1). Clearly, $u_0 \neq 0$, because $v_0 \neq 0$ and $E_1(v_0) = M_0 > 0$. The proof of Theorem 4.2.1 is complete.

4.2.4. *The case* $1 < q < 2$. Here we consider again the boundary value problem (4.2.1), but with $1 < q < 2$. In this case we have the same representation (4.2.2) for the Euler functional $E(u)$. Let us consider the behavior of $\tilde{E}$ with respect to $t > 0$ for a fixed v. If $E_1(v) < 0$, then $\tilde{E}$ is decreasing in t, anyway. If $E_1(v) > 0$, then $\tilde{E}$ is eventually increasing or eventually decreasing in t for $2 < q < 2^*$ or $1 < q < 2$, respectively.

We get the following result.

THEOREM 4.2.4. *Assume* (A1)–(A3) *and* $1 < q < 2$. *Then there exists a nonnegative* (*nontrivial*) *solution of* (4.2.1).

PROOF. The proof is completely coincident with that of Theorem 4.2.1. □

4.3. *The case of the* p*-Laplacian*

In this subsection we generalize the previous results to the case of existence of positive solutions for the boundary value problem

$$\begin{cases} \Delta_p u + \langle \nabla\psi, \nabla u\rangle |\nabla u|^{p-2} + a(x)|u|^{q-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.1}$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$, and $\psi \in C^1(\overline{\Omega})$. As before, we consider this problem under assumptions (A1)–(A3):

- (A1) $a \in L^\infty(\Omega)$;
- (A2) $a^+(x)$ is not identically zero;
- (A3) $\int_\Omega a(x)\rho(x)\,dx < 0$, where $\rho(x) := e^{\psi(x)}$.

Concerning q we suppose

$$1 < q < p^*, \quad q \neq p, \quad p^* = \begin{cases} \frac{Np}{N-p} & \text{for } p < N, \\ +\infty & \text{for } p \geqslant N. \end{cases} \tag{4.3.2}$$

Then we have the following result.

THEOREM 4.3.1. *Let assumptions* (A1)–(A3) *and* (4.3.2) *be satisfied. Then the boundary value problem* (4.2.1) *admits a nonnegative* (*nontrivial*) *solution*

$$u \in W_p^1(\Omega) \cap L^\infty(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega).$$

The proof is divided into several steps.

4.3.1. *Step 1.* Similarly to the previous subsection,

$$E(u) := -\frac{1}{p}\int_\Omega |\nabla u|^p \rho\,dx + \frac{1}{q}\int_\Omega a|u|^q \rho\,dx$$

is the Euler functional associated with (4.3.1). From our assumptions it follows that $E(u)$ is well defined on the Sobolev space

$$W = W_{\rho}^{1,2}(\Omega), \quad \|u\|_W = \left(\int_{\Omega} |\nabla u|^p \rho \, dx\right)^{1/p}.$$

Following the fibering method we set $u(x) = t \cdot v(x)$ with $t \neq 0$ and $v \in W$. Under the (spherical) fibering constraint

$$H(u) := \int_{\Omega} |\nabla v|^p \rho \, dx = 1$$

the functional E reduces to

$$\tilde{E}(t, v) = -\frac{|t|^p}{p} + \frac{|t|^q}{q} E_1(v), \tag{4.3.3}$$

where

$$E_1(v) := \int_{\Omega} a(x) \big|v(x)\big|^q \rho(x) \, dx. \tag{4.3.4}$$

From the bifurcation equation

$$\tilde{E}_t' = -|t|^{p-2} t + |t|^{q-2} t E_1(v) = 0$$

we find for $t \neq 0$

$$\big|t(v)\big| = \big[E_1(v)\big]^{\frac{1}{p-q}} \tag{4.3.5}$$

with the necessary condition $E_1(v) > 0$. Now, by substituting (4.3.5) into (4.3.4), we obtain

$$\hat{E}(v) = \left(\frac{1}{q} - \frac{1}{p}\right)\big[E_1(v)\big]^{\frac{p}{p-q}}. \tag{4.3.6}$$

4.3.2. *Step 2.* Now we search an extremal point of E_1, i.e. of $\hat{E}$, under the fibering constraint $H(v) = 1$.

LEMMA 4.3.2. *The variational problem*

$$M_1 := \sup_{v \in W} \big\{E_1(v) \;\big|\; H(v) \leqslant 1\big\} \tag{4.3.7}$$

admits a nonnegative maximizer $\bar{v} \in W$ *with* $H(\bar{v}) = 1$ *and* $E_1(\bar{v}) = M_1 > 0$.

PROOF. The proof follows exactly the same lines as that of Lemma 4.2.2 together with the proof of Lemma 4.2.3. □

4.3.3. *Step 3.* By virtue of the fibering method we derive that $\bar{u}(x) = \bar{t} \cdot \bar{v}(x)$, with

$$\bar{t} = t(\bar{v}) = \left[E_1(\bar{v})\right]^{-\frac{1}{p-q}}$$

(see equality (4.3.5)), is a nonnegative solution of (4.3.1). Clearly $\bar{u}$ is nontrivial, since $\bar{v} \neq 0$. By the bootstrap argument used, e.g., in [18] we can prove that $\bar{u} \in L^\infty(\Omega)$. Then from the result of Tolksdorff [55] it follows that $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. Theorem 4.3.1 is proved.

4.4. *The nonhomogeneous Neumann problem*

Let us consider now a more difficult case, namely, the Neumann problem for the p-Laplacian with *nonlinear nonhomogeneous* terms. We shall study the existence of positive solutions for the variational boundary value problem

$$\begin{cases} \Delta_p u + \langle \nabla\psi, \nabla u\rangle \nabla u|^{p-2} + a(x)|u|^{q-2}u + b(x)|u|^{s-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4.1}$$

with $\psi \in C^1(\overline{\Omega})$ and

$$1 < p < s < q < p^*. \tag{4.4.2}$$

We consider this problem under assumptions (A1)–(A3) and
- (B1) $b \in L^\infty(\Omega)$;
- (B2) $b \leqslant 0$ in Ω.

THEOREM 4.4.1 (Pohozaev and Tesei). *Let assumptions* (4.4.2), (A1)–(A3), (B1), *and* (B2) *be satisfied. Then problem* (4.4.1) *has a nonnegative nontrivial solution*

$$u \in W^1_p(\Omega) \cap L^\infty(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega).$$

The proof is again based on the fibering method and is divided into several steps.

4.4.1. *Step 1.* We have an Euler functional

$$E(u) := -\frac{1}{p}\int_\Omega |\nabla u|^p \rho\, dx + \frac{1}{q}\int_\Omega a(x)|u|^q \rho\, dx + \frac{1}{s}\int_\Omega b(x)|u|^s \rho\, dx$$

associated with problem (4.4.1). The indicated assumptions imply that $E(u)$ is well defined on the Sobolev space $W := W^{1,p}_\rho(\Omega)$.

Following the fibering method, we set

$$u(x) = tv(x)$$

with $t \in \mathbb{R}$ and $v \in W^{1,p}(\Omega)$.

Then we get

$$E := -\frac{|t|^p}{p} \int_\Omega |\nabla v|^p \rho\, dx + \frac{|t|^q}{q} \int_\Omega a(x)|v|^q \rho\, dx + \frac{|t|^s}{s} \int_\Omega b(x)|v|^s \rho\, dx.$$

We introduce, as above, the fibering functional

$$H(v) := \|v\|_W^p = \int_\Omega |\nabla v|^p \rho\, dx = 1.$$

Then

$$\tilde{E}(t, v) = -\frac{|t|^p}{p} + \frac{|t|^q}{q} \int_\Omega a(x)|v|^q \rho\, dx + \frac{|t|^s}{s} \int_\Omega b(x)|v|^s \rho\, dx, \tag{4.4.3}$$

and the bifurcation equation has the form

$$\frac{d\tilde{E}}{dt} = 0: \quad -|t|^{p-2}t + |t|^{q-2}t \int_\Omega a(x)|v|^q \rho\, dx + |t|^{s-2}t \int_\Omega b(x)|v|^s \rho\, dx = 0. \tag{4.4.4}$$

For $t \neq 0$ we have

$$|t|^{q-p} \int_\Omega a(x)|v|^q \rho\, dx + |t|^{s-p} \int_\Omega b(x)|v|^s \rho\, dx = 1.$$

Thus, for v such that

$$\int_\Omega a(x)|v|^q \rho\, dx > 0 \tag{4.4.5}$$

this equation has only one positive solution $t = t(v)$ with the derivative $t'(v)$.

Substituting this solution into (4.4.3), we get the functional $\hat{E}$:

$$\hat{E}(v) := \tilde{E}\big(t(v), v\big).$$

It follows from our assumptions that $\hat{E}(v)$ is a weakly continuous functional defined for $v \in W_\rho^{1,p}(\Omega)$ under condition (4.4.5).

4.4.2. *Step 2.* Let us consider the variational problem

$$\sup\left\{ \hat{E}(v) \;\middle|\; \int_\Omega |\nabla v|^p \rho\, dx \leqslant 1,\ v \text{ satisfies (4.4.5)} \right\}. \tag{4.4.6}$$

We have $\hat{E}(v) < 0$.

We shall consider a maximizing sequence $\{v_n\}$ for (4.4.6). Reasoning as above, by virtue of (4.4.5) we obtain the boundedness of $\{v_n\}$.

Finally, using the weak continuity of $\hat{E}$, we establish the existence of a function v_0, which provides the maximum for problem (4.4.6).

REMARK 4.4.2. Due to (B2), we have

$$\hat{E}(v) \leqslant \hat{E}(v)|_{b=0} = \left(\frac{1}{q} - \frac{1}{p}\right)\left[E_0(v)\right]^{\frac{p}{p-q}}$$

and hence

$$\max \hat{E}(v) \leqslant \max\left(\frac{1}{q} - \frac{1}{p}\right)\left[E_0(v)\right]^{\frac{p}{p-q}} = \left(\frac{1}{q} - \frac{1}{p}\right)\left[E_0(v_0)\right]^{\frac{p}{p-q}} < 0,$$

because in our case $q > p$.

REMARK 4.4.3. Let us consider (4.4.6) under condition (4.4.5). Since

$$\sup\left\{\hat{E}(v) \;\Big|\; \int_\Omega |\nabla v|^p \rho\, dx \leqslant 1,\ v \text{ satisfies (4.4.5)}\right\} > -\infty,$$

it follows that

$$\max \hat{E}(v_0) > -\infty. \tag{4.4.7}$$

This implies that v_0 satisfies (4.4.5), because, if we assume the contrary, i.e. that v_0 *does not satisfy* (4.4.5), then we have

$$E(v_0) = -\infty.$$

Thus, we have got the function v_0:

$$\hat{E}(v_0) = \sup\left\{\hat{E}(v) \;\Big|\; \int_\Omega |\nabla v|^p \rho\, dx \leqslant 1,\ v \text{ satisfies (4.4.5)}\right\}.$$

Now we shall prove the equality

$$\int_\Omega |\nabla v_0|^p \rho\, dx = 1. \tag{4.4.8}$$

We assume the contrary, i.e., that

$$\int_\Omega |\nabla v_0|^p \rho\, dx < 1. \tag{4.4.9}$$

Note that

$$0 < \int_\Omega |\nabla v_0|^p \rho \, dx,$$

because otherwise, if

$$\int_\Omega |\nabla v_0|^p \rho \, dx = 0, \tag{4.4.10}$$

$v_0 \equiv c_0$ is a constant, and

$$\hat{E}_0(v_0) = \hat{E}_0(c_0) = -\infty$$

by virtue of (A3), and this contradicts the boundedness condition.

Due to (4.4.9) and (4.4.10), there exists $k_0 > 1$ such that

$$\tilde{v}_0(x) = k_0 v_0(x)$$

satisfies

$$\int_\Omega |\nabla \tilde{v}_0|^p \rho \, dx = 1$$

and (4.4.5).

For the functional $\hat{E}$, we have

$$\hat{E}(v) = \min_{t>0} \tilde{E}(tv) = \min_{t>0} \left\{ -\frac{t^p}{p} + \frac{t^q}{q} \int_\Omega a(x)|v|^q \rho \, dx + \frac{t^s}{s} \int_\Omega b(x)|v|^s \rho \, dx \right\}$$

for v satisfying (4.4.5). Thus,

$$\begin{aligned} \hat{E}(v_0) &= \min_{t>0} \tilde{E}(tv_0) \\ &= \min_{t>0} \left\{ -\frac{t^p}{p} + \frac{t^q}{q} \int_\Omega a(x)|v_0|^q \rho \, dx + \frac{t^s}{s} \int_\Omega b(x)|v_0|^s \rho \, dx \right\} \end{aligned}$$

and v_0 satisfies (4.4.5).

For $\tilde{v}_0 = k_0 v_0$, we have ($t' = tk_0$)

$$\begin{aligned} \hat{E}(\tilde{v}_0) &= \min_{t>0} \left\{ -\frac{t^p}{p} + \frac{t^q}{q} k_0^q \int_\Omega a(x)|v_0|^q \rho \, dx + \frac{t^s}{s} k_0^s \int_\Omega b(x)|v_0|^s \rho \, dx \right\} \\ &= \min_{t'>0} \left\{ -\frac{t'^p}{k_0^p p} + \frac{t'^q}{q} \int_\Omega a(x)|v_0|^q \rho \, dx + \frac{t'^s}{s} \int_\Omega b(x)|v_0|^s \rho \, dx \right\} \end{aligned}$$

$$
\begin{aligned}
&= \min_{t>0}\left\{-\frac{t^p}{k_0^p p} + \frac{t^q}{q}\int_\Omega a(x)|v_0|^q \rho\,dx + \frac{t^s}{s}\int_\Omega b(x)|v_0|^s \rho\,dx\right\} \\
&= \min_{t>0}\left\{\left(1-\frac{1}{k_0^p}\right)\frac{t^p}{p} - \frac{t^p}{p} + \frac{t^q}{q}\int_\Omega a(x)|v_0|^q \rho\,dx \right.\\
&\quad \left. + \frac{t^s}{s}\int_\Omega b(x)|v_0|^s \rho\,dx\right\} \\
&= \min_{t>0}\left\{\left(1-\frac{1}{k_0^p}\right)\frac{t^p}{p} + \left[-\frac{t^p}{p} + \frac{t^q}{q}\int_\Omega a(x)|v_0|^q \rho\,dx \right.\right.\\
&\quad \left.\left. + \frac{t^s}{s}\int_\Omega b(x)|v_0|^s \rho\,dx\right]\right\} \\
&> \min_{t>0}\left[-\frac{t^p}{p} + \frac{t^q}{q}\int_\Omega a(x)|v_0|^q \rho\,dx + \frac{t^s}{s}\int_\Omega b(x)|v_0|^s \rho\,dx\right] = \hat{E}(v_0).
\end{aligned}
$$

Thus,

$$\hat{E}(\tilde{v}_0) > \hat{E}(v_0).$$

This inequality contradicts the definition of v_0 as $\sup \hat{E}(v)$.

Thus, we have obtained a solution to the variational problem

$$\hat{E}(v_0) = \sup\left\{\hat{E}(v) \;\Big|\; \int_\Omega |\nabla v|^p \rho\,dx \leqslant 1,\ v \text{ satisfies (4.4.5)}\right\}.$$

The fibering method implies that

$$u_0(x) = t_0 v_0(x),$$

where t_0 is a positive solution to the bifurcation equation (4.4.4) and

$$E(t_0 v_0) = \min_{t>0} E(t v_0) \quad \text{for } v_0 \text{ satisfying (4.4.5).}$$

Next, repeating the reasoning from Subsection 4.2, we find that

$$u_0(x) \geqslant 0$$

and $u_0 \not\equiv 0$ because $\hat{E}(v_0) < 0$ and $t_0 > 0$.

Again using the reasoning from the last stage of the proof of Theorem 4.2.1, we arrive at the required statement.

Theorem 4.4.1 is proved.

4.5. *The case* $k \equiv 0 \equiv c$

In this subsection we consider the following boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u + a(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.5.1}$$

with p and q in the range

$$1 < p < q < p^* \tag{4.5.2}$$

and $\lambda > 0$. Define the following functionals on $W^{1,p}(\Omega)$:

$$H_\lambda(u) = \frac{1}{p}\int_\Omega \big(|\nabla u|^p - \lambda b(x)|u|^p\big)\,dx, \tag{4.5.3}$$

$$A(u) = \frac{1}{q}\int_\Omega a(x)|u|^q\,dx, \tag{4.5.4}$$

$$E_A(u) = -H_\lambda(u) + A(u). \tag{4.5.5}$$

Solutions of (4.5.1) will be obtained as critical points of E_A. By the fibering method we set $u(x) = sv(x)$, where $s \in \mathbb{R}$ and $v \in W^{1,p}(\Omega)$. Then

$$E_A(sv) = -|s|^p H_\lambda(v) + |s|^q A(v). \tag{4.5.6}$$

By the chain rule theorem, if $u = sv$ is a critical point of A, then $\frac{\partial E_A(sv)}{\partial s} = 0$, which yields

$$\frac{\partial E_A(sv)}{\partial s} = -p|s|^{p-2}sH_\lambda(v) + q|s|^{q-2}sA(v) = 0.$$

Therefore

$$\begin{cases} \partial E_A/\partial s = 0, \\ s \neq 0 \end{cases} \quad\Rightarrow\quad |s|^{q-p} = |s_*|^{q-p} = \frac{pH_\lambda(v)}{qA(v)} = \frac{\int_\Omega (|\nabla v|^p - \lambda b|v|^p)\,dx}{\int_\Omega a(x)|v|^q\,dx} \tag{4.5.7}$$

provided $A(v) \neq 0$. If $H_\lambda(v)A(v) \neq 0$, then $H_\lambda(v)$ and $A(v)$ have the same sign, and for this value s_* of s we obtain the reduced functional

$$\tilde{E}_A(v) = \frac{p-q}{p}H_\lambda(v)\left(\frac{pH_\lambda(v)}{qA(v)}\right)^{p/(q-p)}, \tag{4.5.8}$$

for which we have two sign possibilities since $\lambda > 0$:

$$\tilde{E}_A(v) < 0 \quad \text{or} \quad \tilde{E}_A(v) > 0. \tag{4.5.9}$$

If $H_\lambda(v) < 0$, s_* corresponds to a maximum, and if $H_\lambda(v) > 0$, s_* corresponds to a minimum. By taking $H_\lambda(v)$ constant, we can therefore examine the following two problems:

PROBLEM 1. $\sup\{A(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1\}$.

PROBLEM 2. $\inf\{A(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = -1\}$.

We introduce the following assumptions on $a = a_+ - a_-$ and b:
(A1) $\int_\Omega a_+(x)\,dx > 0$,
(A2) $\int_\Omega a(x)\,dx < 0$,
(B) $\int_\Omega b(x)\,dx > 0$.

If $\phi \in L^1(\Omega)$, we set $\bar{\phi} = |\Omega|^{-1} \int_\Omega \phi(x)\,dx$ its average on Ω. Let $\mu = \mu(N, p) > 0$ be the best constant of the Poincaré inequality in $W^{1,p}(\Omega)$, that is,

$$\mu = \inf\left\{\int_\Omega |\nabla\phi|^p\,dx\colon \phi \in W^{1,p}(\Omega),\ \int_\Omega |\phi - \bar{\phi}|^p\,dx = 1\right\}. \tag{4.5.10}$$

If $p < N$, we denote by $\mu_* = \mu_*(N, p)$ the best constant of the Poincaré–Sobolev inequality in $W^{1,p}(\Omega)$, defined by

$$\mu = \inf\left\{\int_\Omega |\nabla\phi|^p\,dx\colon \phi \in W^{1,p}(\Omega),\ \int_\Omega |\phi - \bar{\phi}|^{p^*}\,dx = 1\right\}. \tag{4.5.11}$$

LEMMA 4.5.1. *Assume that* (A2) *holds. Then there exists a positive constant* $\lambda_* = \lambda_*(|\Omega|, p, q, \|a\|_{L^\infty}, \|b\|_{L^\infty})$ *such that if* $0 < \lambda < \lambda_*$, *any sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$ *such that* $\{A(u_n)\}$ *is positive and* $\{H_\lambda(u_n)\}$ *is bounded from above, has the property that it remains bounded in* $W^{1,p}(\Omega)$. *Moreover, it satisfies*

$$\lim_{n\to\infty} \sup A(u_n) =: m < \infty.$$

PROOF. Set $\alpha_n := |\Omega|^{-1} \int_\Omega u_n\,dx$, $u_n = \tilde{u}_n + \alpha_n$. Then there exists a constant M such that

$$H_\lambda(u_n) \leqslant M \quad \Leftrightarrow \quad \int_\Omega |\nabla\tilde{u}_n|^p\,dx - \lambda \int_\Omega b|\alpha_n + \tilde{u}_n|^p\,dx \leqslant M, \tag{4.5.12}$$

and in particular

$$\int_\Omega |\nabla\tilde{u}_n|^p\,dx = M + \lambda \int_\Omega b|\alpha_n + \tilde{u}_n|^p\,dx.$$

Since for any $\epsilon \in (0,1)$ there holds

$$|x+y|^p \leqslant (1+\epsilon)|x|^p + C_{\epsilon,p}|y|^p \quad \big(\forall (x,y) \in \mathbb{R}\times\mathbb{R}\big), \tag{4.5.13}$$

where

$$C_{\epsilon,p} = \frac{(1+\epsilon)^{p/(p-1)} - 1}{((1+\epsilon)^{1/(p-1)} - 1)^p} \leqslant C_p \epsilon^{1-p}, \tag{4.5.14}$$

it follows by inequality (4.5.10)

$$\begin{aligned} \int_\Omega |\nabla \tilde{u}_n|^p\,dx &\leqslant M + (\lambda+\epsilon)\int_\Omega b|\tilde{u}_n|^p\,dx + \lambda\|b\|_{L^\infty}|\Omega| C_p \epsilon^{1-p}|\alpha_n|^p \\ &\leqslant M + \frac{\lambda+\epsilon}{\mu}\|b\|_{L^\infty}\int_\Omega |\nabla \tilde{u}_n|^p\,dx + \lambda\|b\|_{L^\infty}|\Omega|\; C_p\epsilon^{1-p}|\alpha_n|^p. \end{aligned} \tag{4.5.15}$$

Consequently,

$$\frac{\mu - (\lambda+\epsilon)\|b\|_{L^\infty}}{\mu}\int_\Omega |\nabla \tilde{u}_n|^p\,dx \leqslant \lambda\|b\|_{L^\infty}|\Omega| C_p\epsilon^{1-p}|\alpha_n|^p + M. \tag{4.5.16}$$

As a first choice we impose

$$0 < \lambda < \mu/\|b\|_{L^\infty} \quad \text{and} \quad 0 < \epsilon < \mu/\|b\|_{L^\infty} - \lambda. \tag{4.5.17}$$

Step 1. We claim that α_n remains bounded.

Arguing by contradiction, we suppose that $|\alpha_n| \to \infty$. If we set $w_n = u_n/\alpha_n = 1 + \tilde{w}_n$, then

$$\frac{\mu - (\lambda+\epsilon)\|b\|_{L^\infty}}{\mu}\int_\Omega |\nabla \tilde{w}_n|^p\,dx \leqslant \lambda\|b\|_{L^\infty}|\Omega| C_p\epsilon^{1-p} + M|\alpha_n|^{-p}, \tag{4.5.18}$$

because the integral mean value $\overline{w}_n = 1$. It follows from (4.5.10) and (4.5.18) that $\{w_n\}$ is bounded in $W^{1,p}(\Omega)$. Moreover,

$$\begin{aligned} \int_\Omega |\nabla \tilde{w}_n|^q\,dx &\leqslant \mu_*^{-q/p^*}|\Omega|^{1-q/p^*}\left(\int_\Omega |\nabla \tilde{w}_n|^p\,dx\right)^{q/p} \\ &\leqslant \big(\lambda\|b\|_{L^\infty}\big)^{q/p}\mu_*^{-q/p^*}|\Omega|^{1-q/N}\left(\frac{\mu}{\mu - \|b\|_{L^\infty}(\lambda+\epsilon)}\right)^{q/p} \\ &\quad \times C_p^{q/p}\epsilon^{q(1-p)/p} + M|\alpha_n|^{-p} = C_{\epsilon,p,q}\lambda^{q/p} + M|\alpha_n|^{-p} \end{aligned} \tag{4.5.19}$$

is derived from (4.5.18) and (4.5.11). But

$$|1+\tilde{w}_n|^q - 1 = \int_0^1 \frac{d}{ds}|1+s\tilde{w}_n|^q\,ds = q\tilde{w}_n\int_0^1 |1+s\tilde{w}_n|^{q-2}(1+s\tilde{w}_n)\,ds, \tag{4.5.20}$$

then

$$\begin{aligned}\int_\Omega \big||1+\tilde{w}_n|^q - 1\big|\,dx &\leqslant q\left(\int_\Omega |\tilde{w}_n|^q\,dx\right)^{1/q}\left(\int_\Omega \big(1+|\tilde{w}_n|\big)^q\,dx\right)^{1-1/q}\\ &\leqslant q\left(\int_\Omega |\tilde{w}_n|^q\,dx\right)^{1/q}\left(|\Omega|^{1/q} + \left(\int_\Omega |\tilde{w}_n|^q\,dx\right)^{1/q}\right)^{q-1}\\ &\leqslant qc_q\left(\int_\Omega |\tilde{w}_n|^q\,dx\right)^{1/q}\\ &\quad\times\left(|\Omega|^{1-1/q} + \left(\int_\Omega |\tilde{w}_n|^q\,dx\right)^{1-1/q}\right),\end{aligned} \tag{4.5.21}$$

where $c_q = \max(1, 2^{q-2})$. Consequently,

$$\begin{aligned}\int_\Omega a|1+\tilde{w}_n|^q\,dx &= \int_\Omega a\,dx + \int_\Omega a\big(|1+\tilde{w}_n|^q - 1\big)\,dx\\ &\leqslant \int_\Omega a\,dx + qc_q\|a\|_{L^\infty}\big(C_{\epsilon,p,q}\lambda^{1/p} + M|\alpha_n|^{-1}\big)\\ &\quad\times\big(|\Omega|^{1/q} + C_{\epsilon,p,q}\lambda^{1/p} + M|\alpha_n|^{-1}\big)^q.\end{aligned} \tag{4.5.22}$$

We first denote by Λ_ϵ the expression

$$\begin{aligned}\Lambda_\epsilon = \sup\Big\{&\lambda \in \big(0, \mu/\|b\|_{L^\infty} - \epsilon\big):\\ &\int_\Omega a\,dx + qc_q\|a\|_{L^\infty}C_{\epsilon,p,q}\rho^{1/p}\big(|\Omega|^{1/q} + C_{\epsilon,p,q}\rho^{1/p}\big)^q < 0\\ &\forall\rho \in (0,\lambda)\Big\}.\end{aligned} \tag{4.5.23}$$

Such an expression defines a positive real number because of the assumption (A2), and we set

$$\lambda_* = \max\big\{\Lambda_\epsilon:\ 0 < \epsilon < \mu/\|b\|_{L^\infty}\big\} \tag{4.5.24}$$

(notice that λ_* is achieved at some $\epsilon = \epsilon_*$). Since $|\alpha_n| \to \infty$, it follows from the definition of λ_* and (4.5.23) that for any $\lambda \in (0, \lambda_*)$ and $\epsilon = \epsilon_*$ there exists an n large enough such that

$$A(u_n) = \int_\Omega a|u_n|^q \, dx = |\alpha_n|^q \int_\Omega a|1 + \tilde{w}_n|^q \, dx < 0, \tag{4.5.25}$$

contradiction.

Step 2. End of the proof.

Since $\{\alpha_n\}$ is bounded, it follows from (4.5.16) that the same is true for $\{\|\nabla u_n\|_{L^p}\}$, since $\nabla u_n = \nabla \tilde{u}_n$. From (4.5.11),

$$\int_\Omega |u_n - \alpha_n|^p \, dx \leqslant \mu^{-1} \int_\Omega |\nabla u_n|^p \, dx. \tag{4.5.26}$$

Therefore $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$ and also in $L^q(\Omega)$ by the Sobolev inequality (actually it is relatively compact there, since $q < p^*$), and the last statement of the lemma follows. □

LEMMA 4.5.2. *For any λ, the set $\mathcal{A}_+ := \{\zeta \in W^{1,p}(\Omega)\colon\ A(\zeta) > 0,\ H_\lambda(\zeta) = 1\}$ is nonempty under assumption* (A1).

PROOF. For $\delta > 0$ we set $B_\delta(x_0) := \{x \in \Omega\colon\ |x - x_0| < \delta\}$, and denote by ν_δ the first eigenvalue of the p-Laplace operator in $W_0^{1,p}(B_\delta(x_0))$ (it is independent of x_0 provided that $B_\delta(x_0) \subset \Omega$), and ϕ_{δ,x_0} the corresponding eigenfunction with the following normalization:

$$\int_{B_\delta(x_0)} \phi_{\delta,x_0}^q \, dx = 1. \tag{4.5.27}$$

The function ϕ_{δ,x_0} is positive, and $\phi_{\delta,x_0}(x) = \phi_{\delta,y_0}(x + y_0 - x_0)$ for any y_0 such that $B_\delta(y_0) \subset \Omega$. Moreover,

$$\lim_{\delta \to 0} \nu_\delta = \infty, \tag{4.5.28}$$

since $\nu_{\alpha\delta} = \alpha^{-p}\nu_\delta$. Therefore there exists $\delta_0 > 0$ such that, for every $0 < \delta \leqslant \delta_0$, and any x_0 such that $B_\delta(x_0) \subset \Omega$, $H_\lambda(\phi_{\delta,x_0}) \geqslant 1$. If we assume now that $\mathcal{A}_+$ is empty, then for every x_0 as above

$$\int_{B_\delta(x_0)} a\phi_{\delta,x_0}^q \, dx \leqslant 0 \tag{4.5.29}$$

(here we extend the function ϕ_{δ,x_0} by zero outside $B_\delta(x_0)$ and get a $W^{1,p}(\Omega)$ function). But $\phi_{\alpha\delta,x_0}(x) = \alpha^{-N+1}\phi_{\delta,x_0}(\alpha^{-1}(x-x_0)+x)$, therefore, by Lebesgue's differentiation theorem,

$$\lim_{\delta\to 0}\int_{B_\delta(x_0)} a\phi_{\delta,x_0}^q\,dx = a(x_0) \leqslant 0 \tag{4.5.30}$$

for almost all $x_0 \in \Omega$, contradicting (A1). □

LEMMA 4.5.3. *Assume that* (A2) *holds. Then for every* $\lambda \in (0,\lambda_*)$ *and* $v \in W^{1,p}(\Omega)$, *the inequality* $H_\lambda(v) \leqslant 0$ *with* v *not identically zero implies* $A(v) < 0$.

PROOF. Since $H_\lambda(v) = H_\lambda(|v|)$ and $A(v) = A(|v|)$, we can assume that v is nonnegative and not identically zero, otherwise the result follows. The relation $H_\lambda(v) \leqslant 0$ means

$$\int_\Omega |\nabla v|^p\,dx \leqslant \lambda \int_\Omega b|v|^p\,dx. \tag{4.5.31}$$

We write again $v = \alpha + \tilde{v}$ with $\alpha = |\Omega|^{-1}\int_\Omega v\,dx$. We derive from (4.5.16) with $M = 0$, the same C_p, $\epsilon = \epsilon_*$, and $0 < \lambda < \lambda_*$,

$$\frac{\mu - (\lambda+\epsilon)\|b\|_{L^\infty}}{\mu}\int_\Omega |\nabla\tilde{v}|^p\,dx \leqslant \lambda\|b\|_{L^\infty}|\Omega|C_p\epsilon^{1-p}|\alpha|^p. \tag{4.5.32}$$

Since estimates (4.5.20)–(4.5.22) are still valid with $M = 0$, we obtain

$$\begin{aligned}\int_\Omega a|v|^q\,dx &= \alpha^q\left(\int_\Omega a\,dx + \int_\Omega a\big(|1+\tilde{w}|^q - 1\big)\,dx\right)\\ &\leqslant \alpha^q\left(\int_\Omega a\,dx + qc_q\|a\|_{L^\infty}C_{\epsilon,p,q}\lambda^{1/p}\big(|\Omega|^{1/q} + C_q\lambda^{1/p}\big)^q\right).\end{aligned} \tag{4.5.33}$$

The expression is negative by (4.5.23)–(4.5.24) since $\lambda < \lambda_*$. □

PROPOSITION 4.5.4. *Assume that* (A1) *and* (A2) *hold. Then for every* $0 < \lambda < \lambda_*$ *there exists a positive solution* u *of* (4.5.1) *such that* $A(u) > 0$.

PROOF. **Step 1.** The supremum is achieved with the constraint.

Let $\{u_n\}$ be a maximizing sequence for Problem 1, that is,

$$H_\lambda(u_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} A(u_n) = m_1 = \sup\big\{A(\phi):\ H_\lambda(\phi) = 1\big\}. \tag{4.5.34}$$

Clearly, $m_1 \in (0,\infty]$, since $\mathcal{A}_+$ is nonempty. From Lemma 4.5.1, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, and m_1 is finite and positive. There exist a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and

$u \in W^{1,p}(\Omega)$ such that $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$, $u_{n_k} \to u$ strongly in $L^q(\Omega)$. Therefore $A(u) = m_1$ and $H_\lambda(u) \leqslant 1$ by lower semicontinuity. If $0 < H_\lambda(u) < 1$, then for some $k > 1$ we would have $H_\lambda(ku) = 1$ and $A(ku) = k^q m_1 > m_1$, contradicting the definition of m_1. If $H_\lambda(u) \leqslant 0$, it follows from Lemma 4.5.3 that $A(u) < 0$, contradicting $A(u) = m_1 > 0$.

In order to complete the proof, we must show that equation (4.5.1) is verified.

Step 2. Equation (4.5.1) is satisfied by a positive function u with $A(u) > 0$.

We can assume that u is positive by replacing u_n with $|u_n|$. Moreover, Step 1 implies that $\lim_{n_k\to\infty} \int_\Omega |\nabla u_{n_k}|^p\,dx = \int_\Omega |\nabla u|^p\,dx$, and there exists a Lagrange multiplier σ such that

$$DA(u) = \sigma DH_\lambda(u) \quad \Leftrightarrow \quad \begin{cases} a(x)u^{q-1} = \sigma(-\Delta_p u - \lambda b(x)u^{p-1}) & \text{in } \Omega, \\ -|\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5.35}$$

These equations have to be understood in the weak sense, that is, in $(W^{1,p}(\Omega))^*$, which is enough for the duality argument we use. From (4.5.35), we have by taking u as test function

$$\int_\Omega a(x)u^q\,dx = \sigma \int_\Omega \left(|\nabla u|^p - \lambda b(x)u^p\right)dx, \tag{4.5.36}$$

and $m_1 = \sigma$. Since σ is positive, we obtain a positive solution of (4.5.1) satisfying $A(u) > 0$ by replacing u with $\sigma^{1/(q-p)}u$. □

We turn now to Problem 2. If a and b satisfy (A2) and (B) respectively, then $H_\lambda(k) = -1$ and $A(k) < 0$ hold for some positive constant k, and $\mathcal{A}_- = \{\zeta \in W^{1,p}(\Omega)\colon A(\zeta) < 0,\ H_\lambda(\zeta) = -1\}$ is not an empty set. Therefore

$$\inf\bigl\{A(\zeta)\colon \zeta \in W^{1,p}(\Omega),\ H_\lambda(\zeta) = -1\bigr\} =: m_2 \in [-\infty, 0). \tag{4.5.37}$$

However, for this problem, it is not possible to prove that minimizing sequences are bounded, but because of homogeneity, solving Problem 2 is equivalent to solving

PROBLEM 2′. $\quad \inf\bigl\{H_\lambda(\phi)\colon \phi \in W^{1,p}(\Omega),\ A(\phi) = -1\bigr\}.$ (4.5.38)

We start with the following counterpart of Lemma 4.5.1 and improvement of Lemma 4.5.3.

LEMMA 4.5.5. *Assume that* (A2) *holds. If* $0 < \lambda < \lambda_*$, *any sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$ *such that* $\{A(u_n)\}$ *is bounded and* $\{H_\lambda(u_n)\}$ *is negative, has the property that it remains bounded in* $W^{1,p}(\Omega)$.

PROOF. As in Lemma 4.5.1, we set $\alpha_n := |\Omega|^{-1} \int_\Omega u_n\,dx$ and $u_n = \tilde{u}_n + \alpha_n$. By assumption,

$$\int_\Omega |\nabla \tilde{u}_n|^p\,dx = \lambda \int_\Omega b|\alpha_n + \tilde{u}_n|^p\,dx. \tag{4.5.39}$$

We first prove by contradiction that $\{\alpha_n\}$ is bounded. As in Lemmas 4.5.1 and 4.5.3, for the same choice of $\epsilon = \epsilon_*$ and constants, we have

$$\frac{\mu - (\lambda + \epsilon)\|b\|_{L^\infty}}{\mu} \int_\Omega |\nabla \tilde{u}_n|^p \, dx \leqslant \lambda \|b\|_{L^\infty} |\Omega| C_p \epsilon^{1-p} |\alpha_n|^p. \tag{4.5.40}$$

If $|\alpha_n| \to \infty$, we set $v_n = u_n/\alpha_n = 1 + \tilde{v}_n$ and

$$\int_\Omega |\nabla \tilde{v}_n|^p \, dx \leqslant \frac{\mu\lambda}{\mu - (\lambda + \epsilon)\|b\|_{L^\infty}} |\Omega| C_p \epsilon^{1-p} \|b\|_{L^\infty} |\alpha_n|^p. \tag{4.5.41}$$

On the other hand,

$$\begin{aligned} \int_\Omega a|\tilde{v}_n|^q \, dx = O\big(|\alpha_n|^{-q}\big) &= \int_\Omega a \, dx + \int_\Omega a\big(|1 + \tilde{v}_n|^q - 1\big) \, dx \\ &\leqslant \int_\Omega a \, dx + q c_q \|a\|_{L^\infty} C_{\epsilon,p,q} \lambda^{1/p} \big(|\Omega|^{1/q} + C_q \lambda^{1/p}\big)^q. \end{aligned} \tag{4.5.42}$$

But for $0 < \lambda < \lambda_*$ and $\epsilon = \epsilon_*$

$$\int_\Omega a \, dx + q c_q \|a\|_{L^\infty} C_{\epsilon,p,q} \lambda^{1/p} \big(|\Omega|^{1/q} + C_q \lambda^{1/p}\big)^q < 0, \tag{4.5.43}$$

which infers a contradiction for n large enough. Therefore $\{\alpha_n\}$ is bounded. The same holds for $\{\|\nabla u_n\|_{L^p}\}$ by (4.5.40) and for $\{\|u_n\|_{L^p}\}$ by (4.5.10). □

PROPOSITION 4.5.6. *Assume that* (A2) *and* (B) *hold. Then for every* $0 < \lambda < \lambda_*$ *there exists a positive solution* u^* *of* (4.5.1) *such that* $A(u^*) < 0$.

PROOF. Let $\{u_n^*\}$ be a maximizing sequence for Problem 2′, that is, a sequence such that

$$A\big(u_n^*\big) = -1 \quad \text{and} \quad \lim_{n\to\infty} H_\lambda\big(u_n^*\big) = m_3 = \inf\big\{H_\lambda(\phi){:}\ A(\phi) = -1\big\}. \tag{4.5.44}$$

By Lemma 4.5.5, the sequence $\{u_n^*\}$ is bounded in $W^{1,p}(\Omega)$. Then there exist a subsequence $\{u_{n_k}^*\} \subset \{u_n^*\}$ and $u^* \in W^{1,p}(\Omega)$ such that $\nabla u_{n_k}^* \rightharpoonup \nabla u^*$ weakly in $L^p(\Omega)$, $u_{n_k}^* \to u^*$ strongly in $L^p(\Omega)$ and in $L^q(\Omega)$. Therefore $A(u^*) = -1$ and $H_\lambda(u^*) \leqslant m_3$. If $H_\lambda(u^*) = m_3$, the proof is completed. Otherwise we suppose that $H_\lambda(u^*) = m^* < m_3$. Thus there would exist $v \in W^{1,p}(\Omega)$ such that $A(v) = -1$ and $H_\lambda(v) < m_3$, contradicting the minimality of m_3. Consequently, $H_\lambda(u^*) = m_3$, and we finish the proof as in Proposition 4.5.4. □

By combining Propositions 4.5.4 and 4.5.6 the following multiple solutions existence result is derived.

THEOREM 4.5.7. *Assume that* (A1), (A2), *and* (B) *hold. Then for every* $0 < \lambda < \lambda_*$ *there exist two positive solutions* u *and* u^* *of* (4.5.1) *such that* $A(u) > 0$ *and* $A(u^*) < 0$, *respectively.*

4.6. *The case $a \equiv 0 \equiv c$*

In this subsection we consider the following boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ -|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{r-2}u & \text{on } \partial\Omega \end{cases} \tag{4.6.1}$$

with $\lambda > 0$ and the exponents p and r in the following range:

$$1 < p, r < \tilde{p} \quad \text{and} \quad p \neq r. \tag{4.6.2}$$

We set

$$K(u) = \frac{1}{r}\int_{\partial\Omega} k(x)|u|^r \, d\sigma, \tag{4.6.3}$$

which is well defined on $W^{1,p}(\Omega)$ since $r < \tilde{p}$, and

$$E_K(u) = -H_\lambda(u) + K(u), \tag{4.6.4}$$

where H_λ has been defined in Subsection 4.5. Let $q > 0$ be

$$q = \min\left\{C > 0\colon \int_\Omega |\phi|^p \, dx \leqslant C\left(\int_\Omega |\nabla\phi|^p \, dx + \int_{\partial\Omega} |\phi|^p \, d\sigma\right), \right.$$
$$\left. \forall\phi \in W^{1,p}(\Omega)\right\}. \tag{4.6.5}$$

Although the following Poincaré boundary trace and Poincaré–Sobolev boundary trace inequalities are well known, we recall their proofs for the sake of completeness. For any $\phi \in L^1(\Omega)$, we denote $\bar{\phi} = |\Omega|^{-1}\int_\Omega \phi \, dx$.

LEMMA 4.6.1. *There exist two positive constants ν and ν_* such that*

$$\inf\left\{\int_\Omega |\nabla\phi|^p \, dx\colon \phi \in W^{1,p}(\Omega), \int_{\partial\Omega} |\phi - \bar{\phi}|^p \, d\sigma = 1\right\} = \nu, \tag{4.6.6}$$

$$\inf\left\{\int_\Omega |\nabla\phi|^p \, dx\colon \phi \in W^{1,p}(\Omega), \int_{\partial\Omega} |\phi - \bar{\phi}|^{\tilde{p}} \, d\sigma\right\} = \nu^*. \tag{4.6.7}$$

PROOF. We recall the classical Poincaré inequality

$$\int_\Omega |\nabla\phi|^p \, dx = \int_\Omega \left|\nabla(\phi - \bar{\phi})\right|^p dx \geqslant \mu \int_\Omega |\phi - \bar{\phi}|^p \, dx, \tag{4.6.8}$$

and the trace estimate

$$\begin{aligned}\|\phi-\bar{\phi}\|^p_{W^{1-1/p,p}(\partial\Omega)} &\leqslant M\|\phi-\bar{\phi}\|^p_{W^{1,p}(\Omega)}\\ &\leqslant M(1+1/\mu)\int_\Omega |\nabla\phi|^p\,dx. \end{aligned} \tag{4.6.9}$$

Since

$$\begin{aligned}\|\psi\|^p_{L_p(\partial\Omega)} &\leqslant |\partial\Omega|^{1-p/\tilde{p}}\|\psi\|^p_{L_{\tilde{p}}(\partial\Omega)}\\ &\leqslant |\partial\Omega|^{1-p/\tilde{p}}M'\|\psi\|^p_{W^{1-1/p,p}(\partial\Omega)}\end{aligned} \tag{4.6.10}$$

for any $\psi\in W^{1-1/p,p}(\partial\Omega)$ (see [1]), we get (4.6.6) and (4.6.7). □

REMARK 4.6.2. There exists a version of these inequalities involving the boundary average $\hat{\psi}$ of any $\psi\in L^1(\partial\Omega)$ defined by $\hat{\psi}=|\partial\Omega|^{-1}\int_{\partial\Omega}\psi\,dx$. Namely, for any $p\leqslant r<\tilde{p}$ there exists a positive constant $\hat{\nu}$ such that

$$\inf\left\{\int_\Omega|\nabla\phi|^p\,dx\colon\ \phi\in W^{1,p}(\Omega),\ \int_{\partial\Omega}|\phi-\hat{\phi}|^p\,d\sigma=1\right\}=\hat{\nu}. \tag{4.6.11}$$

The proof is easily derived by contradiction and using (4.6.7).

We turn out now to equation (4.6.1) and define the fibering functional as in Subsection 4.5: for $v\in W^{1,p}(\Omega)$ and $s\in\mathbb{R}$ we write

$$E_K(sv)=-|s|^pH_\lambda(v)+|s|^rK(v), \tag{4.6.12}$$

$$\begin{cases}\partial E_K/\partial s=0,\\ s\neq 0\end{cases}\quad\Rightarrow\quad |s|^{r-p}=\frac{pH_\lambda(v)}{rK(v)}=\frac{\int_\Omega(|\nabla v|^p-\lambda b|v|^p)\,dx}{\int_{\partial\Omega}k|v|^r\,d\sigma}. \tag{4.6.13}$$

Of course, the interest relies in the nonspectral case $p\neq q$. Since two signs are possible, we consider two extremal problems.

PROBLEM 3. $\sup\{K(\phi)\colon\ \phi\in W^{1,p}(\Omega),\ H_\lambda(\phi)=1\}$.

PROBLEM 4. $\inf\{K(\phi)\colon\ \phi\in W^{1,p}(\Omega),\ H_\lambda(\phi)=-1\}$.

We introduce the following assumptions on $k=k_+-k_-$.

(K1) $\int_{\partial\Omega}k_+\,d\sigma>0$,

(K2) $\int_{\partial\Omega}k\,d\sigma<0$.

LEMMA 4.6.3. *Assume that* (K2) *holds. Then there exists*

$$\lambda^* = \lambda^*\big(\Omega, p, r, \|b\|_{L^\infty}, \|k\|_{L^\infty}\big) > 0$$

with the following property: if $0 < \lambda < \lambda^*$ *and* $\{u_n\}$ *is any sequence in* $W^{1,p}(\Omega)$ *such that* $\{K(u_n)\}$ *is positive and* $\{H_\lambda(u_n)\}$ *is bounded from above, then* $\{u_n\}$ *remains bounded in* $W^{1,p}(\Omega)$. *Moreover, it satisfies*

$$\lim_{n\to\infty} \sup K(u_n) = m' < \infty.$$

PROOF. As in Lemma 4.6.1, we shall first prove that the sequence $\{\alpha_n\}$ (where α_n is the average of u_n on Ω) is bounded. By contradiction, let us suppose that $|\alpha_n| \to \infty$ and set $u_n = \tilde{u}_n + \alpha_n$ and $w_n = u_n/\alpha_n = 1 + \tilde{w}_n$. Since

$$\int_\Omega |\nabla \tilde{u}_n|^p\, dx - \lambda \int_\Omega b|\alpha_n + \tilde{u}_n|^p\, dx \leqslant M$$

for some constant M, with ϵ satisfying (4.5.17) this implies

$$\frac{\mu - (\lambda+\epsilon)\|b\|_{L^\infty}}{\mu} \int_\Omega |\nabla \tilde{w}_n|^p\, dx \leqslant \lambda \|b\|_{L^\infty} |\Omega| C_p \epsilon^{1-p} + M|\alpha_n|^{-p}$$

(inequalities (4.6.13)–(4.6.18) remain valid), and we get

$$\begin{aligned}\int_{\partial\Omega} |\tilde{w}_n|^r\, dS &\leqslant \nu_*^{-r/\tilde{p}} |\partial\Omega|^{1-r/\tilde{p}} \left(\int_\Omega |\nabla \tilde{w}_n|^p\, dx\right)^{r/p} \\ &\leqslant \big(\lambda\|b\|_{L^\infty}\big)^{r/p} \nu_*^{-r/\tilde{p}} |\partial\Omega|^{1-r/N} \left(\frac{\mu}{\mu - \|b\|_{L^\infty}(\lambda+\epsilon)}\right)^{r/p} \\ &\quad \times C_p^{r/p} \epsilon^{r(1-p)/p} + M|\alpha_n|^{-p} \\ &=: D_{\epsilon,p,r} \lambda^{r/p} + M|\alpha_n|^{-p}. \end{aligned} \tag{4.6.14}$$

The following inequalities are derived from (4.6.20) (with q replaced by r):

$$\begin{aligned} &\int_{\partial\Omega} \big||1+\tilde{w}_n|^r - 1\big|\, d\sigma \\ &\quad \leqslant r\left(\int_{\partial\Omega} |\tilde{w}_n|^r\, d\sigma\right)^{1/r} \left(\int_{\partial\Omega} \big(1+|\tilde{w}_n|\big)^r\, d\sigma\right)^{1-1/r} \\ &\quad \leqslant r\left(\int_{\partial\Omega} |\tilde{w}_n|^r\, d\sigma\right)^{1/r} \left(|\partial\Omega|^{1/r} + \left(\int_{\partial\Omega} |\tilde{w}_n|^r\, d\sigma\right)^{1/r}\right)^{r-1} \\ &\quad \leqslant r c_r \left(\int_{\partial\Omega} |\tilde{w}_n|^r\, d\sigma\right)^{1/r} \left(|\partial\Omega|^{1-1/r} + \left(\int_{\partial\Omega} |\tilde{w}_n|^r\, d\sigma\right)^{1-1/r}\right), \end{aligned} \tag{4.6.15}$$

where $c_r = \max(1, 2^{r-2})$. Therefore,

$$\int_{\partial\Omega} k|1+\tilde{w}_n|^r\, d\sigma = \int_{\partial\Omega} k\, d\sigma + \int_{\partial\Omega} k\big(|1+\tilde{w}_n|^r - 1\big)\, d\sigma$$
$$\leqslant \int_{\partial\Omega} k\, d\sigma + rc_r\|k\|_{L^\infty}\big(D_{\epsilon,p,r}\lambda^{1/p} + M|\alpha_n|^{-1}\big)$$
$$\times \big(|\partial\Omega|^{1/r} + D_{\epsilon,p,r}\lambda^{1/p} + M|\alpha_n|^{-1}\big)^r. \tag{4.6.16}$$

We denote by Λ^ϵ the following expression:

$$\Lambda^\epsilon = \sup\bigg\{\lambda \in \big(0, \mu/\|b\|_{L^\infty} - \epsilon\big):$$
$$\int_{\partial\Omega} k\, d\sigma + rc_r\|k\|_{L^\infty} D_{\epsilon,p,r}\rho^{1/p}\big(|\partial\Omega|^{1/r} + D_{\epsilon,p,r}\rho^{1/p}\big)^r < 0,$$
$$\forall\rho \in (0,\lambda)\bigg\}. \tag{4.6.17}$$

Such an expression defines a positive real number because of the assumption (K2). If we set

$$\lambda^* = \max\big\{\Lambda^\epsilon\colon\ 0 < \epsilon < \mu/\|b\|_{L^\infty}\big\}, \tag{4.6.18}$$

thus maximum is achieved for some $\epsilon = \epsilon^* \in (0, \mu/\|b\|_{L^\infty})$. Since $|\alpha_n| \to \infty$, it follows from the definition of λ^* and (4.6.17) that for any $\lambda \in (0,\lambda^*)$ and $\varepsilon = \varepsilon^*$ there exists an n large enough such that

$$K(u_n) = \int_{\partial\Omega} k|u_n|^r\, d\sigma = |\alpha_n|^r \int_{\partial\Omega} k|1+\tilde{w}_n|^r\, d\sigma < 0, \tag{4.6.19}$$

contradiction.

Since $\{\alpha_n\}$ is bounded, the same is true for $\{\|\nabla u_n\|_{L^p}\}$ by (4.6.16), then for $\{\int_{\partial\Omega} k|u_n|^p\, d\sigma\}$ by (4.6.6) and finally for $\{\|u_n\|_{L^p}\}$ by (4.6.5). Consequently, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. This implies

$$\|u_n\|_{L^r(\partial\Omega)} \leqslant |\partial\Omega|^{1/r-1/\tilde{p}}\|u_n\|_{L^{\tilde{p}}(\partial\Omega)} \leqslant M_1\|u_n\|_{W^{1-1/p,p}(\partial\Omega)}$$
$$\leqslant M_2\|u_n\|_{W^{1,p}(\Omega)}. \tag{4.6.20}$$

Thus the last statement follows (actually $\{u_n|_{\partial\Omega}\}$ is relatively compact in $L^r(\partial\Omega)$). □

LEMMA 4.6.4. *For any* λ, *the set* $\mathcal{B}_+ = \{\zeta \in W^{1,p}(\Omega)\colon\ K(\zeta) > 0,\ H_\lambda(\zeta) = 1\}$ *is non-empty under assumption* (K1).

PROOF. We proceed essentially almost as in Lemma 4.5.2: for any $x_0 \in \partial\Omega$ we consider $\hat{\phi}_{\delta,x_0}$ the first positive eigenfunction of the p-Laplace operator in $W_0^{1,p}(B_\delta(x_0))$, with the normalization

$$\int_{H_\delta(x_0)} \hat{\phi}^r_{\delta,x_0}\, d\sigma_{N-1} = 1, \tag{4.6.21}$$

where $H_\delta(x_0)$ is the intersection of $B_\delta(x_0)$ with any hyperplane H containing x_0 (the first eigenfunction is radial), and $d\sigma$ is the $(N-1)$-dimensional Lebesgue measure. We extend $\hat{\phi}_{\delta,x_0}$ by zero outside Ω and call $\hat{\phi}_{\delta,x_0,\Omega}$ its restriction to Ω. Then $\hat{\phi}_{\delta,x_0,\Omega}$ belongs to $W^{1,p}(\Omega)$. Since $\partial\Omega$ is smooth and $\hat{\phi}_{\alpha\delta,x_0}(x) = \alpha^{-N+1}\hat{\phi}_{\delta,x_0}(\alpha^{-1}(x-x_0)+x)$, we have

$$\begin{aligned}\int_{H_\delta(x_0)} |\nabla\hat{\phi}_{\delta,x_0,\Omega}|^p\, d\sigma &= 2^{-1}\int_{B_\delta(x_0)} |\nabla\hat{\phi}_{\delta,x_0}|^p\, dx\big(1+o(1)\big)\\ &= 2^{-1}\nu_\delta \int_{B_\delta(x_0)} |\hat{\phi}_{\delta,x_0}|^p\, dx\big(1+o(1)\big)\end{aligned} \tag{4.6.22}$$

and

$$\int_{\partial\Omega\cap B_\delta(x_0)} \hat{\phi}^r_{\delta,x_0,\Omega}\, d\sigma = 1+o(1)$$

as $\delta \to 0$, uniformly with respect to $x_0 \in \partial\Omega$. Clearly,

$$\lim_{\delta\to 0} H_\lambda(\hat{\phi}_{\delta,x_0,\Omega}) = \infty, \tag{4.6.23}$$

and the following concentration occurs:

$$\lim_{\delta\to 0} K(\hat{\phi}_{\delta,x_0,\Omega}) = k(x_0). \tag{4.6.24}$$

The end of the proof is as in Lemma 4.5.2, and we conclude that $\mathcal{B}_+$ is not empty if the function k satisfies (K1). □

LEMMA 4.6.5. *Assume that* (K2) *holds. Then for every* $\lambda \in (0,\lambda^*)$ *and* $v \in W^{1,p}(\Omega)$, $H_\lambda(v) \leqslant 0$ *and* v *not identically zero implies* $K(v) < 0$.

PROOF. We can assume that $v = \alpha + \tilde{v}$ (with $\alpha = |\Omega|^{-1}\int_\Omega v\, dx$) is nonnegative. Since $H_\lambda(v) \leqslant 0$, we have (4.5.32) as in Lemma 4.5.3. If we take $\epsilon = \epsilon^*$, $0 < \lambda < \lambda^*$ and set $\overline{w} = \bar{v}/\alpha$, inequality (4.6.33) is replaced by

$$\begin{aligned}&\int_{\partial\Omega} k|\alpha+\tilde{v}|^r\, d\sigma\\ &\quad = \alpha^r\left(\int_{\partial\Omega} k\, d\sigma + \int_\Omega k\big(|1+\tilde{w}|^r - 1\big)\, d\sigma\right)\end{aligned}$$

$$\leqslant \alpha^r\left(\int_{\partial\Omega} k\,d\sigma + rc_r\|k\|_{L^\infty} D_{\epsilon,p,r}\lambda^{1/p}\left(|\partial\Omega|^{1/r} + D_{\epsilon,p,r}\lambda^{1/p}\right)^r\right). \quad (4.6.25)$$

This expression is negative because of the choice of λ. □

PROPOSITION 4.6.6. *Assume that* (K1) *and* (K2) *hold. Then for every* $0 < \lambda < \lambda^*$ *there exists a positive solution* u *of* (4.6.1) *such that* $K(u) > 0$.

PROOF. Let $\{u_n\}$ be a maximizing sequence for Problem 3, that is,

$$H_\lambda(u_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} K(u_n) = m_4 = \sup\{K(\phi)\colon\ H_\lambda(\phi) = 1\}. \quad (4.6.26)$$

As in Proposition 4.5.4, the first step is to prove that the supremum is achieved with the constraint. Clearly, $m_4 \in (0,\infty]$, since $\mathcal{B}_+$ is nonempty. From Lemma 4.6.3, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, and m_4 is finite and positive. There exist a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in W^{1,p}(\Omega)$ such that $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$, $u_{n_k}|_{\partial\Omega} \to u|_{\partial\Omega}$ strongly in $L^r(\partial\Omega)$. Thus $K(u) = m_4$ and $H_\lambda(u) \leqslant 1$ by lower semicontinuity. If $0 < H_\lambda(u) < 1$, then for some $k > 1$ we would have $H_\lambda(ku) = 1$ and $K(ku) = k^r m_4 > m_4$, contradicting the definition of m_4. If $H_\lambda(u) \leqslant 0$, it follows from Lemma 4.6.4 that $K(u) < 0$, contradicting $K(u) = m_4 > 0$. Moreover, it can be assumed that u_n and then u are nonnegative, since $H_\lambda(v) = H_\lambda(|v|)$ and $K(v) = K(|v|)$. Consequently, there exists a Lagrange multiplier μ such that

$$DK(u) = \mu DH_\lambda(u) \quad (4.6.27)$$

in $(W^{1,p}(\Omega))'$. This means

$$\int_{\partial\Omega} k(x)u^{r-1}\eta\,dx = \mu\int_\Omega \left(|\nabla u|^{p-2}\nabla u\cdot\nabla\eta - \lambda b u^{p-1}\eta\right)dx \quad (4.6.28)$$

for every $\eta \in W^{1,p}(\Omega)$. Taking $u = \eta$ gives $\int_{\partial\Omega} k(x)u^r\,dx = \mu = m_4$. Replacing u with $\mu^{1/(r-p)}u$ gives the result. □

We turn now to Problem 4. If k and b satisfy (K2) and (B) respectively, then, for some positive constant c, $H_\lambda(c) = -1$ and $K(c) < 0$, and the set $\mathcal{B}_- = \{\zeta \in W^{1,p}(\Omega)\colon K(\zeta) < 0,\ H_\lambda(\zeta) = -1\}$ is not empty. Therefore

$$\inf\{K(\zeta)\colon\ \zeta \in W^{1,p}(\Omega)\colon\ H_\lambda(\zeta) = -1\} = m_5 \in [-\infty, 0). \quad (4.6.29)$$

As in Subsection 4.5, it is not possible to prove that minimizing sequences are bounded. Therefore we replace Problem 4 with an equivalent one,

PROBLEM 4′. $\quad \inf\{H_\lambda(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ K(\phi) = -1\}. \quad (4.6.30)$

We start with the following improvement of Lemma 4.6.4.

LEMMA 4.6.7. *Assume that* (K2) *holds. If* $0 < \lambda < \lambda_*$, *then any sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$ *such that* $\{K(u_n)\}$ *is bounded and* $\{H_\lambda(u_n)\}$ *is negative, has the property that it remains bounded in* $W^{1,p}(\Omega)$.

PROOF. Set $\alpha_n := |\Omega|^{-1} \int_\Omega u_n\,dx$, $u_n = \tilde{u}_n + \alpha_n$ and $\tilde{v}_n = \tilde{u}_n/\alpha_n$. By assumption,

$$\int_\Omega |\nabla \tilde{u}_n|^p\,dx \leqslant \lambda \int_\Omega b|\alpha_n + \tilde{u}_n|^p\,dx. \tag{4.6.31}$$

Let us suppose that $\{\alpha_n\}$ is not bounded. Then for $\epsilon = \epsilon_*$ and λ satisfying (4.6.17), we have

$$\begin{aligned}\int_{\partial\Omega} k|v_n|^r\,d\sigma = O\big(|\alpha_n|^{-r}\big) &= \int_{\partial\Omega} k\,d\sigma + \int_{\partial\Omega} k\big(|1+\tilde{v}_n|^r - 1\big)\,d\sigma \\ &\leqslant \int_{\partial\Omega} k\,d\sigma + rc_r\|k\|_{L^\infty} D_{\epsilon,p,r}\lambda^{1/p}\big(|\partial\Omega|^{1/r} + D_{\epsilon,p,r}\lambda^{1/p}\big)^r.\end{aligned} \tag{4.6.32}$$

If we assume that $|\alpha_n| \to \infty$, the right-hand side of (4.6.32) becomes negative for n large enough, which is a contradiction. Therefore $\{\alpha_n\}$ is bounded, and the same holds for $\{\|\nabla u_n\|_{L^p}\}$. We end the proof as in Lemma 4.6.3. □

PROPOSITION 4.6.8. *Assume that* (K2) *and* (B) *hold. Then for every* $0 < \lambda < \lambda_*$ *there exists a positive solution* u^* *of* (4.6.1) *such that* $K(u^*) < 0$.

PROOF. Let $\{u_n^*\}$ be a minimizing sequence for Problem 4′, that is, a sequence such that

$$K\big(u_n^*\big) = -1 \quad \text{and} \quad \lim_{n\to\infty} H_\lambda\big(u_n^*\big) = m_5 = \inf\big\{H_\lambda(\phi)\colon\ K(\phi) = -1\big\}. \tag{4.6.33}$$

From Lemma 4.6.7, the sequence $\{u_n^*\}$ is bounded in $W^{1,p}(\Omega)$. Then there exist a subsequence $\{u_{n_k}^*\} \subset \{u_n^*\}$ and $u^* \in W^{1,p}(\Omega)$ such that $\nabla u_{n_k}^* \rightharpoonup \nabla u^*$ weakly in $L^p(\Omega)$ and $u_{n_k}^*|_{\partial\Omega} \to u^*|_{\partial\Omega}$ strongly in $L^r(\partial\Omega)$. Therefore $K(u^*) = -1$ and $H_\lambda(u^*) \leqslant m_5$. If $H_\lambda(u^*) = m_5$, the proof is completed. Otherwise we suppose that $H_\lambda(u^*) = m^* < m_5$. Thus there would exist $v \in W^{1,p}(\Omega)$ such that $K(v) = -1$ and $H_\lambda(v) < m_5$, contradiction. Consequently, $H_\lambda(u^*) = m_5$, and the completion of the proof is as in Proposition 4.6.6. □

By combining Propositions 4.6.6 and 4.6.8 we obtain

THEOREM 4.6.9. *Assume that* (K1), (K2), *and* (B) *hold. Then for every* $0 < \lambda < \lambda_*$ *there exist two positive solutions* u *and* u^* *of* (4.6.1) *such that* $K(u) > 0$ *and* $K(u^*) < 0$, *respectively.*

4.7. *The general case with $r < q$*

In this subsection we are interested in the following boundary value problem:

$$\begin{cases} -\Delta_p u - \lambda b(x)|u|^{p-2}u = a(x)|u|^{q-2}u + c(x)|u|^{s-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{r-1}u & \text{on } \partial\Omega, \end{cases} \tag{4.7.1}$$

where $\lambda > 0$ and the exponents p, q, s and r satisfy

$$\begin{cases} \text{(i)} \quad 1 < p < s < q < p^* - 1, \\ \text{(ii)} \quad 1 < r < \min(q, \tilde{p} - 1). \end{cases} \tag{4.7.2}$$

When $\phi \in W^{1,p}(\Omega)$ and the exponents are as above, the following expressions are well defined:

$$H_\lambda(\phi) = \frac{1}{p} \int_\Omega \big(|\nabla\phi|^p - \lambda b(x)|\phi|^p\big)\, dx, \tag{4.7.3}$$

$$A(\phi) = \frac{1}{q} \int_\Omega a(x)|\phi|^q\, dx, \tag{4.7.4}$$

$$K(\phi) = \frac{1}{r} \int_{\partial\Omega} k(x)|\phi|^r\, d\sigma, \tag{4.7.5}$$

$$C(\phi) = \frac{1}{s} \int_\Omega c(x)|\phi|^s\, dx. \tag{4.7.6}$$

The solutions u of (4.7.1) are positive critical points of the functional

$$E(\phi) = -H_\lambda(\phi) + A(\phi) + C(\phi) + K(\phi). \tag{4.7.7}$$

If $\phi \in W^{1,p}(\Omega)$ and $\phi = t\psi$, then

$$E(t\psi) = \tilde{E}(t, \psi) = -|t|^p H_\lambda(\psi) + |t|^q A(\psi) + |t|^s C(\psi) + |t|^r K(\psi). \tag{4.7.8}$$

The assumptions on the functions a, b, k, and c, besides measurability and essential boundedness, are the following ones, which have been partially used in the previous subsections:

(A1) $\displaystyle\int_\Omega a_+(x)\,dx > 0.$

(A2) $\displaystyle\int_\Omega a(x)\,dx < 0.$

(B) $\displaystyle\int_\Omega b(x)\,dx > 0.$

(K3) $k(x) \leqslant 0$ a.e. in $\partial\Omega$.

(C) $c(x) \leqslant 0$ a.e. in Ω.

REMARK 4.7.1. The functional $\tilde{E}$ is C^1 on $W^{1,p}(\Omega) \times \mathbb{R}$, and at this point it is important to notice that $C(\psi) \leqslant 0$ and $K(\psi) \leqslant 0$, but $H_\lambda(\psi)$ and $A(\psi)$ have no prescribed sign independent of ψ. We denote

$$\mathcal{C}_+ = \bigl\{\phi \in W^{1,p}(\Omega)\colon\ H_\lambda(\phi) > 0,\ A(\phi) > 0\bigr\}. \tag{4.7.9}$$

Under assumption (A1) it follows from Lemma 4.5.2 that the set $\mathcal{C}_+$ is a nonempty open subset of $W^{1,p}(\Omega)$. From (4.7.8) we have

$$\begin{aligned}\frac{\partial \tilde{E}}{\partial t}(t,\psi) &= -p|t|^{p-2}tH_\lambda(\psi) + q|t|^{q-2}tA(\psi) + s|t|^{s-2}tC(\psi)\\ &\qquad + r|t|^{r-2}tK(\psi)\\ &= \tilde{l}(t,\psi).\end{aligned} \tag{4.7.10}$$

The following is a straightforward consequence of the above expression.

LEMMA 4.7.2. *Assume that* (A1) *holds and* $\psi \in \mathcal{C}_+$. *Then the function* $t \mapsto l(t,\psi)$ *admits a unique zero on* $(0,\infty)$ *that we shall denote* $t_* = t_*(\psi)$.

LEMMA 4.7.3. *Assume that* (A1) *holds and* $\psi \in \mathcal{C}_+$. *Then*

$$t_*(\psi) \geqslant t_*^0(\psi) = \bigl(pH_\lambda(\psi)/\bigl(qA(\psi)\bigr)\bigr)^{1/(q-p)}, \tag{4.7.11}$$

and

$$\tilde{E}(t_*,\psi) \leqslant \tilde{E}_0(t_*,\psi) \leqslant \frac{p-q}{q}H_\lambda(\psi)\left(\frac{pH_\lambda(\psi)}{qA(\psi)}\right)^{p/(q-p)}, \tag{4.7.12}$$

where

$$\tilde{E}_0(t,\psi) := -|t|^p H_\lambda(\psi) + |t|^q A(\psi) =: E_A(t\psi).$$

PROOF. By definition of $\tilde{E}_0$, we have

$$\frac{\partial \tilde{E}_0(t,\psi)}{\partial t} = -p|t|^{p-2} t H_\lambda(\psi) + q|t|^{q-2} t A(\psi) = \tilde{l}^0(t,\psi).$$

Then $\tilde{l}^0(t,\psi) = 0$ where $t_*^0 = t_*^0(\psi) = (pH_\lambda(\psi)/(qA(\psi)))^{1/(q-p)}$. Since $C(\psi) \leqslant 0$ and $K(\psi) \leqslant 0$, there always holds

$$\tilde{E}(t,\psi) \leqslant \tilde{E}_0(t,\psi) \quad \text{and} \quad \tilde{l}(t,\psi) \leqslant \tilde{l}^0(t,\psi) \tag{4.7.13}$$

for $t > 0$. This implies $l(t_*^0 \psi) \leqslant 0$ and (4.7.11). Since $t \mapsto \tilde{E}(t,\psi)$ is decreasing on $(0, t_*)$, we have

$$\begin{aligned} \tilde{E}(t_*,\psi) &\leqslant \tilde{E}\big(t_*^0,\psi\big) \\ &= -\big(t_*^0\big)^p H_\lambda(\psi) + \big(t_*^0\big)^q A(\psi) + \big(t_*^0\big)^s C(\psi) + \big(t_*^0\big)^r K(\psi) \\ &\leqslant -\big(t_*^0\big)^p H_\lambda(\psi) + \big(t_*^0\big)^q A(\psi) \\ &= \frac{p-q}{q} H_\lambda(\psi) \left(\frac{pH_\lambda(\psi)}{qA(\psi)}\right)^{p/(q-p)}, \end{aligned} \tag{4.7.14}$$

which is (4.7.12). □

We denote by E_* the *reduced functional* which is defined on $\mathcal{C}_*$ by

$$E_*(\psi) = \tilde{E}(t_*,\psi). \tag{4.7.15}$$

From the definition of $t_* = t_*(\psi)$,

$$E_*(\psi) = \inf_{t>0} E(t\psi) = E_*(\tau\psi) \quad (\forall \tau > 0). \tag{4.7.16}$$

LEMMA 4.7.4. *The mapping $\phi \mapsto E_*(\phi)$ from $\mathcal{C}_+$ into $\mathbb{R}$ is C^1 in the strong $W^{1,p}(\Omega)$-topology.*

PROOF. We have to check that the implicit function theorem applies to $(t,\psi) \mapsto \tilde{l}(t,\psi)$ at $t = t_*$. Since t_* is the positive zero of

$$t \mapsto \tilde{l}_r(t,\psi) = -pH_\lambda(\psi) + qt^{q-p} A(\psi) + st^{s-p} C(\psi) + rt^{r-p} K(\psi), \tag{4.7.17}$$

we can look for the t-partial derivative of the above new function, namely

$$\begin{aligned} \frac{\partial \tilde{l}_r}{\partial t}(t,\psi) &= q(q-p)t^{q-p-1} A(\psi) + s(s-p)t^{s-p-1} C(\psi) \\ &\quad + r(r-p)t^{r-p-1} K(\psi). \end{aligned}$$

Consequently,

$$\tilde{l}_r(t,\psi) - \frac{t_*}{q-p}\frac{\partial \tilde{l}_r}{\partial t}(t_*,\psi)$$
$$= -pH_\lambda(\psi) + s\left(1 - \frac{s-p}{q-p}\right)t_*^{s-p}C(\psi) + r\left(1 - \frac{r-p}{q-p}\right)t_*^{r-p}K(\psi) < 0,$$

and $\frac{\partial \tilde{l}_r}{\partial t}(t_*,\psi) > 0$. It follows that $\psi \mapsto t_*(\psi)$ is C^1, and the same holds by composition for $\phi \mapsto E_*(\phi)$. □

Suppose now that $u \in W^{1,p}(\Omega)$ is a critical point of E such that $H_\lambda(u) > 0$, then necessarily $A(u) > 0$. If we set $u = tv$, $t > 0$ is uniquely determined by the condition $H_\lambda(v) = 1$, and $\frac{d}{ds}E(sv)|_{s=t} = 0$. The only critical point of $s \mapsto E(sv)$ is achieved at $t = t_*(v)$, in which case $E(t_*(v)v) = E_*(v) < 0$. Therefore, we introduced the new problem.

PROBLEM 5.

$$\sup\bigl\{E_*(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1,\ A(\phi) > 0\bigr\}. \tag{4.7.18}$$

PROPOSITION 4.7.5. *Assume* (A1), (A2), (K3), *and* (C) *hold. Then for every* $0 < \lambda < \lambda_*$ (*which has been defined in Subsection* 4.5) *there exists a positive solution* $u \in W^{1,p}(\Omega)$ *of* (4.7.1) *with the property that* $A(u) > 0$.

PROOF. Let $\{u_n\}$ be a maximizing (and nonnegative without any loss of generality) sequence for Problem 5, that is,

$$A(u_n) > 0, \quad H_\lambda(u_n) = 1,$$
$$\lim_{n\to\infty} E_*(u_n) =: m_6 = \sup\bigl\{E_*(\phi)\colon\ H_\lambda(\phi) = 1,\ A(\phi) > 0\bigr\}. \tag{4.7.19}$$

Step 1. From Lemma 4.5.1, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. If $t_* = t_*(u_n)$, Lemma 4.7.3 implies

$$\tilde{E}(t_*,u_n) =: E_*(u_n) \leqslant \frac{p-q}{q}\left(\frac{p}{q}\right)^{p/(q-p)}\bigl(A(u_n)\bigr)^{-p/(q-p)}. \tag{4.7.20}$$

Since $1 < p < q$ and $\{E_*(u_n)\}$ is increasing and negative, the right-hand side of (4.7.20) is minorized by some negative number. It follows that $\{A(u_n)\}$ is bounded from below by some positive constant σ.

From the Sobolev imbedding theorem and the trace theorem, there exist a subsequence $\{u_{n_l}\}$ and a function $u \in W^{1,p}(\Omega)$ such that $\nabla u_{n_l} \rightharpoonup \nabla u$ in $L^p(\Omega)$, $u_{n_l} \to u$ in $L^p(\Omega) \cap L^s(\Omega) \cap L^q(\Omega)$ and $u_{n_r}|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^r(\partial\Omega)$.

Step 2. We claim that $E_*(u) = m_6$ and $H_\lambda(u) = 1$.

If we impose $H_\lambda(u_n) = 1$ in (4.7.17), then $t_{*,n} = t_*(u_n)$ is defined as the positive zero of

$$t \mapsto \tilde{l}_r(t, u_n) = -p + qt^{q-p} A(u_n) + st^{s-p} C(u_n) + rt^{r-p} K(u_n).$$

Since the coefficients $A(u_{n_l})$, $C(u_{n_l})$, and $K(u_{n_l})$ converge respectively to $A(u)$, $C(u)$, and $K(u)$, with $A(u_{n_l}) \geqslant \sigma > 0$, it follows that $t_{*,n_l} = t_*(u_{n_l}) \to t_*(u)$ as $n_l \to \infty$. Therefore

$$E_*(u_{n_l}) = -t^p_{*,n_l} + t^q_{*,n_l} A(u_{n_l}) + t^s_{*,n_l} C(u_{n_l}) + t^r_{*,n_l} K(u_{n_l}) \to E_*(u). \quad (4.7.21)$$

Therefore $E_*(u) = m_6$.

If $0 < H_\lambda(u) < 1$, we replace u by σu for some $\sigma > 1$ such that $H_\lambda(\sigma u) = 1$. Then $\sigma u \in \mathcal{C}_+$, and if we set $t^\sigma_* = t_*(\sigma u)$,

$$E_*(\sigma u) = E_*(u) = \sup\big\{E_*(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1,\ A(\phi) > 0\big\}. \quad (4.7.22)$$

Therefore we can assume that $H(u) = 1$ by replacing u with σu. If $H_\lambda(u) \leqslant 0$, we have

$$\int_\Omega a(x)|u|^q\,dx < 0 \quad (4.7.23)$$

from Lemma 4.5.3 whenever $\lambda \in (0, \lambda_*)$.

Step 3. *End of the proof.* In Steps 1–2 we have shown the existence of a critical value of E_* under the constraint $\{\phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1,\ A(\phi) > 0\}$. Therefore there exists a Lagrange multiplier τ such that $DE_*(u) = \tau DH_\lambda(u)$. But $E_*(u) = \tilde{E}(t_*, u)$. Since $\frac{\partial \tilde{E}}{\partial t}(t_*, u) = 0$, the Lagrange multipliers equation is reduced to

$$D_u \tilde{E}(t_*, u) = \tau D_u H_\lambda(u), \quad (4.7.24)$$

or equivalently,

$$\begin{cases} t^q_* a(x)u^{q-1} + t^s_* c(x)u^{s-1} = -\tau\Delta_p u - \lambda b(x)\tau u^{p-1} & \text{in } \Omega, \\ \tau|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = t^r_* k(x)u^{r-1} & \text{on } \partial\Omega. \end{cases} \quad (4.7.25)$$

Therefore

$$-p\tau H_\lambda(u) + qt^q_* A(u) + st^s_* C(u) + rt^r_* K(u) = 0. \quad (4.7.26)$$

But $-pt^p_* H_\lambda(\psi) + qt^q_* A(\psi) + st^s_* C(\psi) + rt^r_* K(\psi) = 0$ from the bifurcation equation $\tilde{l}(t_*, u) = 0$. Therefore $\tau = t^p_*$, and by homogeneity we have found a positive solution of (4.7.1) with the property that $A(u) > 0$ and $H_\lambda(u) > 0$. □

REMARK 4.7.6. We could also have looked for critical points of E in $\mathcal{C}_+$ as infimum of the reduced functional, that is,

$$\inf\{E_*(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1,\ A(\phi) > 0\}. \tag{4.7.27}$$

However, in this case the infimum is zero and it is not achieved. Therefore the solution u we have obtained by Proposition 4.7.5 is a saddle point for E.

Next, we look for critical points under negativity assumptions on A and H_λ. We denote

$$\mathcal{C}_- = \{\phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) < 0,\ A(\phi) < 0\}. \tag{4.7.28}$$

Such a set is open in $W^{1,p}(\Omega)$.

LEMMA 4.7.7. *Suppose $\psi \in \mathcal{C}_-$. Then*

(i) *The function $t \mapsto l(t\psi)$ admits a unique zero on $(0,\infty)$ that we shall denote $t^* = t^*(\psi)$.*

(ii) *The following estimates hold*:

$$t^*(\psi) \leqslant t_0^*(\psi) = \left(\frac{pH_\lambda(\psi)}{qA(\psi)}\right)^{1/(q-p)} \tag{4.7.29}$$

and

$$0 < \tilde{E}(t^*, \psi) \leqslant \tilde{E}^0(t_0^*, \psi) = \frac{p-q}{q} H_\lambda(\psi)\left(\frac{pH_\lambda(\psi)}{qA(\psi)}\right)^{p/(q-p)}, \tag{4.7.30}$$

where

$$\tilde{E}^0(t,\psi) := -|t|^p H_\lambda(\psi) + |t|^q A(\psi) = E_A(t\psi).$$

(iii) *The mapping $\psi \mapsto t^*(\psi)$ is C^1 from $\mathcal{C}_-$ into $(0,\infty)$.*

(iv) *If $\psi, \phi \in \mathcal{C}_-$ are such that $A(\phi) = A(\psi)$, $C(\phi) = C(\psi)$, $K(\phi) = K(\psi)$ and $H_\lambda(\phi) < H_\lambda(\psi) < 0$, then $t^*(\phi) > t^*(\psi)$ and $\tilde{E}(t^*(\phi),\phi) > \tilde{E}(t^*(\psi),\psi)$.*

PROOF. The first statement is clear from (4.7.8) since $\psi \in \mathcal{C}_-$, $C(\psi) \leqslant 0$ and $A(\psi) \leqslant 0$. As in the positive case, there always holds

$$\tilde{E}(t,\psi) \leqslant \tilde{E}^0(t,\psi) = -t^p H_\lambda(\psi) + t^q A(\psi) \tag{4.7.31}$$

and

$$\frac{\partial \tilde{E}(t,\psi)}{\partial t} = \tilde{l}(t,\psi) \leqslant \frac{\partial \tilde{E}^0}{\partial t} = \tilde{l}^0(t,\psi) = -pt^{p-1}H_\lambda(\psi) + qt^{q-1}A(\psi)$$

for $t \geqslant 0$. Since $\tilde{l}^0(t,\psi) = 0$ for positive t if and only if $t = t_0^*(\psi) = (pH_\lambda(\psi)/(qA(\psi)))^{1/(q-p)}$, we have (4.7.29), and (4.7.30) follows from (4.7.31). The statement (iii) is obtained by the same algebraic computation as the one in Lemma 4.7.4. For proving (iv) we recall that $t^* = t^*(\psi)$ is defined as being the positive zero of (4.7.17). If we write $X(\psi) = H_\lambda(\psi)$, $A(\psi) = \mathbf{A}$, $C(\psi) = \mathbf{C}$ and $K(\psi) = \mathbf{K}$, then $t^* = t^*(\psi)$ satisfies

$$-pX(\psi) + q(t^*)^{q-p}\mathbf{A} + s(t^*)^{s-p}\mathbf{C} + r(t^*)^{r-p}\mathbf{K} = 0 \tag{4.7.32}$$

and

$$-p + \big(q(q-p)(t^*)^{q-p-1}\mathbf{A} + s(s-p)(t^*)^{s-p-1}\mathbf{C} + r(r-p)(t^*)^{r-p-1}\mathbf{K}\big)\frac{\partial t^*(\psi)}{\partial X(\psi)} = 0. \tag{4.7.33}$$

Since $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{K}$ are fixed and negative, $\frac{\partial t^*(\psi)}{\partial X(\psi)} < 0$. Thus $H_\lambda(\phi) < H_\lambda(\psi) < 0$ implies $t^*(\phi) > t^*(\psi)$. Because the function $t \mapsto E(t\phi)$ is increasing on $(0, t^*(\phi))$, and $A(\phi) = A(\psi)$, $C(\phi) = C(\psi)$, and $K(\phi) = K(\psi)$ we finally deduce $\tilde{E}(t^*(\phi),\phi) > \tilde{E}(t^*(\psi),\psi)$.

Defining the new reduced functional E^* by

$$E^*(\psi) = \tilde{E}(t^*,\psi), \tag{4.7.34}$$

we get

$$E^*(\psi) = \sup_{t>0} E(t\psi) = E^*(\tau\psi) \quad (\forall \tau > 0). \tag{4.7.35}$$

It follows from Lemma 4.7.7(iii) that the mapping E^* is C^1 on $\mathcal{C}_-$. As in the positive case, we look for a critical point of E in $\mathcal{A}_-$ under the form □

PROBLEM 6.

$$\sup\big\{E^*(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) < 0,\ A(\phi) = -1\big\}. \tag{4.7.36}$$

PROPOSITION 4.7.8. *Assume* (A2), (K3), *and* (C) *hold. Then for every* $0 < \lambda < \lambda_*$, *there exists a positive solution* $u \in W^{1,p}(\Omega)$ *of* (4.7.1) *with the property that* $A(u) < 0$.

PROOF. Let $\{u_n\}$ be a nonnegative maximizing sequence for E^* in $\mathcal{C}_-$. Then $A(u_n) = -1$, $H_\lambda(u_n) < 0$ and

$$\lim_{n\to\infty} E^*(u_n) =: m_7 = \sup\big\{E^*(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) < 0,\ A(\phi) = -1\big\}. \tag{4.7.37}$$

But

$$
\begin{aligned}
E^*(u_n) &\leqslant E_0^*(u_n)\\
&\leqslant \sup\bigl\{E_0^*(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) < 0,\ A(\phi) = -1\bigr\}\\
&= \frac{q-p}{q}\left(\frac{p}{q}\right)^{p/(q-p)} \sup\bigl\{\bigl(-H_\lambda(\phi)\bigr)^{q/(q-p)}\colon\ \phi \in W^{1,p}(\Omega),\ A(\phi) = -1\bigr\}\\
&= \frac{q-p}{q}\left(\frac{p}{q}\right)^{p/(q-p)} \bigl(-\inf\bigl\{H_\lambda(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ A(\phi) = -1\bigr\}\bigr)^{q/(q-p)}\\
&= \frac{q-p}{q}\left(\frac{p}{q}\right)^{p/(q-p)} (-m_3)^{q/(q-p)}, \qquad (4.7.38)
\end{aligned}
$$

where m_3 was defined and proved to be finite in Proposition 4.5.6. Therefore m_7 is positive and finite. From Lemma 4.5.1 $H(u_n) \leqslant 0$ and $A(u_n) = -1$ imply that $\{u_n\}$ remains bounded in $W^{1,p}(\Omega)$, provided that $0 < \lambda < \lambda^*$. Then there exist a subsequence $\{u_{n_l}\}$ and a function $u \in W^{1,p}(\Omega)$ such that $\nabla u_{n_l} \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$, $u_{n_l} \to u$ in $L^p(\Omega) \cap L^s(\Omega) \cap L^q(\Omega)$, and $u_{n_l}|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^r(\partial\Omega)$.

By semicontinuity $H_\lambda(u) \leqslant \lim_{n\to\infty} \inf H_\lambda(u_n) \leqslant 0$. Because the exponents q, r, s are subcritical,

$$
\lim_{n_l\to\infty} A(u_{n_l}) = A(u) = -1, \quad \lim_{n_l\to\infty} C(u_{n_l}) = C(u) \quad \text{and}
$$
$$
\lim_{n_l\to\infty} K(u_{n_l}) = K(u).
$$

Suppose that $H_\lambda(u) \leqslant \lim_{n\to\infty} \inf H_\lambda(u_n) = \mathbf{H} \leqslant 0$. Since $t_n^* = t^*(u_n)$ satisfies

$$
-pH_\lambda(u_n) - q\bigl(t_n^*\bigr)^{q-p} + s\bigl(t_n^*\bigr)^{s-p} C(u_n) + r\bigl(t_n^*\bigr)^{r-p} K(u_n) = 0, \qquad (4.7.39)
$$

it follows by the implicit function theorem that $t_{n_l}^* \to t_\infty$, where

$$
-p\mathbf{H} - q(t_\infty)^{q-p} + s(t_\infty)^{s-p} C(u) + r(t_\infty)^{r-p} K(u) = 0. \qquad (4.7.40)
$$

As a consequence of Lemma 4.7.7(iv), $t_\infty < t^* = t^*(u)$, which is defined by

$$
-pH_\lambda(u) - q(t^*)^{q-p} + s(t^*)^{s-p} C(u) + r(t^*)^{r-p} K(u) = 0 \qquad (4.7.41)
$$

and

$$
E^*(u) > \lim_{n\to\infty} E^*(u_n) = m_7, \qquad (4.7.42)
$$

contradicting the definition of m_7. Therefore $H_\lambda(u) = \lim_{n\to\infty}\inf H_\lambda(u_n) \leqslant 0$. If $H_\lambda(u) = 0$, (4.7.39) and (4.7.40) imply that $t^*_{n_l} \to 0$ and $\lim_{n\to\infty} E^*(u_n) = 0$, contradicting again the definition of m_7. Therefore $H_\lambda(u) < 0$, $A(u) = -1$ and

$$E^*(u) = m_7. \tag{4.7.43}$$

As in Proposition 4.5.4, (4.5.28) holds with $\tau = (t^*)^p$ from the bifurcation equation $\tilde{l}(t^*, u) = 0$. This completes the proof. □

The following result follows from Propositions 4.7.5–4.7.8.

THEOREM 4.7.9. *Assume that* (A1), (A2), (K3), (C), *and* (*B*) *hold*. *Then for every* $0 < \lambda < \lambda_*$ *there exist two positive solutions* u *and* u^* *of* (4.7.1) *in* $W^{1,p}(\Omega)$ *such that* $A(u) > 0$ *and* $A(u^*) < 0$, *respectively.*

4.8. *The general case with $r = q$*

In this subsection we first consider the following boundary value problem:

$$\begin{cases} -\Delta_p u - \lambda b(x)|u|^{p-2}u = a(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{4.8.1}$$

where $\lambda > 0$ and the exponents p and q satisfy

$$1 < p < q < \tilde{p} - 1. \tag{4.8.2}$$

The functional corresponding to this problem is again

$$E(\phi) = -H_\lambda(\phi) + A(\phi) + K(\phi),$$

where A and K have been defined in Subsection 4.7. However, since the exponents involved in A and K are the same, the natural assumption has to be made directly on the new functional

$$\phi \mapsto G(\phi) = \frac{1}{q}\int_\Omega a(x)|\phi|^q\,dx + \frac{1}{q}\int_{\partial\Omega} k(x)|\phi|^q\,d\sigma, \tag{4.8.3}$$

which is well defined when $\phi \in W^{1,p}(\Omega)$. The solutions u of (4.8.1) are positive critical points of

$$E_G(\phi) = -H_\lambda(\phi) + G(\phi). \tag{4.8.4}$$

We introduced global assumptions on G, namely,

(G1) $G(\varphi_0) > 0$ for some $\varphi_0 \in W^{1,p}(\Omega)$,

(G2) $G(1) < 0$.

If $\varphi \in W^{1,p}(\Omega)$ and $\varphi = t\psi$,

$$E_G(t\psi) = \tilde{E}_G(t, \psi) = -|t|^p H_\lambda(\psi) + |t|^q G(\psi). \tag{4.8.5}$$

Therefore

$$\begin{cases} \partial E_G/\partial s = 0, \\ s \neq 0 \end{cases} \quad \Rightarrow \quad |s|^{q-p} = |s_*|^{q-p} = \frac{pH_\lambda(v)}{qG(v)}$$
$$= \frac{\int_\Omega (|\nabla v|^p - \lambda b|v|^p)\, dx}{\int_\Omega a|v|^q\, dx + \int_{\partial\Omega} k|v|^q\, dS}, \tag{4.8.6}$$

provided that $G(v) \neq 0$. If $H_\lambda(v)G(v) \neq 0$, we define the reduced functional

$$\tilde{E}_G(v) = \frac{p-q}{q} H_\lambda(v) \left(\frac{pH_\lambda(v)}{qG(v)} \right)^{p/(q-p)}, \tag{4.8.7}$$

for which we have the two sign possibilities since $\lambda > 0$: $\tilde{E}_G(v) < 0$ or $\tilde{E}_G(v) > 0$. If $H_\lambda(v) < 0$, s_* corresponds to a maximum, and if $H_\lambda(v) > 0$, s_* corresponds to a minimum. Because of homogeneity we can take $H_\lambda(v)$ constant, which reduces the question of finding critical points to the following problems:

PROBLEM 7. $\sup\{G(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = 1\}$.

PROBLEM 8. $\inf\{G(\phi)\colon \phi \in W^{1,p}(\Omega),\ H_\lambda(\phi) = -1\}$.

The next result plays a fundamental role in deriving bounds for minimizing or maximizing sequences as Lemmas 4.5.1 and 4.6.3 do in Subsections 4.5 and 4.6.

LEMMA 4.8.1. *Assume that* (G2) *holds*. *There exists*

$$\tilde{\lambda}\big(|\Omega|, p, q, \|a\|_{L^\infty}, \|k\|_{L^\infty}, \|b\|_{L^\infty}\big) > 0,$$

such that if $0 < \lambda < \tilde{\lambda}$, *then any sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$ *such that* $\{G(u_n)\}$ *is positive and* $\{H_\lambda(u_n)\}$ *is bounded from above, has the property that it remains bounded in* $W^{1,p}(\Omega)$. *Moreover, it satisfies*

$$\lim_{n\to\infty} \sup G(u_n) =: m < \infty.$$

PROOF. Our technique mixes the ones introduced for proving Lemmas 4.5.1 and 4.6.3. We set $\alpha_n := |\Omega|^{-1} \int_\Omega u_n\, dx$, $u_n = \tilde{u}_n + \alpha_n$. Since $H_\lambda(u_n) \leqslant M$, we derive inequality (4.5.16), i.e.

$$\frac{\mu - (\lambda + \epsilon)\|b\|_{L^\infty}}{\mu} \int_\Omega |\nabla \tilde{u}_n|^p\, dx \leqslant \lambda \|b\|_{L^\infty} |\Omega| C_p \epsilon^{1-p} |\alpha_n|^p + M, \tag{4.8.8}$$

with the following choice for ϵ:

$$0 < \lambda < \mu/\|b\|_{L^\infty} \quad \text{and} \quad 0 < \epsilon < \mu/\|b\|_{L^\infty} - \lambda. \tag{4.8.9}$$

Assume first that $\{\alpha_n\}$ is not bounded, $|\alpha_n| \to \infty$ for example, and set $w_n = u_n/\alpha_n = 1 + \tilde{w}_n$, then

$$\int_\Omega |\tilde{w}_n|^q \, dx \leqslant C_{\epsilon,p,q} \lambda^{q/p} + M|\alpha_n|^{-p} \tag{4.8.10}$$

and

$$\int_{\partial\Omega} |\tilde{w}_n|^q \, dS \leqslant D_{\epsilon,p,q} \lambda^{q/p} + M|\alpha_n|^{-p} \tag{4.8.11}$$

from (4.5.19) and (4.6.14). Moreover,

$$\begin{aligned} \int_\Omega \big||1+\tilde{w}_n|^q - 1\big| \, dx &\leqslant q c_q \left(\int_\Omega |\tilde{w}_n|^q \, dx \right)^{1/q} \\ &\quad \times \left(|\Omega|^{1-1/q} + \left(\int_\Omega |\tilde{w}_n|^q \, dx \right)^{1-1/q} \right) \end{aligned} \tag{4.8.12}$$

and

$$\begin{aligned} \int_{\partial\Omega} \big||1+\tilde{w}_n|^q - 1\big| \, dS &\leqslant q c_q \left(\int_{\partial\Omega} |\tilde{w}_n|^q \, dS \right)^{1/q} \\ &\quad \times \left(|\partial\Omega|^{1-1/q} + \left(\int_{\partial\Omega} |\tilde{w}_n|^q \, dS \right)^{1-1/q} \right) \end{aligned} \tag{4.8.13}$$

from (4.5.21) and (4.6.15). Therefore, using (4.8.10)–(4.8.13),

$$\begin{aligned} \alpha_n^{-q} G(u_n) &= \int_\Omega a|1+\tilde{w}_n|^q \, dx + \int_{\partial\Omega} k|1+\tilde{w}_n|^q \, dS \\ &= G(1) + \int_\Omega a\big(|1+\tilde{w}_n|^q - 1\big) \, dx + \int_{\partial\Omega} k\big(|1+\tilde{w}_n|^q - 1\big) \, dS \\ &\leqslant G(1) + q c_q \|a\|_{L^\infty} \big(C_{\epsilon,p,q} \lambda^{1/p} + M|\alpha_n|^{-1}\big)\big(|\Omega|^{1/q} \\ &\quad + C_{\epsilon,p,q} \lambda^{1/p} + M|\alpha_n|^{-1}\big)^q + q c_q \|k\|_{L^\infty} \big(D_{\epsilon,p,q} \lambda^{1/p} + M|\alpha_n|^{-1}\big) \\ &\quad \times \big(|\partial\Omega|^{1/q} + D_{\epsilon,p,q} \lambda^{1/p} + M|\alpha_n|^{-1}\big)^q. \end{aligned} \tag{4.8.14}$$

Define

$$\begin{aligned} \Lambda_\epsilon = \sup\bigl\{\lambda \in \bigl(0, \mu/\|b\|_{L^\infty} - \epsilon\bigr)\colon\ & G(1) \\ & + q c_q \|a\|_{L^\infty} C_{\epsilon,p,q} \rho^{1/p}\bigl(|\Omega|^{1/q} + C_{\epsilon,p,q}\rho^{1/p}\bigr)^q \\ & + q c_q \|k\|_{L^\infty} D_{\epsilon,p,q} \rho^{1/p}\bigl(|\partial\Omega|^{1/q} + D_{\epsilon,p,q}\rho^{1/p}\bigr)^q < 0\ \forall \rho \in (0,\lambda)\bigr\}. \end{aligned} \tag{4.8.15}$$

Such an expression defines a positive real number because of assumption (G2). If we set

$$\tilde{\lambda} = \max\bigl\{\Lambda_\epsilon\colon\ 0 < \epsilon < \mu/\|b\|_{L^\infty}\bigr\} \tag{4.8.16}$$

and take $0 < \lambda < \tilde{\lambda}$, it follows that $G(u_n) < 0$ for n large enough, contradiction. The remainder of the proof is as in Lemmas 4.5.1 and 4.6.3. □

In the same way the following result is a straightforward adaptation of Lemmas 4.5.3 and 4.6.5 to the framework of the functional G.

LEMMA 4.8.2. *Assume that* (G2) *holds. If* $\lambda \in (0, \tilde{\lambda})$, *then* $v \in W^{1,p}(\Omega)$, *not identically zero, and* $H_\lambda(v) \leqslant 0$ *imply* $G(v) < 0$.

Since assumption (G1) implies that the set $\mathcal{G}_+ := \{\zeta \in W^{1,p}(\Omega)\colon\ G(\zeta) > 0,\ H_\lambda(\zeta) = 1\}$ is not empty, the following existence result is proved similarly to Proposition 4.5.4.

PROPOSITION 4.8.3. *Assume that* (G1) *and* (G2) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exists a positive solution* u *of* (4.8.1) *with the property that* $G(u) > 0$.

For the last problem the main observation is that $\mathcal{G}_- := \{\zeta \in W^{1,p}(\Omega)\colon\ G(\zeta) < 0,\ H_\lambda(\zeta) = -1\}$ is not empty under assumption (G2), and we can replace Problem 8 by

PROBLEM 8′. $\inf\{H_\lambda(\phi)\colon\ \phi \in W^{1,p}(\Omega),\ G(\phi) = -1\}$.

The key lemma for the existence of a minimizer is an adaptation of Lemmas 4.5.5 and 4.6.7 that we state below without proof since it is straightforward.

LEMMA 4.8.4. *Assume that* (G2) *holds. Then if* $\lambda \in (0, \tilde{\lambda})$, *any sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$ *such that* $\{G(u_n)\}$ *is bounded and* $\{H_\lambda(u_n)\}$ *is negative has the property that it remains bounded in* $W^{1,p}(\Omega)$.

As a consequence the second existence result follows.

PROPOSITION 4.8.5. *Assume that* (G2) *and* (B) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exists a positive solution* u^* *of* (4.8.1) *with the property that* $G(u^*) < 0$.

Combining Propositions 4.8.3 and 4.8.5 yields

THEOREM 4.8.6. *Assume that* (G1), (G2) *and* (B) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exist two positive solutions* u *and* u^* *of* (4.8.1) *such that* $G(u) > 0$ *and* $G(u^*) < 0$ *respectively.*

The complete fibering technique of Subsection 4.7 can be easily adapted to study of the more inhomogeneous problem

$$\begin{cases} -\Delta_p u - \lambda b(x)|u|^{p-2}u = c(x)|u|^{s-2}u + a(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{4.8.17}$$

where $\lambda > 0$ and the exponents p, s and q satisfy

$$1 < p < s < q < \tilde{p} - 1. \tag{4.8.18}$$

Since the proofs of the results below are mere adaptations of the previous ones, we just state them.

PROPOSITION 4.8.7. *Assume that* (G1), (G2) *and* (C) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exists a positive solution* u *of* (4.8.17) *with the property that* $G(u) > 0$.

PROPOSITION 4.8.8. *Assume that* (B), (G2) *and* (C) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exists a positive solution* u^* *of* (4.8.17) *with the property that* $G(u^*) < 0$.

THEOREM 4.8.9. *Assume that* (G1), (G2), (C) *and* (B) *hold. Then for every* $0 < \lambda < \tilde{\lambda}$ *there exist two positive solutions* u *and* u^* *of* (4.8.17) *in* $W^{1,p}(\Omega)$ *such that* $G(u) > 0$ *and* $G(u^*) < 0$ *respectively.*

4.9. *Positive solutions with more general nonlinear boundary conditions*

We discuss the following problem:

$$\Delta_p u = 0, \quad u > 0 \quad \text{in} \quad \Omega \tag{4.9.1}$$

$$|\nabla u|^{p-2}\partial_\nu u - \mu b(x')|u|^{p-2}u = k(x')|u|^{r-2}u \quad \text{on } \partial\Omega \tag{4.9.2}$$

with

$$\Delta_p u = \nabla\big(|\nabla u|^{p-2}\nabla u\big) = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \quad \text{and} \quad 1 < p < r < p_* \quad \text{and} \quad \mu > 0.$$

Here b and k are measurable bounded functions, satisfying some assumptions stated below.
In this case we have

$$H(v) = \int_{\Omega} |\nabla v|^p \, dx - \mu \int_{\partial\Omega} b|v|^p \, d\sigma',$$

$$K(v) = \frac{1}{r} \int_{\partial\Omega} k|v|^r \, d\sigma'.$$

Hence

$$E(tv) = \frac{|t|^p}{p} H(v) - |t|^r K(v).$$

We now assume that

$$b > 0. \tag{B.1}$$

About function k we assume that there is a function $v_0 \in W^{1,p}(\Omega)$ such that

$$H(v_0) > 0 \quad \text{and} \quad K(v_0) > 0. \tag{K.1}$$

Next we suppose that

$$\int_{\partial\Omega} k \, d\sigma' < 0. \tag{K.2}$$

Let us consider the first case:

$$H(v) = +1 \quad \text{and} \quad K(v) > 0.$$

Then

$$E_1(v) = \max_{t \geqslant 0} E(tv) = \frac{r-p}{rp} \big(rK(v)\big)^{-\frac{p}{r-p}}$$

and the variational problem is equivalent to the following

$$\sup\big\{K(v) \bigm| H(v) = 1\big\}. \tag{4.9.3}$$

For this problem there is $\mu^* > 0$ such that if $\mu < \mu^*$ then there is a positive solution v_1 of (4.9.3).

Hence there is a positive solution u_1 of (4.9.1)–(4.9.2), namely,

$$u_1(x) = t_1 v_1(x)$$

with

$$t_1 = \bigl(rK(v_1)\bigr)^{-\frac{1}{r-p}}.$$

For obtaining the second solution we consider the functional

$$E_2(v) = \min_{t\geqslant 0} E(tv)$$

for the case $H(v) < 0$ and $K(v) < 0$.

Then we get

$$E_2(v) = \frac{r-p}{pr} H(v)\biggl(\frac{H(v)}{rK(v)}\biggr)^{\frac{p}{r-p}}.$$

This functional is homogeneous of zero order,

$$E_2(sv) = E_2(v), \quad \forall s \in \mathbb{R}.$$

Due to this fact we can consider

$$\inf\bigl\{E_2(v) \bigm| K(v) = -1\bigr\} \tag{4.9.4}$$

since

$$\bigl(K'(v), v\bigr) = \int_{\partial\Omega} k|v|^r\, dx' = -r \neq 0.$$

This variational problem is equivalent to the following

$$\inf\bigl\{c_{p,r} \cdot H(v)\bigl|H(v)\bigr|^{\frac{p}{r-p}} \bigm| K(v) = -1\bigr\} < 0 \tag{4.9.5}$$

with

$$c_{p,r} = \frac{r-p}{p} \cdot r^{\frac{r}{p-r}} > 0.$$

For this problem there is $\mu_* > 0$ such that if $0 < \mu < \mu_*$ then there is a positive minimizer v_2 of (4.9.5).

Hence there exists a positive solution u_2 of (4.9.1)–(4.9.2), namely,

$$u_2(x) = t_2 v_2(x)$$

with

$$t_2 = \bigl(r^{-1} \cdot \bigl|H(v_2)\bigr|\bigr)^{\frac{1}{r-p}}.$$

Thus we obtain the following result.

PROPOSITION 4.9.1. *Assume that* (B.1), (K.1) *and* (K.2) *hold. Then there is* $\mu_0 = \min\{\mu^*, \mu_*\}$ *such that for any* $0 < \mu < \mu_0$ *there exist positive solutions* u_1 *and* u_2 *in* $W^{1,p}(\Omega)$ *of* (4.9.1).

Moreover, $H(u_1) > 0$ *and* $H(u_2) < 0$.

5. Multiplicity for some quasilinear equations (Yu. Bozhkov and E. Mitidieri)

The present section contains results from [14]. We investigate the existence of positive solutions of the following quasilinear problem:

$$\begin{cases} Lu := -(r^\alpha |u'(r)|^\beta u'(r))' = \lambda a(r) r^\gamma |u|^\beta u + b(r) r^\gamma |u|^{q-2} u & \text{in } (0, R), \\ u'(0) = u(R) = 0, \\ u > 0 & \text{in } (0, R). \end{cases} \tag{5.0.1}$$

Here $\alpha, \beta, \gamma, \lambda, R$ $(0 < R < \infty)$, q are real numbers and $a(r), b(r)$ are given functions.

Some problems associated to the operator L have been studied in [16]. The parameters α, β and γ are supposed to satisfy some relations which will be specified below. This class of quasilinear equations is interesting and sufficiently general since, as one can observe, it may involve as special cases the following operators acting on radial functions defined in a ball of radius R centered at the origin:

(1) Laplace operator if $\alpha = \gamma = N - 1,\ \beta = 0$;

(2) p-Laplace operator if $\alpha = \gamma = N - 1,\ \beta = p - 2$;

(3) k-Hessian operator if $\alpha = N - k,\ \gamma = N - 1,\ \beta = k - 1$. (5.0.2)

The main technique we shall use to study (5.0.1) is the fibering method introduced and developed in the previous sections, which provides a powerful tool for proving existence theorems, in particular for problems that obey a certain kind of homogeneity. In the study of (5.0.1) we are motivated by the observation that the critical exponents for L were found in [16], where we can find, among other things, an inequality, which ensures that the main embeddings that we need between the natural functional spaces associated to the problem under consideration are compact. Moreover, the properties of the first eigenvalue of the corresponding eigenvalue problem with $a(r) = 1$ are also established in [16].

This section is organized as follows. In Subsection 5.1 we collect some basic facts about the operator L and state the hypotheses that we shall assume throughout the chapter. In Subsection 5.2 we study the corresponding eigenvalue problem with a more general a and prove the existence, simplicity and isolation of the first eigenvalue in this case. Then a version of the fibering method adapted to our specific problem is described in Subsection 5.3. The main results on existence and multiplicity of radial solutions for (5.0.1) are proved in Subsections 5.4–5.6. Finally, in Subsection 5.7 we comment on nonexistence results for classical solutions of quasilinear equations, in particular we establish a nonexistence result for k-Hessian equations in a general bounded smooth domain.

5.1. *Preliminaries*

In this subsection we present some preliminaries, notations and assumptions which will be used throughout the section.

To begin with, let

$$\alpha > 0, \quad \beta > -1, \quad \alpha - \beta - 1 > 0, \quad \gamma > 0, \quad 0 < R < \infty. \tag{5.1.1}$$

For parameters satisfying (5.1.1) it was found in [16] that the critical exponent for (5.0.1) with $a(r) = b(r) = 1$ has the value

$$l^* = \frac{(\gamma + 1)(\beta + 2)}{\alpha - \beta - 1}. \tag{5.1.2}$$

Let

$$X_R := \Big\{ u \in C^2(0, R) \cap C^1\big([0, R)\big) \,\Big|\, \int_0^R r^\alpha \big|u'(r)\big|^{\beta+2}\, dr < \infty,$$
$$u(R) = 0,\ u'(0) = 0 \Big\},$$

with the norm

$$\|u\|_{X_R} = \left\{ \int_0^R r^\alpha \big|u'(r)\big|^{\beta+2}\, dr \right\}^{\frac{1}{\beta+2}}.$$

In this way X_R becomes a Banach space. We also define the space

$$L^s_\gamma\big((0, R); a\big) = \left\{ u \in L^s(0, R) \,\Big|\, 0 < \int_0^R a(r)|u|^s r^\gamma\, dr < \infty \right\}.$$

With this at hand, the following inequality holds:

$$\left(\int_0^R |u|^s r^\gamma\, dr \right)^{\frac{1}{s}} \leqslant c \left(\int_0^R r^\alpha |u'|^{\beta+2}\, dr \right)^{\frac{1}{\beta+2}} \tag{5.1.3}$$

where $s < l^*$—the critical exponent given by (5.1.2). See [16]. The latter inequality implies that the space X_R is compactly embedded in $L^s_\gamma((0, R); a)$ provided $s < l^*$ if $\alpha - \beta - 1 > 0$. See [16] for more details.

Now we shall state the assumptions. Let $\lambda, q \in \mathbb{R}$. In addition to (5.1.1) we shall suppose that

$$1 < \beta + 2 < q < l^*, \tag{5.1.4}$$

where l^* is defined in (5.1.2).

The functions $a(r)$ and $b(r)$ are supposed to be bounded in $(0, R)$:

$$a, b \in L^{\infty}(0, R), \tag{5.1.5}$$

and

$$a(r) = a_1(r) - a_2(r), \quad \text{where } a_1, a_2 \geqslant 0,\ a_1(r) \not\equiv 0. \tag{5.1.6}$$

We claim that (5.1.4) and (5.1.5) imply that

$$\int_0^R a(r)|u|^{\beta+2} r^{\gamma}\, dr,$$

and

$$\int_0^R b(r)|u|^{q} r^{\gamma}\, dr,$$

are finite for $u \in X_R$. Indeed, by (5.1.3) and $\beta + 2 < l^*$ we have

$$\left| \int_0^R a(r)|u|^{\beta+2} r^{\gamma}\, dr \right| \leqslant \|a\|_{L^{\infty}((0,R))} \|u\|^{\beta+2}_{L^{\beta+2}_{\gamma}((0,R);1)} \leqslant c\|u\|^{\beta+2}_{X_R} < \infty,$$

and by (5.1.4) it follows that

$$\left| \int_0^R b(r)|u|^{q} r^{\gamma}\, dr \right| \leqslant \|b\|_{L^{\infty}((0,R))} \|u\|^{q}_{L^{q}_{\gamma}((0,R);1)} \leqslant c\|u\|^{q}_{X_R} < \infty.$$

This proves the claim.

Now we can define the following functionals on X_R:

$$f_1(u) = \int_0^R a(r)|u|^{\beta+2} r^{\gamma}\, dr, \tag{5.1.7}$$

and

$$f_2(u) = \int_0^R b(r)|u|^{q} r^{\gamma}\, dr. \tag{5.1.8}$$

It is standard to check that f_1 and f_2 are weakly lower semicontinuous.

We shall also suppose that

$$b^{+}(r) \not\equiv 0, \tag{5.1.9}$$

and

$$\int_0^R b(r)|u_1|^{q} r^{\gamma}\, dr < 0, \tag{5.1.10}$$

where $u_1(r)$ is the positive eigenfunction associated to the operator L (see Theorem 5.2.1 in the next subsection).

DEFINITION 5.1.1. We say that $u \in X_R$ is a weak solution of (5.0.1) if

$$\int_0^R r^\alpha |u'|^\beta v'\,dr = \lambda \int_0^R a(r)|u|^\beta u v r^\gamma\,dr + \int_0^R b(r)|u|^{q-2} u v r^\gamma\,dr, \quad (5.1.11)$$

for all $v \in X_R$.

5.2. *The eigenvalue problem for L*

In this subsection we consider the eigenvalue equation

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = \lambda a(r) r^\gamma |u|^\beta u & \text{in } (0, R), \\ u'(0) = u(R) = 0, \end{cases} \quad (5.2.1)$$

where $a \in L^\infty((0, R))$ satisfies (5.1.6), in a weak sense, that is,

$$\begin{cases} \int_0^R r^\alpha |u'(r)|^\beta u' v'\,dr = \lambda \int_0^R a(r)|u|^\beta u v r^\gamma\,dr & \text{in } (0, R), \\ u'(0) = u(R) = 0 \end{cases}$$

for any $v \in X_R$. Problem (5.2.1) is closely related to (5.0.1). Moreover, to apply the fibering method to problem (5.0.1), we need the following result, which ensures simplicity and isolation of the first eigenvalue λ_1 of the operator L.

THEOREM 5.2.1. *There exists a number $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf \frac{\|u\|_{X_R}^{\beta+2}}{\int_0^R a(r)|u|^{\beta+2} r^\gamma\,dr},$$

where the infimum is taken over $u \in X_R$ such that $\int_0^R a(r)|u|^{\beta+2} r^\gamma\,dr > 0$, and a satisfies condition (5.1.6) *above. Moreover,*

(i) *there exists a positive function $u_1 \in X_R$ which is a weak solution of* (5.2.1) *with $\lambda = \lambda_1$,*

(ii) *λ_1 is simple, in the sense that any two eigenfunctions corresponding to λ_1 differ by a positive constant multiplier*;

(iii) *λ_1 is isolated, which means that there are no eigenvalues less than λ_1 and no eigenvalues in the interval $(\lambda_1, \lambda_1 + \delta)$ for some $\delta > 0$ sufficiently small.*

PROOF. By inequality (5.1.3), we have

$$\begin{aligned} 0 < \int_0^R a(r)|u|^{\beta+2} r^\gamma\,dr &\leqslant \|a\|_{L^\infty((0,R))} \int_0^R |u|^{\beta+2} r^\gamma\,dr \\ &\leqslant C\|a\|_{L^\infty((0,R))} \|u\|_{X_R} \end{aligned}$$

since $\beta + 2 < l^*$ and $a \in L^\infty$. Therefore the infimum under consideration exists.

Let $E(u) := \|u\|_{X_R}^{\beta+2}$ and denote by M the manifold defined by

$$M = \left\{ u \in X_R \;\Big|\; \int_0^R a(r)|u|^{\beta+2} r^\gamma \, dr = 1 \right\}.$$

Restricting E to M we shall use the method of Lagrange multipliers to obtain a weak solution. By (5.1.1) and (5.1.3) it follows that X_R is compactly embedded in $L^\infty((0, R); a)$ for any $s < l^*$. Therefore M is weakly closed in X_R. The functional E is coercive with respect to X_R. Using the definition of E and (5.1.3), one can verify that E is a weakly lower semicontinuous functional. Then by standard variational arguments it follows that E attains its infimum at a point $u^* \in M$.

Since $E \in C^1(X_R)$, an easy calculation gives

$$E'(u)v = (\beta + 2) \int_0^R r^\alpha \left|u'(r)\right|^\beta u'v' \, dr,$$

where $E'(u)v$ is the Gateaux derivative of E at $u \in X_R$ in the direction of v. Denote

$$H(u) = \int_0^R a(r)|u|^{\beta+2} r^\gamma \, dr - 1.$$

Then

$$H'(u)v = (\beta + 2) \int_0^R a(r)|u|^\beta u v r^\gamma \, dr$$

and in particular for $u = v$:

$$H'(u)u = (\beta + 2) \int_0^R |u|^{\beta+2} r^\gamma \, dr = \beta + 2 > 0$$

for $u \in M$.

Now we look for a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$E'(u^*)v - \lambda H'(u^*)v = (\beta + 2) \int_0^R r^\alpha \left|u^{*\prime}(r)\right|^\beta u^{*\prime} v' \, dr - \lambda(\beta + 2) \int_0^R a(r)|u^*|^\beta u^* v \, r^\gamma \, dr = 0.$$

Let $v = u^*$. Then

$$0 < (\beta + 2) \int_0^R r^\alpha \left|u^{*\prime}(r)\right|^{\beta+2} dr = \lambda(\beta + 2) \int_0^R a(r)|u^*|^{\beta+2} \, r^\gamma \, dr = \lambda(\beta + 2).$$

Thus $\lambda > 0$ and

$$(\beta+2)\int_0^R r^\alpha \left|u^{*\prime}(r)\right|^\beta u^{*\prime} v'\,dr = \lambda(\beta+2)\int_0^R a(r)|u^*|^\beta u^* v\, r^\gamma\,dr.$$

Therefore u^* is a weak solution of the eigenvalue problem (5.2.1). Moreover, by the arguments in [16] it follows that u^* does not vanish in $[0, R)$ and hence we can choose a certain $u_1 = cu^* > 0$.

(ii) We shall prove the simplicity of λ. Let u and v be two weak eigenfunctions, that is,

$$\begin{aligned}
&\int_0^R r^\alpha |u'|^\beta u' w'\,dr = \lambda \int_0^R r^\gamma a(r)|u|^\beta u w\,dr,\\
&\int_0^R r^\alpha |v'|^\beta v' w'\,dr = \lambda \int_0^R r^\gamma a(r)|v|^\beta v w\,dr,\\
&u(R) = u'(0) = v(R) = v'(0) = 0,
\end{aligned}$$

for any w such that $w(R) = w'(0) = 0$ and $\int_0^R r^\gamma |w'|^{\beta+2}\,dr < \infty$. Set $u_\varepsilon = u + \varepsilon$ and $v_\varepsilon = v + \varepsilon$. Then substituting $w = u_\varepsilon - v_\varepsilon^{\beta+2} u_\varepsilon^{-\beta-1}$ into the first equation above and $w = v_\varepsilon - u_\varepsilon^{\beta+2} v_\varepsilon^{-\beta-1}$ into the second, we get that

$$\begin{aligned}
&\lambda_1 \int_0^R r^\gamma a(r)\left(\frac{|u|^{\beta+1}}{|u_\varepsilon|^{\beta+1}} - \frac{|v|^{\beta+1}}{|v_\varepsilon|^{\beta+1}}\right)\left(|u_\varepsilon|^{\beta+2} - |v_\varepsilon|^{\beta+2}\right)dr\\
&\quad = \int_0^R r^\alpha \left(\left[1 + (\beta+1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{\beta+2}\right]\left|u_\varepsilon'\right|^{\beta+2}\right.\\
&\qquad \left. + \left[1 + (\beta+1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{\beta+2}\right]\left|v_\varepsilon'\right|^{\beta+2}\right)dr\\
&\qquad - (\beta+2)\int_0^R r^\alpha \left(\left[\frac{v_\varepsilon}{u_\varepsilon}\right]^{\beta+1}\left|u_\varepsilon'\right|^\beta u_\varepsilon' v_\varepsilon' + \left[\frac{u_\varepsilon}{v_\varepsilon}\right]^{\beta+1}\left|v_\varepsilon'\right|^\beta v_\varepsilon' u_\varepsilon'\right)dr.
\end{aligned}$$

Denote $z_\varepsilon = \log|u_\varepsilon|$ and $w_\varepsilon = \log|v_\varepsilon|$. With this notation, the last equality can be written in the following way:

$$\begin{aligned}
&\lambda_1 \int_0^R r^\gamma a(r)\left(\frac{|u|^{\beta+1}}{|u_\varepsilon|^{\beta+1}} - \frac{|v|^{\beta+1}}{|v_\varepsilon|^{\beta+1}}\right)\left(|u_\varepsilon|^{\beta+2} - |v_\varepsilon|^{\beta+2}\right)dr\\
&\quad = \int_0^R r^\alpha \left(\left(|u_\varepsilon|^{\beta+2} - |v_\varepsilon|^{\beta+2}\right)\left(\left|z_\varepsilon'\right|^{\beta+2} - \left|w_\varepsilon'\right|^{\beta+2}\right)\right)dr\\
&\qquad - (\beta+2)\int_0^R r^\alpha \left(|v_\varepsilon|^{\beta+2}\left|z_\varepsilon'\right|^\beta z_\varepsilon'\left(w_\varepsilon' - z_\varepsilon'\right)\right.\\
&\qquad \left. + |u_\varepsilon|^{\beta+2}\left|w_\varepsilon'\right|^\beta w_\varepsilon'\left(z_\varepsilon' - w_\varepsilon'\right)\right)dr.
\end{aligned}$$

By inequality (4.3) [26, p. 163],

$$|w_2|^{\beta+2} \geqslant |w_1|^{\beta+2} + (\beta+2)|w_1|^{\beta} w_1(w_2 - w_1) + \frac{|w_2 - w_1|^{\beta+2}}{2^{\beta+1} - 1}.$$

We choose $w_1 = w_\varepsilon'$ and $w_2 = z_\varepsilon'$ in this inequality. Therefore

$$|u_\varepsilon|^{\beta+2}\big(\big|z_\varepsilon'\big|^{\beta+2} - \big|w_\varepsilon'\big|^{\beta+2}\big) - (\beta+2)|u_\varepsilon|^{\beta+2}\big|w_\varepsilon'\big|^{\beta} w_\varepsilon'\big(z_\varepsilon' - w_\varepsilon'\big) \geqslant \frac{1}{2^{\beta+1}-1}\frac{|u_\varepsilon v_\varepsilon' - v_\varepsilon u_\varepsilon'|^{\beta+2}}{|v_\varepsilon|^{\beta+2}}$$

and similarly

$$|v_\varepsilon|^{\beta+2}\big(\big|w_\varepsilon'\big|^{\beta+2} - \big|z_\varepsilon'\big|^{\beta+2}\big) - (\beta+2)|v_\varepsilon|^{\beta+2}\big|z_\varepsilon'\big|^{\beta} z_\varepsilon'\big(w_\varepsilon' - z_\varepsilon'\big) \geqslant \frac{1}{2^{\beta+1}-1}\frac{|u_\varepsilon v_\varepsilon' - v_\varepsilon u_\varepsilon'|^{\beta+2}}{|u_\varepsilon|^{\beta+2}}.$$

Integrating the last two inequalities and combining them with the relations above, we finally obtain

$$\begin{aligned} 0 &\leqslant \frac{1}{2^{\beta+1}-1}\int_0^R r^\alpha\left(\frac{1}{|u_\varepsilon|^{\beta+2}} + \frac{1}{|v_\varepsilon|^{\beta+2}}\right)\big|u_\varepsilon v_\varepsilon' - v_\varepsilon u_\varepsilon'\big|^{\beta+2}\,dr \\ &\leqslant \lambda_1\int_0^R r^\gamma a(r)\left(\frac{|u|^{\beta+1}}{|u_\varepsilon|^{\beta+1}} - \frac{|v|^{\beta+1}}{|v_\varepsilon|^{\beta+1}}\right)\big(|u_\varepsilon|^{\beta+2} - |v_\varepsilon|^{\beta+2}\big)\,dr. \end{aligned}$$

The latter integral tends to zero when $\varepsilon \to 0$. Hence and by Fatou's lemma we conclude that $uv' = vu'$ which implies that $v = lu$ for some $l \in \mathbb{R}$.

(iii) Let $v \in X_R$ be an eigenfunction corresponding to eigenvalue λ. By the definition of λ_1 it follows that

$$\lambda_1\int_0^R a(r)|v|^{\beta+2}r^\gamma\,dr \leqslant \int_0^R r^\alpha|v'|^{\beta+2}\,dr = \lambda\int_0^R a(r)|v|^{\beta+2}r^\gamma\,dr.$$

Hence $\lambda \geqslant \lambda_1$, that is, λ_1 is isolated from below.

We normalize v:

$$\int_0^R r^\alpha|v'|^{\beta+2}\,dr = 1.$$

Let $u > 0$ be the eigenfunction associated to λ_1 and

$$\int_0^R r^\alpha|u'|^{\beta+2}\,dr = 1.$$

Following [6] consider

$$I(u,v)=\int_0^R r^{\alpha}|u'|^{\beta}u'\big(u-v^{\beta+2}/u^{\beta+1}\big)'\,dr$$
$$+\int_0^R r^{\alpha}|v'|^{\beta}v'\big(v-u^{\beta+2}/v^{\beta+1}\big)'\,dr.$$

Using an argument, similar to that of [6] (or to that in the proof of (ii)), it can be proved that $I(w_1,w_2)\geqslant 0$ for any w_i, $i=1,2$, such that $\int_0^R r^{\alpha}|w_i'|^{\beta+2}dr<\infty$ and $w_i/w_j\in L^{\infty}((0,R))$, $i,j=1,2$. By the normalization and the fact that u and v are eigenfunctions with eigenvalues λ_1 and λ respectively, we obtain

$$0\leqslant I(u,v)=\int_0^R r^{\gamma}a(r)\big(|u|^{\beta+2}-|v|^{\beta+2}\big)\,dr=-(\lambda_1-\lambda)^2/(\lambda_1\lambda)<0$$

if $\lambda_1\neq\lambda$. This contradiction implies that $\lambda=\lambda_1$ and $v=lu$. □

5.3. *The fibering method*

Now we shall present the cornerstone of the fibering method adapted to problem (5.0.1). For this purpose, we consider the following functional:

$$J_{\lambda}(u):=\frac{1}{\beta+2}\int_0^R r^{\alpha}\big|u'(r)\big|^{\beta+2}\,dr$$
$$-\frac{\lambda}{\beta+2}\int_0^R a(r)|u|^{\beta+2}r^{\gamma}\,dr-\frac{1}{q}\int_0^R b(r)|u|^q r^{\gamma}\,dr. \tag{5.3.1}$$

Clearly $J_{\lambda}\in C^1(X_R)$. Critical points of J_{λ} are then weak solutions of problem (5.0.1).

Further, following the main point of the previous sections, for $u\in X_R$ we set

$$u(x)=tz(x), \tag{5.3.2}$$

where $t\neq 0$ is a real number and $z\in X_R$. Substituting (5.3.2) into (5.3.1), we obtain

$$J_{\lambda}(tz)=\frac{|t|^{\beta+2}}{\beta+2}\int_0^R r^{\alpha}\big|z'(r)\big|^{\beta+2}\,dr-\frac{\lambda|t|^{\beta+2}}{\beta+2}\int_0^R a(r)|z|^{\beta+2}r^{\gamma}\,dr$$
$$-\frac{|t|^q}{q}\int_0^R b(r)|z|^q r^{\gamma}\,dr. \tag{5.3.3}$$

If $u\in X_R$ is a critical point of J_{λ}, then

$$\frac{\partial J_{\lambda}}{\partial t}(tz)=0,$$

which is equivalent to

$$|t|^{\beta} t \int_0^R r^{\alpha} \left|z'(r)\right|^{\beta+2} dr - \lambda |t|^{\beta} t \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr$$
$$- |t|^{q-2} t \int_0^R b(r)|z|^q r^{\gamma}\, dr = 0. \tag{5.3.4}$$

Assuming that

$$\int_0^R r^{\alpha} \left|z'(r)\right|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr \neq 0$$

and

$$\int_0^R b(r)|z|^q r^{\gamma}\, dr \neq 0,$$

from (5.3.4) we get that

$$|t|^{q-\beta-2} = \frac{\int_0^R r^{\alpha}|z'(r)|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr}{\int_0^R b(r)|z|^q r^{\gamma}\, dr} > 0. \tag{5.3.5}$$

In Subsections 5.4 and 5.5 we shall suppose that

$$\int_0^R r^{\alpha} \left|z'(r)\right|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr > 0,\ \int_0^R b(r)|z|^q r^{\gamma}\, dr > 0. \tag{5.3.6}$$

In Subsection 5.6 we shall admit

$$\int_0^R r^{\alpha} \left|z'(r)\right|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr < 0,\ \int_0^R b(r)|z|^q r^{\gamma}\, dr < 0. \tag{5.3.7}$$

Thus, in both cases (5.3.6) and (5.3.7), the function $t = t(z)$ is well defined. Now we insert into (5.3.3) the expression for $t = t(z)$, determined by (5.3.5). In this way we obtain a functional $I_{\lambda}(z) = J_{\lambda}(t(z)z)$ given by

$$I_{\lambda}(z) = \sigma\left(\frac{1}{\beta+2} - \frac{1}{q}\right) \frac{\left|\int_0^R r^{\alpha}|z'(r)|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^{\gamma}\, dr\right|^{q/(q-\beta-2)}}{\left|\int_0^R b(r)|z|^q r^{\gamma}\, dr\right|^{(\beta+2)/(q-\beta-2)}}, \tag{5.3.8}$$

where

$$\sigma = \operatorname{sgn}\left(\int_0^R b(r)|z|^q r^\gamma \, dr\right).$$

Therefore, provided z satisfies (5.3.6) or (5.3.7), we have

$$\frac{d}{dt}\big(J_\lambda(tz)\big)\Big|_{t=t(z)} = 0.$$

It is clear that the following lemma holds.

LEMMA 5.3.1.

(i) *For every $z \in X_R$ such that $\int_\Omega b(x)|z|^q \, dx \neq 0$ and every $p > 0$ one has*

$$I_\lambda(pz) = I_\lambda(z),$$

that is, the functional I_λ is homogeneous of degree 0.

(ii) $I'_\lambda(z)z = 0$, *where $I'_\lambda(z)z$ is the Gâteaux derivative of I_λ at $z \in X_R$ in the direction of z. If z is a critical point of I_λ, then $|z|$ is also a critical point.*

Hence, as in [23], one can assume that the critical points of I_λ are nonnegative. The next two lemmas are direct consequences of the general fibering method described in the previous sections.

LEMMA 5.3.2. *Let $z \in X_R$ be a critical point, which satisfies* (5.3.6) *or* (5.3.7). *Then the function*

$$u(r) = tz(r),$$

where $t > 0$ is determined by (5.3.5), *is a critical point of J_λ.*

LEMMA 5.3.3. *Let us consider a constraint*

$$E(z) = c = \text{const},$$

where $E : X_R \to \mathbb{R}$ is a C^1 functional. If

$$E'(z)z \neq 0 \quad \textit{and} \quad E(z) = c,$$

then every critical point of I_λ with the constraint $E(z) = c$ is a critical point of I_λ.

Our first aim is to prove the existence of a critical point of I_λ with an appropriate condition $E = c$, which in turn will be an actual critical point of I_λ and hence a critical point of J_λ—the weak solution of (5.0.1).

This general scheme was suggested in [23]. In the next subsections we shall adapt the ideas of [23] to our specific problem.

5.4. *Existence for* $\lambda \in [0, \lambda_1)$

If we look at the functional $J_\lambda(u)$ given by (5.3.1), we can observe that its first two terms form a $(\beta+2)$-homogeneous expression with respect to u. It is then natural to denote by E_λ the functional

$$E_\lambda(z) = \int_0^R r^\alpha \left|z'(r)\right|^{\beta+2} dr - \lambda \int_0^R a(r)|z|^{\beta+2} r^\gamma \, dr \tag{5.4.1}$$

and to consider it as a possible functional generating the constraint $E_\lambda(u) = c$, for which we would apply Lemma 5.3.3. Then Theorem 5.2.1 implies that $E_\lambda(z) \geqslant 0$ for every $z \in X_R$. We calculate easily that the Gâteaux derivative of E_λ at $z \in X_R$ in the direction of z is

$$E'_\lambda(z)z = (\beta+2)E_\lambda(z).$$

Hence if

$$E_\lambda(z) = 1,$$

then $E'_\lambda(z)z = \beta + 2 > 0$ and the conditions on E_λ in Lemma 5.3.3 are satisfied. Moreover, since we are assuming $E_\lambda(z) = 1$, by (5.3.5) we can see that we are in the case (5.3.6), that is, $\int_\Omega b(x)|z|^q \, dx > 0$. Further, the functional $I_\lambda(z)$ (see (5.3.8)) becomes

$$I_\lambda(z) = \left(\frac{1}{\beta+2} - \frac{1}{q}\right) \frac{1}{(\int_0^R b(r)|z|^q r^\gamma \, dr)^{(\beta+2)/(q-\beta-2)}}. \tag{5.4.2}$$

The main result of this subsection is the following

THEOREM 5.4.1. *Suppose that conditions* (5.1.1), (5.1.4)–(5.1.9) *hold and that, in addition,* $\lambda \in [0, \lambda_1)$. *Then problem* (5.0.1) *has a positive weak solution* $u \in X_R$.

PROOF. Formula (5.4.1) suggests that we consider an auxiliary problem of finding a minimizer z^* of

$$\sup\left\{ \int_0^R b(r)|z|^q r^\gamma \, dr \;\middle|\; E_\lambda(z) = 1,\ \int_0^R b(r)|z|^q r^\gamma \, dr > 0 \right\} = M_\lambda > 0. \tag{5.4.3}$$

We claim that problem (5.4.3) has a solution. Indeed, the set

$$Y_\lambda = \left\{ z \in X_R \;\middle|\; E_\lambda(z) = 1 \right\}$$

is nonempty $(0 \leqslant \lambda < \lambda_1)$. By Theorem 5.2.1 we have that for any $z \in Y_\lambda$:

$$\int_0^R r^\alpha \left|z'(r)\right|^{\beta+2} dr = \lambda \int_0^R a(r)|z|^{\beta+2} r^\gamma \, dr + 1 \leqslant \frac{\lambda}{\lambda_1} \int_0^R r^\alpha \left|z'(r)\right|^{\beta+2} dr + 1$$

and hence

$$\int_0^R r^\alpha \left|z'(r)\right|^{\beta+2} dr \leqslant \frac{\lambda_1}{\lambda_1 - \lambda}$$

since $0 \leqslant \lambda < \lambda_1$. Therefore a minimizing sequence z_n for (5.4.3) is bounded in X_R. Thus we can suppose that z_n converges weakly in X_R to some z^*. By (5.1.4) and (5.1.5),

$$\int_0^R b(r)|z_n|^q r^\gamma \, dr \to \int_0^R b(r)|z^*|^q r^\gamma \, dr = M_\lambda > 0.$$

In particular, $z^* \not\equiv 0$, and by Lemma 5.3.1 we may assume $z^* > 0$.

The weak lower semicontinuity of $\|\cdot\|_{X_R}$, (5.1.4), and $E_\lambda(z_n) = 1$ imply that

$$\int_0^R r^\alpha |z^{*\prime}|^{\beta+2} \, dr \leqslant \lim_{n\to\infty} \inf \|z_n\|_{X_R}^{\beta+1}$$

and therefore

$$E_\lambda(z^*) \leqslant \lim_{n\to\infty} \inf E_\lambda(z_n) = 1.$$

If $E_\lambda(z^*) < 1$, then there exists a number $t > 1$ such that $E_\lambda(tz^*) = 1$. Set $z_1 = tz^*$. We have $z_1 \in Y_\lambda$ and

$$\int_0^R b(r)|z_1|^q r^\gamma \, dr = t^q \int_0^R b(r)|z^*|^q r^\gamma \, dr = t^q M_\lambda > M_\lambda,$$

a contradiction. This contradiction shows that $E_\lambda(z^*) = 1$ and therefore $z^* \in Y_\lambda$ is a solution of (5.4.3). By Lemma 5.3.3 it is a critical point of I_λ, and by Lemma 5.3.2, $u(r) = tz^*(r)$ is a critical point of J_λ. Thus $u \in X_R$ is a weak positive solution of (5.0.1). This completes the proof. □

5.5. *The eigenvalue case* $\lambda = \lambda_1$

We consider problem (5.4.3) with $\lambda = \lambda_1$. In this case the corresponding set Y_λ is not bounded in X_R. Therefore we have to impose an additional condition on our data. Henceforth we shall suppose that condition (5.1.10) is fulfilled.

THEOREM 5.5.1. *Suppose that conditions* (5.1.1), (5.1.4)–(5.1.10) *hold and* $\lambda = \lambda_1$. *Then problem* (5.0.1) *has a positive weak solution* $u \in X_R$.

PROOF. The arguments of the proof of this theorem would be the same as those of Theorem 5.4.1 if we could prove that problem (5.4.3) with $\lambda = \lambda_1$ has a solution.

Let z_n be a maximizing sequence such that

$$E_{\lambda_1}(z_n) = 1, \quad \int_0^R b(r)|z_n|^q r^\gamma\, dr = m_n \to M_{\lambda_1} > 0.$$

(The positivity of M_{λ_1} follows from (5.1.9).) If it is unbounded, we can suppose that $\|z_n\|_{X_R} \to \infty$. We set $w_n = z_n/\|z_n\|_{X_R}$. Obviously $\|w_n\|_{X_R} = 1$. Then

$$E_{\lambda_1}(z_n) = \|z_n\|_{X_R}^{\beta+2}\left(\|w_n\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|w_n|^{\beta+2} r^\gamma\, dr\right) = 1.$$

Hence

$$0 \leqslant \|w_n\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|w_n|^{\beta+2} r^\gamma\, dr = 1/\|z_n\|_{X_R}^{\beta+2}. \tag{5.5.1}$$

Therefore

$$\lim_{n\to\infty} \lambda_1 \int_0^R a(r)|w_n|^{\beta+2} r^\gamma\, dr = 1.$$

We may assume that w_n converges weakly in X_R to some w^*. Then

$$\int_0^R a(r)|w^*|^{\beta+2} r^\gamma\, dr = 1/\lambda_1,$$

which means that $w^* \not\equiv 0$. Furthermore,

$$\|w^*\|_{X_R}^{\beta+2} \leqslant \lim_{n\to\infty} \inf \|w_n\|_{X_R}^{\beta+2} = 1,$$

and from (5.5.1) we deduce that

$$0 \leqslant \|w^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|w^*|^{\beta+2} r^\gamma\, dr \leqslant 0.$$

Therefore w^* is an eigenfunction of L, and by Theorem 5.2.1 there exists a number $p > 0$ such that

$$w^*(r) = p u_1(r).$$

Since

$$\int_0^R b(r)|z_n|^q r^\gamma\, dr = \|z_n\|_{X_R}^q \int_0^R b(r)|w_n|^q r^\gamma\, dr = m_n \to M_{\lambda_1} > 0,$$

we conclude that

$$\int_0^R b(r)|w^*|^q r^\gamma\, dr \geqslant 0,$$

and therefore

$$\int_0^R b(r)|u_1|^q r^\gamma\, dr \geqslant 0,$$

which contradicts (5.1.10). Hence we can assume that z_n is bounded and $\lim_{n\to\infty} z_n = z^*$ weakly in X_R. Thus

$$\int_0^R b(r)|z_n|^q r^\gamma\, dr \to \int_0^R b(r)|z^*|^q r^\gamma\, dr = M_{\lambda_1} > 0,$$

therefore $z^* \not\equiv 0$. Furthermore,

$$0 \leqslant \|z^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|z^*|^{\beta+2} r^\gamma\, dr \leqslant 1.$$

In what follows we shall verify various claims. First, if

$$0 = \|z^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|z^*|^{\beta+2} r^\gamma\, dr,$$

then by Theorem 5.2.1 necessarily $z^* = pu_1,\ p > 0$, and

$$\int_0^R b(r)|z^*|^q r^\gamma\, dr = p^q \int_0^R b(r)|u_1|^q r^\gamma\, dr = M_{\lambda_1} > 0,$$

which contradicts (5.1.10). Therefore

$$0 < \|z^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|z^*|^{\beta+2} r^\gamma\, dr \leqslant 1.$$

Further suppose that we have strict inequalities

$$0 < \|z^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|z^*|^{\beta+2} r^\gamma\, dr < 1.$$

Then one can find a $t > 1$ such that

$$E_{\lambda_1}(tz^*) = 1$$

and

$$\int_0^R b(r)|tz^*|^q r^\gamma\, dr = t^q \int_0^R b(r)|z^*|^q r^\gamma\, dr$$
$$= t^q M_{\lambda_1} > M_{\lambda_1} = \sup\left\{\int_0^R b(r)|z|^q r^\gamma\, dr \;\middle|\; E_{\lambda_1}(z) = 1\right\},$$

a contradiction. Therefore

$$\|z^*\|_{X_R}^{\beta+2} - \lambda_1 \int_0^R a(r)|z^*|^{\beta+2} r^\gamma\, dr = 1.$$

Hence z^* is the maximizer of problem (5.4.3) with $\lambda = \lambda_1$, and the rest of the proof is the same as that of Theorem 5.4.1. This completes the proof. □

5.6. *Existence of two distinct solutions for $\lambda > \lambda_1$*

THEOREM 5.6.1. *Suppose that conditions* (5.1.1), (5.1.4)–(5.1.10) *hold and $\lambda > \lambda_1$. Then there exists a number $\delta > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ problem* (5.0.1) *has two distinct positive weak solutions in X_R.*

PROOF. Consider the following two variational problems:

(I) Find a maximizer $z_1 \in X_R$ of the problem

$$M_\lambda = \sup\left\{\int_0^R b(r)|z|^q r^\gamma\, dr \;\middle|\; E_\lambda(z) = 1\right\}. \tag{5.6.1}$$

(II) Find a minimizer $z_2 \in X_R$ of the problem

$$m_\lambda = \inf\left\{E_\lambda(z) \;\middle|\; \int_0^R b(r)|z|^q r^\gamma\, dr = -1\right\}. \tag{5.6.2}$$

The proof is divided into several steps.

STEP 1. Suppose (5.1.6) and (5.1.9). Then (5.6.1) is equivalent to the problem of finding a maximizer $z_1^* \in X_R$ of

$$0 < M_\lambda^* = \sup\left\{\int_0^R b(r)|z|^q r^\gamma\, dr \;\middle|\; E_\lambda(z) \leqslant 1\right\}. \tag{5.6.3}$$

PROOF. It is obvious that any maximizer of (5.6.1) is a maximizer of (5.6.3). If $z_1^* \in X_R$ is a maximizer of (5.6.3) and $E_\lambda(z_1^*) < 1$, then we can find $p > 1$ such that $E_\lambda(pz_1^*) = 1$. Then

$$\int_0^R b(r)\left|pz_1^*\right|^q r^\gamma\, dr = p^q \int_0^R b(r)\left|z_1^*\right|^q r^\gamma\, dr = p^q M_\lambda^* > M_\lambda^*,$$

which is a contradiction. Therefore $E_\lambda(z_1^*) = 1$, that is, any maximizer of (5.6.3) is a maximizer of (5.6.1). □

STEP 2. Let (5.1.1), (5.1.4)–(5.1.10) hold. Then there is a number $\delta_1 > 0$ such that for any $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ problem (5.6.1) has a solution $z_1 \in X_R$.

PROOF. From Step 1, we shall deduce the existence of $\delta_1 > 0$ corresponding to problem (5.6.3). Suppose that this is not true, that is, there exists a sequence $\delta_s \to 0$, $\delta_s > 0$, such that problem (5.6.3) with $\lambda = \lambda^s = \lambda_1 + \delta_s$ does not have a solution. Fix an integer s and consider (5.6.3) with λ^s. Denoting by z_n^s the corresponding maximizing sequence, we have

$$\|z_n^s\|_{X_R}^{\beta+2} - \lambda^s \int_0^R a(r)\left|z_n^s\right|^{\beta+2} r^\gamma\, dr \leqslant 1,$$
$$\lim_{n\to\infty} \int_0^R b(r)\left|z_n^s\right|^q r^\gamma\, dr = M_{\lambda_s}^* > 0. \tag{5.6.4}$$

If z_n^s is bounded, we may assume that z_n^s converges weakly in X_R to some z_0^s as $n \to \infty$. Then by the same argument as in Theorem 5.5.1 we can conclude that z_0^s is a solution of (5.6.3)—a contradiction. Thus we may consider z_n^s to be unbounded. Let $w_n^s = z_n^s / \|z_n^s\|_{X_R}$. Since $\|w_n^s\|_{X_R} = 1$, we may assume that $\lim_{n\to\infty} w_n^s = w_0^s$ weakly in X_R. Then

$$\int_0^R b(r)\left|z_n^s\right|^q r^\gamma\, dr = \|z_n^s\|_{X_R}^q \int_0^R b(r)\left|w_n^s\right|^q r^\gamma\, dr \to M_{\lambda_s}^* > 0,$$

therefore

$$\int_0^R b(r)\left|w_0^s\right|^q r^\gamma\, dr \geqslant 0. \tag{5.6.5}$$

By (5.6.4) we also have

$$\int_0^R r^\alpha \left|w_n^{s\prime}\right|^{\beta+2} dr - \lambda^s \int_0^R a(r)\left|w_n^s\right|^{\beta+2} r^\gamma\, dr \leqslant 1/\|z_n^s\|_{X_R}^{\beta+2}. \tag{5.6.6}$$

By letting $n \to \infty$ we get

$$\int_0^R r^\alpha \left|w_0^{s\prime}(r)\right|^{\beta+2} dr - \lambda^s \int_0^R a(r)\left|w_0^s\right|^{\beta+2} r^\gamma\, dr \leqslant 0. \tag{5.6.7}$$

From (5.6.6) there follows

$$\lambda^s \int_0^R a(r)\left|w_n^s\right|^{\beta+2} r^\gamma\, dr \geqslant \int_0^R r^\alpha \left|w_n^{s\prime}(r)\right|^{\beta+2} dr - \frac{1}{\|z_n^s\|_{X_R}^{\beta+2}}.$$

By letting $n \to \infty$, we obtain

$$\lambda^s \int_0^R a(r) \left| w_0^s \right|^{\beta+2} r^\gamma \, dr \geqslant 1. \tag{5.6.8}$$

From the weak lower semicontinuity of $\| \cdot \|_{X_R}$ and $\|w_n^s\| = 1$ we get

$$\int_0^R r^\alpha \left| w_0^{s\prime} \right|^{\beta+2} dr \leqslant 1.$$

This inequality allows us to suppose that w_0^s converges weakly in X_R to some w^*. Letting $s \to \infty$ in (5.6.8), we get that

$$\lambda_1 \int_0^R a(r) |w^*|^{\beta+2} r^\gamma \, dr \geqslant 1.$$

Hence $w^* \not\equiv 0$. By inequality (5.6.7) we obtain

$$0 \leqslant \int_0^R r^\alpha \left| w^{*\prime}(r) \right|^{\beta+2} dr - \lambda_1 \int_0^R a(r) |w^*|^{\beta+2} r^\gamma \, dr \leqslant 0.$$

The latter and Theorem 5.2.1 imply that $w^* = t u_1$, $t > 0$. By (5.6.5) we get that

$$\int_0^R b(r) |w^*|^q r^\gamma \, dr \geqslant 0$$

and thus

$$t^q \int_0^R b(r) |u_1|^q r^\gamma \, dr \geqslant 0,$$

a contradiction to (5.1.10).

Therefore there is a number $\delta_1 > 0$ such that the variational problem (5.6.3) has a solution $z_1 \in X_R$ for any $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$. By Step 1 z_1 is a solution of (5.6.1). □

STEP 3. The set

$$X_- = \left\{ z \in X_R \;\middle|\; \int_0^R b(r) |z|^q r^\gamma \, dr = -1 \right\}$$

is not empty and $m_\lambda < 0$ for $\lambda < \lambda_1$. (Recall that m_λ is defined in (5.6.2).)

PROOF. By (5.1.10)

$$\int_0^R b(r) |u_1|^q r^\gamma \, dr \leqslant 0.$$

Therefore there is a $t > 0$ such that

$$\int_0^R b(r)|tu_1|^q r^\gamma\, dr = t^q \int_0^R b(r)|u_1|^q r^\gamma\, dr = -1$$

and hence $tu_1 \in X_-$.

Since $\lambda > \lambda_1$, we have

$$E_\lambda(tu_1) = t^{\beta+2}\left(\int_0^R r^\alpha \left|u_1'\right|^{\beta+2} dr - \lambda \int_0^R a(r)|u_1|^{\beta+2} r^\gamma\, dr\right)$$
$$= t^{\beta+2}(\lambda_1 - \lambda)\int_0^R a(r)|u_1|^{\beta+2} r^\gamma\, dr < 0.$$

This inequality implies $m_\lambda < 0$. □

STEP 4. Assume that (5.1.1), (5.1.4)–(5.1.10) hold. Then there is a number $\delta_2 > 0$ such that for any $\lambda \in (\lambda_1, \lambda_1 + \delta_2)$ problem (5.6.2) has a solution $z_2 \in X_R$ such that $E_\lambda(z_2) < 0$.

PROOF. The proof is by contradiction and is analogous to that in Step 2.

Assume that the opposite assertion holds. Then there is a sequence $\delta_s \to 0$, $\delta_s > 0$, such that problem (5.6.2) with $\lambda = \lambda^s = \lambda_1 + \delta_s$ does not have solutions. Fix an integer s and consider (5.6.2) with λ^s. Denoting by z_n^s the corresponding maximizing sequence:

$$\int_0^R b(r)\left|z_n^s\right|^q r^\gamma\, dr = -1.$$

If z_n^s were bounded we could obtain a contradiction as in the proof of Step 2. Suppose that z_n^s is unbounded in X_R. As before, set $w_n^s = z_n^s / \|z_n^s\|_{X_R}$, $\|w_n^s\|_{X_R} = 1$. We have $\lim_{n\to\infty} w_n^s = w_0^s$ weakly in X_R. By

$$\int_0^R b(r)\left|z_n^s\right|^q r^\gamma\, dr = \left\|z_n^s\right\|_{X_R}^q \int_0^R b(r)\left|w_n^s\right|^q r^\gamma\, dr = -1$$

we conclude that

$$\int_0^R b(r)\left|w_0^s\right|^q r^\gamma\, dr = 0.$$

Since $E_{\lambda^s}(z_n^s) < 0$, as in Step 2, we can obtain

$$\int_0^R r^\alpha \left|w_0^{s\prime}(r)\right|^{\beta+2} dr - \lambda^s \int_0^R a(r)\left|w_0^s\right|^{\beta+2} r^\gamma\, dr \leqslant 0,$$

$$\lambda^s \int_0^R a(r)\left|w_0^s\right|^{\beta+2} r^\gamma\, dr \geqslant 1.$$

Analogously to the previous proofs, we can suppose that $w_0^s \rightharpoonup w^*$ weakly in X_R, and letting $s \to \infty$, we get

$$\lambda_1 \int_0^R a(r)|w^*|^{\beta+2} r^\gamma \, dr \geqslant 1,$$

$$0 \leqslant \int_0^R r^\alpha \left| w^{*'}(r) \right|^{\beta+2} dr - \lambda_1 \int_0^R a(r)|w^*|^{\beta+2} r^\gamma \, dr \leqslant 0,$$

$$\int_0^R b(r)|w^*|^q r^\gamma \, dr = \lim_{s\to\infty} \int_0^R b(r)\left| w_0^s \right|^q r^\gamma \, dr = 0.$$

These relations imply that w^* is a multiple of u_1 and therefore

$$\int_0^R b(r)|u_1|^q r^\gamma \, dr = 0,$$

which contradicts (5.1.10). The fact that $E_\lambda(z_2) < 0$ follows from Step 3. □

STEP 5. Denote $\delta = \min(\delta_1, \delta_2)$, where δ_1 is given by Step 2 and δ_2 by Step 4. Let $t_i > 0$, $i = 1, 2$, be the numbers determined by (5.3.5) with $z = z_i$, the solutions obtained in Steps 2 and 4, respectively. Set $u = t_1 z_1$ and $v = t_2 z_2$. Then by Lemma 5.3.3, u and v are weak solutions of (5.0.1). It is easily seen that the first weak solution u satisfies

$$\|u\|_{X_R}^{\beta+2} - \lambda \int_0^R b(r)|u|^q r^\gamma \, dr > 0,$$

while the second one, v, satisfies

$$\|v\|_{X_R}^{\beta+2} - \lambda \int_0^R b(r)|v|^q r^\gamma \, dr \leqslant 0.$$

Therefore u and v are distinct. This completes the proof of Theorem 5.6.1.

□

5.7. *Nonexistence results for classical solutions*

In this subsection we shall comment on nonexistence results for classical solutions of quasilinear equations in a general smooth bounded domain D. However, it is clear that the assumption “the considered solutions are classical” does not seem to be a natural hypothesis for this kind of problem. Indeed, the context of this book suggests that the natural class to consider should be the class of *weak* solutions.

Our argument, which is based on a variational identity [41], enables only to consider classical solutions. With regard to problem (5.0.1), an identity of this type was proved in [16]. It was used to obtain a nonexistence result in the critical case $q = l^*$. In order not to increase the volume of the chapter we shall not present further details here directing the interested reader to [16].

We should mention that Guedda and Véron [26] proved a variational identity for *weak* solutions of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

under some suitable growth assumptions on f. We claim that the argument of Guedda and Véron [26] can be adapted to prove a variational identity for weak solutions of other quasilinear equations, e.g., for the following degenerate problem:

$$\begin{cases} -\operatorname{div}(|x|^{\sigma}|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + b(x)|u|^{q-2}u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

where $\sigma \geqslant 0$ and $D \subset \mathbb{R}^N$ is a bounded domain containing the origin. Clearly this problem is a generalization of that considered in [23] and a slight modification of the arguments in [23] will give the existence and multiplicity of the solutions in this case. We point out that the radial form of this degenerate equation is of type (5.0.1). Problems of this kind will be treated elsewhere in a more general context.

We are confident that a variational identity for weak solutions of quasilinear equations involving, for instance, k-Hessian operators still holds if the potential does not grow very fast. However, in the present book we shall not consider this kind of generalization.

Suppose that D is strictly star-shaped with respect to the origin. This means that

$$(x, \nu) > 0 \tag{5.7.1}$$

for any point $x \in \partial D$, ν being the outer normal to ∂D at x. Suppose also that the functions $a, b \in C^1(\overline{D})$.

We are going to establish a nonexistence result for classical solutions of k-Hessian equations of the following form:

$$-S_k\big(\nabla^2 u\big) = \lambda a(r)|u|^{k-1}u + b(r)|u|^{q-2}u \quad \text{in } D \tag{5.7.2}$$

with the Dirichlet boundary condition

$$u = 0. \tag{5.7.3}$$

Let us recall that the k-Hessian operator S_K, $1 \leqslant k \leqslant N$, is the partial differential operator defined by

$$S_k\big(\nabla^2 u\big) = \sigma_k\big(\lambda(u)\big), \tag{5.7.4}$$

where $u \in C^2$ and $\sigma_k(\lambda(u)) = \sigma_k(\lambda_1, \dots, \lambda_N)$ is the kth elementary symmetric function of the eigenvalues of the Hessian matrix $\nabla^2 u$, whose elements are the second derivatives of u [15]. Observe that the radial form of equation (5.7.2) is of the type (5.0.1).

If $u \in C^3(D)$ is a solution of (5.7.2), (5.7.3) in D, then, following the arguments in [41] (see also Subsection 4.9 here), we obtain that

$$\int_D \left\{ \left[-\frac{\lambda N}{k+1} - c\lambda \right] a(x) - \frac{\lambda}{k+1} \langle \nabla a(x), x \rangle \right\} |u|^{k+1}\, dx$$
$$+ \int_D \left\{ \left[-\frac{N}{q} - c \right] b(x) - \frac{1}{q} \langle \nabla b(x), x \rangle \right\} |u|^q\, dx$$
$$+ \left(\frac{N-2k}{k+1} + c \right) \frac{1}{k} \int_D T_{k-1}(\nabla^2 u)_{ij} u_i u_j\, dx$$
$$= -\frac{1}{k+1} \int_{\partial D} T_{k-1}(\nabla^2 u)_{ij} u_i u_j (x, \nu)\, ds,$$

where c is an arbitrary real number and $T_{k-1}(\nabla^2 u)_{ij}$ is the Newtonian tensor, which is a positive definite matrix if u is any so-called admissible function for S_k [15]. Therefore the inequalities

$$\begin{aligned} &\frac{N-2k}{k+1} + c \geqslant 0, \\ &\left(-\frac{\lambda N}{k+1} - c\lambda \right) a(x) - \frac{\lambda}{k+1} \big(\nabla a(x), x \big) \geqslant 0, \\ &\left(-\frac{N}{q} - c \right) b(x) - \frac{1}{q} \big(\nabla b(x), x \big) \geqslant 0 \end{aligned} \tag{5.7.5}$$

imply the following nonexistence result.

THEOREM 5.7.1. *Assume that D is strictly star-shaped with respect to the origin* (5.7.1) *and $a, b \in C^1(\overline{D})$. Suppose also that for any $x \in D$ and $c \in \mathbb{R}$ inequalities* (5.7.5) *hold. Then* (5.7.2), (5.7.3) *has no nontrivial admissible solution in $C^3(D)$, provided $N > 2k$.*

6. Multiplicity for some quasilinear systems (Yu. Bozhkov and E. Mitidieri)

This section contains results from [13]. Here we shall study some existence and nonexistence results for the following quasilinear system:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ -\Delta_q v = \mu b(x)|v|^{q-2}v + (\beta+1)c(x)|u|^{\alpha+1}|v|^{\beta-1}v. \end{cases} \tag{6.0.1}$$

Here $\alpha, \beta, \lambda, \mu, p > 1, q > 1$ are real numbers, Δ_p and Δ_q are correspondingly the p- and q-Laplace operators, and $a(x), b(x), c(x)$ given functions.

System (6.0.1) will be considered in a bounded domain $\Omega \subset \mathbb{R}^N$ with the homogeneous Dirichlet boundary condition

$$u = v = 0 \quad \text{on } \partial\Omega. \tag{6.0.2}$$

Systems involving quasilinear operators of p-Laplacian type have been studied by various authors [17,34,54]. Among other results, existence and nonexistence theorems were obtained. For such purpose the method of sub- and super-solutions, the blow-up method and the Mountain Pass Theorem have been used.

Our main tool is the fibering method described in the previous sections. Its general nature enables us to prove existence and multiplicity theorems for (6.0.1) and (6.0.2) in a somewhat more constructive and explicit way.

Dealing with existence theorem, the parameters λ and μ, appearing in (6.0.1), will be naturally related to λ_1 and μ_1, the first eigenvalues of $-\Delta_p$ in $W_0^{1,p}$ and $-\Delta_q$ in $W_0^{1,q}$, respectively. The existence and properties of the first eigenvalue of p-Laplacian operators, subject to homogeneous Dirichlet conditions in a bounded domain, are obtained in [6,23, 24,26,32].

This section is organized as follows. In Subsection 6.1 we introduce some notation, define the functional spaces that will be used throughout the chapter and state our basic assumptions. For convenience of the reader we also collect some of the properties of the p-Laplacian eigenvalues and corresponding eigenfunctions. Subsection 6.2 contains a slight modification of the fibering method, adapted for vector-valued problems. The main results of this section, that is, the existence and multiplicity theorems for problems (6.0.1), (6.0.2) are presented in Subsection 6.3. Finally, in Subsection 6.4 we prove a nonexistence result for classical solutions, using a variational identity.

6.1. *The p-Laplacian operator and its eigenvalues*

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 < p, q < \infty$. We define the Sobolev spaces $Y_p = W_0^{1,p}(\Omega)$ and $Y_q = W_0^{1,q}(\Omega)$ equipped with the norms

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}, \qquad \|v\|_q = \left(\int_\Omega |\nabla v|^q\,dx\right)^{1/q}, \tag{6.1.1}$$

respectively. Then we denote $Y = Y_p \times Y_q$ and for $(u, v) \in Y$,

$$\big\|(u, v)\big\| = \|u\|_p^p + \|v\|_q^q. \tag{6.1.2}$$

Now consider the eigenvalue equation for the p-Laplace operator:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.3}$$

where $a \in L^\infty(\Omega)$. Problem (6.1.3) is closely related with our main problem (6.0.1), (6.0.2). Hence we need the following lemma.

LEMMA 6.1.1. (See Anane [6], Drábek and Pohozaev [23] and Lundqvist [32].) *There exists a number $\lambda_1 > 0$ such that*

(1) $$\lambda_1 = \inf \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega a(x)|u|^p dx}, \tag{6.1.4}$$
where the infimum is taken over $u \in Y_p$ such that $\int_\Omega a(x)|u|^p \, dx > 0$;

(2) *there exists a positive function $\phi \in Y_p \cap L^\infty(\overline{\Omega})$ which is a solution of* (6.1.3) *with* $\lambda = \lambda_1$;

(3) λ_1 *is simple, in the sense that any two eigenfunctions corresponding to* λ_1 *differ by a constant multiplier*;

(4) λ_1 *is isolated, which means that there are no eigenvalues less than* λ_1 *and no eigenvalues in the interval* $(\lambda_1, \lambda_1 + \delta)$ *for some* $\delta > 0$ *sufficiently small.*

Note that we consider (6.1.3) in a weak sense, that is,

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla z)\, dx = \lambda \int_\Omega a(x)|u|^{p-2} uz\, dx, \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

for any $z \in Y_p$.

Now we state the hypotheses that we shall assume throughout this section.

Let $\alpha, \beta, \lambda, \mu, p > 1, q > 1$ be real numbers. We shall suppose that

$$1 < p < p^*, \quad 1 < q < q^*, \tag{6.1.5}$$

$$\frac{N-p}{p}(\alpha+1) + \frac{N-q}{q}(\beta+1) < N, \tag{6.1.6}$$

where

$$p^* = Np/(N-p), \qquad q^* = Nq/(N-q)$$

are the well-known critical exponents (see [34,54]). We assume that system (6.0.1) is superhomogeneous in the sense that

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1. \tag{6.1.7}$$

It can be seen that the latter condition is equivalent to

$$d = (\alpha+1)(\beta+1) - (\alpha-p+1)(\beta-q+1) > 0. \tag{6.1.8}$$

Moreover, since (6.1.6) is equivalent to

$$N < \frac{\alpha+\beta+2}{\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1}, \tag{6.1.9}$$

one can observe that our system is subcritical [34], which avoids noncompactness problems. See [34] for more details on this point.

Note that (6.1.6) implies

$$\alpha + 1 < p^*, \quad \beta + 1 < q^*.$$

The functions $a(x)$, $b(x)$ and $c(x)$ are supposed to be bounded in Ω:

$$a, b, c \in L^\infty(\Omega) \tag{6.1.10}$$

and

$$a(x) = a_1(x) - a_2(x); \quad a_1, a_2 \geqslant 0, \ a_1(x) \not\equiv 0, \tag{6.1.11}$$

$$b(x) = b_1(x) - b_2(x); \quad b_1, b_2 \geqslant 0, \ b_1(x) \not\equiv 0. \tag{6.1.12}$$

By the Sobolev inequality it can be easily seen that (6.1.5), (6.1.6), and (6.1.10) imply that the integrals

$$\int_\Omega a(x)|u|^p \, dx$$

and

$$\int_\Omega b(x)|v|^q \, dx$$

are finite for $(u, v) \in Y$. Now we can define the following functionals on Y_p and Y_q:

$$f_1(u) = \int_\Omega a(x)|u|^p \, dx \tag{6.1.13}$$

and

$$f_2(v) = \int_\Omega b(x)|v|^q \, dx. \tag{6.1.14}$$

Since a and b are bounded, it is standard to check that f_1 and f_2 are weakly lower semi-continuous. Similarly, conditions (6.1.5), (6.1.6), and (6.1.10) imply that the functional

$$f_3(u, v) = \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1} \, dx \tag{6.1.15}$$

is weakly lower semicontinuous on Y.

We shall also suppose that

$$c^+(x) \not\equiv 0 \tag{6.1.16}$$

and

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx < 0. \tag{6.1.17}$$

The functions $\varphi \in Y_p$ and $\psi \in Y_q$ above are the first eigenfunctions of Δ_p and Δ_q correspondingly.

We end this subsection with the following

DEFINITION 6.1.2 (*Weak solution*). We say that $(u, v) \in Y$ is a weak solution of (6.0.1) if

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla z)\,dx \\
&\quad = \lambda \int_\Omega a(x)|u|^{p-2}uz\,dx + (\alpha+1)\int_\Omega c(x)|u|^{\alpha-1}u|v|^{\beta+1}z\,dx, \\
&\int_\Omega |\nabla v|^{q-2}(\nabla v, \nabla w)\,dx \\
&\quad = \mu \int_\Omega b(x)|v|^{q-2}vw\,dx + (\beta+1)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}vw\,dx
\end{aligned}$$

for any $(z, w) \in Y$.

6.2. *The fibering method for quasilinear systems*

System (6.0.1) has a variational structure. Indeed, denote

$$F(x, u, v) := \frac{\lambda}{p}a(x)|u|^p + \frac{\mu}{q}b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1} \tag{6.2.1}$$

and consider

$$\mathcal{F}(x, u, v, \nabla u, \nabla v) = \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q - F(x, u, v). \tag{6.2.2}$$

Let $J : Y \to \mathbb{R}$ be defined by

$$J(u, v) := \int_\Omega \mathcal{F}(x, u, v, \nabla u, \nabla v)\,dx,$$

or, in a more detailed form,

$$\begin{aligned}
J(u, v) = {}& \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega a(x)|u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx \\
& - \frac{\mu}{q}\int_\Omega b(x)|v|^q\,dx - \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx.
\end{aligned} \tag{6.2.3}$$

Clearly, the critical points of J are the weak solutions of problem (6.0.1), (6.0.2).

The cornerstone of the fibering method applied to this problem consists of the following. We express $(u, v) \in Y$ in the form

$$u = rz, \qquad v = \rho w, \tag{6.2.4}$$

where the functions $z \in Y_p$, $w \in Y_q$, and r, ρ are real numbers. Since we look for nontrivial solutions, we must assume that $r \neq 0$ and $\rho \neq 0$. Substituting (6.2.4) into (6.2.3), we obtain

$$\begin{aligned} J(rz, \rho w) = \frac{|r|^p}{p} \int_\Omega |\nabla z|^p \, dx - \frac{\lambda |r|^p}{p} \int_\Omega a(x)|z|^p \, dx + \frac{|\rho|^q}{q} \int_\Omega |\nabla w|^q \, dx \\ - \frac{\mu |\rho|^q}{q} \int_\Omega b(x)|w|^q \, dx - |r|^{\alpha+1}|\rho|^{\beta+1} \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} \, dx. \end{aligned} \tag{6.2.5}$$

If $(u, v) \in Y$ is a critical point of J then

$$\frac{\partial J}{\partial r}(rz, \rho w) = 0 \quad \text{and} \quad \frac{\partial J}{\partial \rho}(rz, \rho w) = 0. \tag{6.2.6}$$

Assuming that

$$A := \int_\Omega |\nabla z|^p \, dx - \lambda \int_\Omega a(x)|z|^p \, dx \neq 0, \tag{6.2.7}$$

$$B := \int_\Omega |\nabla w|^q \, dx - \mu \int_\Omega b(x)|w|^q \, dx \neq 0, \tag{6.2.8}$$

$$C := \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} \, dx \neq 0, \tag{6.2.9}$$

we can rewrite (6.2.5) in the following way:

$$J(rz, \rho w) = \frac{|r|^p}{p} A + \frac{|\rho|^q}{q} B - |r|^{\alpha+1}|\rho|^{\beta+1} C. \tag{6.2.10}$$

Conditions (6.2.6) are equivalent to

$$\begin{aligned} \frac{\partial J}{\partial r} = 0 \quad &\Leftrightarrow \quad |r|^{p-2} r A - (\alpha+1)|r|^{\alpha-1} r |\rho|^{\beta+1} C = 0, \\ \frac{\partial J}{\partial \rho} = 0 \quad &\Leftrightarrow \quad |\rho|^{q-2} \rho B - (\beta+1)|r|^{\alpha+1} |\rho|^{\beta-1} \rho C = 0, \end{aligned}$$

that is,

$$\begin{cases} |r|^{p-2}A - (\alpha+1)|r|^{\alpha-1}|\rho|^{\beta+1}C = 0, \\ |\rho|^{q-2}B - (\beta+1)|r|^{\alpha+1}|\rho|^{\beta-1}C = 0. \end{cases} \tag{6.2.11}$$

Resolving system (6.2.11), we obtain as an intermediate step that

$$r^{p-\alpha-1} = |\rho|^{\beta+1}C(\alpha+1)/A.$$

Hence A and C must have the same sign. Analogously,

$$\rho^{q-\beta-1} = |r|^{\alpha+1}C(\beta+1)/B$$

and B and C must also have the same sign. Thus A, B and C must have the same sign! Note that conditions (6.2.7)–(6.2.9) have been essentially used. Hence the solution of (6.2.11) is given by

$$|r| = \left(\frac{(\alpha+1)^{\beta-q+1}|B|^{\beta+1}}{(\beta+1)^{\beta+1}|C|^{q}|A|^{\beta-q+1}}\right)^{1/d}, \tag{6.2.12}$$

$$|\rho| = \left(\frac{(\beta+1)^{\alpha-p+1}|A|^{\alpha+1}}{(\alpha+1)^{\alpha+1}|C|^{p}|B|^{\alpha-p+1}}\right)^{1/d}, \tag{6.2.13}$$

where $d > 0$ is given in (6.1.8).

The fact that A, B, C must have simultaneously the same sign leads us to considering two cases. In the next subsections, we shall assume that

$$A > 0, \quad B > 0, \quad C > 0 \tag{6.2.14}$$

or

$$A < 0, \quad B < 0, \quad C < 0. \tag{6.2.15}$$

Thus, in both cases (6.2.14) and (6.2.15), the functions $r = r(z,w)$ and $\rho = \rho(z,w)$ are well defined. Now we insert the expressions for $r = r(z,w)$ and $\rho = \rho(z,w)$ determined by (6.2.12) and (6.2.13) into (6.2.10). In this way, we obtain a functional

$$I(z,w) = J\big(r(z,w)z, \rho(z,w)w\big), \tag{6.2.16}$$

given by

$$I(z,w) = K\left|\int_\Omega |\nabla z|^p\,dx - \lambda\int_\Omega a(x)|z|^p\,dx\right|^{(\alpha+1)q/d} \times \frac{|\int_\Omega |\nabla w|^q\,dx - \mu\int_\Omega b(x)|w|^q\,dx|^{(\beta+1)p/d}}{|\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx|^{pq/d}}, \tag{6.2.17}$$

where

$$K=\left(\frac{(\alpha+1)^{(\beta-q+1)p/d}}{p(\beta+1)^{(\beta+1)p/d}}+\frac{(\beta+1)^{(\alpha-p+1)q/d}}{q(\alpha+1)^{(\alpha+1)q/d}}\right.\\ \left.-\frac{1}{(\alpha+1)^{(\alpha+1)q/d}(\beta+1)^{(\beta+1)p/d}}\right)\operatorname{sgn}\left(\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx\right).$$

Therefore, provided z and w satisfy (6.2.14) or (6.2.15), we have

$$\left.\frac{\partial J}{\partial r}\right|_{r=r(z,w),\rho=\rho(z,w)}=0 \tag{6.2.18}$$

and

$$\left.\frac{\partial J}{\partial \rho}\right|_{r=r(z,w),\rho=\rho(z,w)}=0. \tag{6.2.19}$$

Next, we introduce the following notation: for any functional $f:Y\to\mathbb{R}$ we denote by

$$f'(z,w)(h_1,h_2),$$

the Gâteaux derivative of f at $(z,w)\in Y$ in direction of $(h_1,h_2)\in Y$.

Let

$$E_1(z)=\int_\Omega|\nabla z|^p\,dx-\lambda\int_\Omega a(x)|z|^p\,dx, \tag{6.2.20}$$

$$E_2(w)=\int_\Omega|\nabla w|^q\,dx-\mu\int_\Omega b(x)|w|^q\,dx \tag{6.2.21}$$

and

$$E_i^{(1)}(z,w)(h_1,h_2)=\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0,\sigma=0}E_i(z+\varepsilon h_1,w+\sigma h_2),$$

$$E_i^{(2)}(z,w)(h_1,h_2)=\left.\frac{\partial}{\partial\sigma}\right|_{\varepsilon=0,\sigma=0}E_i(z+\varepsilon h_1,w+\sigma h_2),$$

$$I^{(1)}(z,w)(h_1,h_2)=\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0,\sigma=0}I(z+\varepsilon h_1,w+\sigma h_2),$$

$$I^{(2)}(z,w)(h_1,h_2)=\left.\frac{\partial}{\partial\sigma}\right|_{\varepsilon=0,\sigma=0}I(z+\varepsilon h_1,w+\sigma h_2).$$

It is easy to see that the following lemma holds. We omit the straightforward details.

LEMMA 6.2.1.

(1) *The functional I is homogeneous of degree 0, that is, for every $z \in Y_p$, $w \in Y_q$ such that $\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx \neq 0$ and every $t \neq 0$ we have*

$$I(tz, tw) = I(z, w).$$

(2) *I is even and*

$$I'(z, w)(z, w) = 0.$$

REMARK 6.2.2. If $(z, w) \in Y$ is a critical point of I, by well-known properties of p-Laplace Dirichlet integral (see [41]), it follows that $(|z|, |w|) \in Y$ is also a critical point of I.

The next two lemmas are direct consequences of the results proved in the previous sections.

LEMMA 6.2.3. *Let (z, w) be a critical point of I, which satisfies* (6.2.14) *or* (6.2.15). *Then the function (u, v) defined by*

$$u(x) = rz(x), \qquad v(x) = \rho w(x),$$

where $r \neq 0$ and $\rho \neq 0$ are determined by (6.2.12) *or* (6.2.13), *is a critical point of J.*

PROOF. Since (z, w) is a critical point of I, we have

$$I'(z, w)(h_1, h_2) = \bigl(I^{(1)}(z, w)(h_1, h_2), I^{(2)}(z, w)(h_1, h_2)\bigr) = 0.$$

Therefore, since

$$\left.\frac{\partial J}{\partial r}\right|_{r=r(z,w),\rho=\rho(z,w)} = \left.\frac{\partial J}{\partial \rho}\right|_{r=r(z,w),\rho=\rho(z,w)} = 0$$

(see (6.2.18) and (6.2.19)), by the chain rule we have

$$\begin{aligned} 0 &= I^{(1)}(z, w)(h_1, h_2) \\ &= r(z, w)J^{(1)}(rz, \rho w)(h_1, h_2) + \left.\frac{\partial J}{\partial r}\right|_{r=r(z,w),\rho=\rho(z,w)} \frac{\partial r}{\partial z} \\ &\quad + \left.\frac{\partial J}{\partial \rho}\right|_{r=r(z,w),\rho=\rho(z,w)} \frac{\partial \rho}{\partial z} = r(z, w)J^{(1)}(rz, \rho w)(h_1, h_2). \end{aligned}$$

Thus $J^{(1)}(u, v) = 0$. Analogously, $J^{(2)}(u, v) = 0$ and therefore $J'(u, v) = 0$. □

LEMMA 6.2.4. *Let E_1 and E_2 be defined by* (6.2.20) *and* (6.2.21). *Consider*

$$E_1(z,w)=c_1 \quad \text{and} \quad E_2(z,w)=c_2,$$

where $c_i \in \mathbb{R}\,(i=1,2)$. Suppose that

$$\det\begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} \neq 0 \quad \text{if } E_1(z,w)=c_1 \text{ and } E_2(z,w)=c_2. \tag{6.2.22}$$

Then every critical point of I with the conditions $E_1(z,w)=c_1$ and $E_2(z,w)=c_2$ is a critical point of I.

PROOF. Let (z,w) be a conditional critical point of I. By the Euler theorem there exist $m_1, m_2 \in \mathbb{R}$ such that

$$I'(z,w)=m_1E_1'(z,w)+m_2E_2'(z,w). \tag{6.2.23}$$

Since by Lemma 6.2.1 we have $I'(z,w)(z,w)=0$, by (6.2.23) we obtain

$$m_1E_1^{(1)}+m_2E_2^{(1)}=0,$$
$$m_1E_1^{(2)}+m_2E_2^{(2)}=0.$$

Now by (6.2.22) we have

$$\det\begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} \neq 0.$$

Therefore $m_1=m_2=0$. Thus $I'(z,w)=0$, that is, (z,w) is a critical point of I.

The last two lemmas are fundamental in what follows. Our first aim is to prove the existence of a critical point of I with appropriate conditions $E_1(z,w)=c_1$ and $E_2(z,w)=c_2$. This in turn will be an actual critical point of I and hence a critical point of J—a weak solution of (6.0.1). □

In the next subsection we follow the pattern as in [23].

6.3. *Existence and multiplicity results*

We have already pointed out that the existence and multiplicity results are in connection with the first eigenvalues λ_1 and μ_1 of the p- and q-Laplacian respectively. We distinguish the following six cases:

(1) $0 \leqslant \lambda < \lambda_1$, $0 \leqslant \mu < \mu_1$,
(2) $0 \leqslant \lambda < \lambda_1$, $\mu = \mu_1$,

(3) $0 \leqslant \lambda < \lambda_1$, $\mu > \mu_1$,
(4) $\lambda = \lambda_1$, $\mu = \mu_1$,
(5) $\lambda = \lambda_1$, $\mu > \mu_1$,
(6) $\lambda > \lambda_1$, $\mu > \mu_1$.
The rest three possible cases can be treated analogously. In order not to increase the volume of the chapter, we shall not present details for cases (2), (3), and (5), merely pointing out that the methods of the next sub-subsections carry out to these cases.

6.3.1. *Existence theorems for* $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$. The form of the functional J suggests that we consider

$$E_1(z) = 1 \quad \text{and} \quad E_2(w) = 1 \tag{6.3.1}$$

as the constraints in Lemma 6.2.4. Indeed, we calculate

$$E_1^{(1)} = pE_1(z) = pA,$$
$$E_1^{(2)} = E_2^{(1)} = 0,$$
$$E_2^{(2)} = qE_2(w) = qB.$$

Therefore

$$\det \begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} = pqAB > 0,$$

and the conditions of Lemma 6.2.4 are fulfilled. Moreover, since we are assuming (6.3.1), inequalities (6.2.14) hold, that is, $1 = E_1 = A > 0$, $1 = E_2 = B > 0$ and

$$C = \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0.$$

Further, the functional I becomes

$$I(z, w) = K\frac{1}{(\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx)^{pq/d}}. \tag{6.3.2}$$

The main result in this sub-subsection is the following

THEOREM 6.3.1. *Suppose that* (6.1.5)–(6.1.16) *hold and that, in addition,* $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$. *Then problem* (6.0.1), (6.0.2) *has at least two positive weak solutions* $(u_i, v_i) \in Y$, $i = 1, 2$.

The proof of this theorem will be a consequence of the next two propositions.

PROPOSITION 6.3.2. *Suppose that* (6.1.5)–(6.1.16) *hold and that, in addition,* $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$. *Then problem* (6.0.1), (6.0.2) *has at least one positive weak solution* $(u_1, v_1) \in Y$.

PROOF. The formulas (6.2.20) and (6.2.21) suggest to consider an auxiliary problem: find a maximizer (z^*, w^*) of

$$0 < M_{\lambda,\mu} := \sup\left\{ \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \;\middle|\; E_1(z) = 1 \text{ and } E_2(w) = 1 \right\}. \tag{6.3.3}$$

We claim that problem (6.3.3) has a solution. Indeed, the sets

$$X_\lambda = \left\{ z \in Y_p \;\middle|\; E_1(z) = 1 \right\},$$

and

$$X_\mu = \left\{ w \in Y_q \;\middle|\; E_2(w) = 1 \right\}$$

are nonempty. By Lemma 6.1.1 we have that for any $z \in X_\lambda$:

$$\|z\|_p^p = \lambda \int_\Omega a(x)|z|^p\,dx + 1 \leqslant \frac{\lambda}{\lambda_1}\|z\|_p^p + 1,$$

that is,

$$\|z\|_p^p \leqslant \frac{\lambda_1}{\lambda_1 - \lambda},$$

and analogously

$$\|w\|_q^q \leqslant \frac{\mu_1}{\mu_1 - \mu}.$$

Since $0 \leqslant \lambda < \lambda_1$ and $0 \leqslant \mu < \mu_1$, we have

$$\big\|(z, w)\big\| = \|z\|_p^p + \|w\|_q^q \leqslant \frac{\lambda_1}{\lambda_1 - \lambda} + \frac{\mu_1}{\mu_1 - \mu}.$$

Therefore a maximizing sequence (z_n, w_n) for (6.3.3) is bounded in Y. Thus we can suppose that (z_n, w_n) converges weakly in Y to some (z^*, w^*). By (6.1.15)

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}\,dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx = M_{\lambda,\mu} > 0.$$

In particular $z^* \not\equiv 0$ and $w^* \not\equiv 0$.

The weakly lower semicontinuity of the corresponding norms, (6.1.5), (6.1.6), and $E_1(z_n) = 1,\ E_2(w_n) = 1$ imply that

$$E_1(z^*) \leqslant 1, \qquad E_2(w^*) \leqslant 1.$$

Indeed,

$$\|z^*\|_p^p \leqslant \lim_{n\to\infty} \inf \|z_n\|_p^p,$$

$$\|w^*\|_q^q \leqslant \lim_{n\to\infty} \inf \|w_n\|_q^q,$$

$$\int_\Omega a(x)|z^*|^p\,dx = \lim_{n\to\infty} \int_\Omega a(x)|z_n|^p\,dx,$$

$$\int_\Omega b(x)|w^*|^q\,dx = \lim_{n\to\infty} \int_\Omega b(x)|w_n|^q\,dx.$$

If $E_1(z^*) < 1$, then there exists a number $t_1 > 1$ such that $E_1(t_1 z^*) = 1$ and hence $t_1 z^* \in X_\lambda$. If $E_2(w^*) < 1$, then there exists a number $t_2 > 1$ such that $E_2(t_2 w^*) = 1$ and hence $t_2 w^* \in X_\mu$. Therefore,

$$\begin{aligned}\int_\Omega c(x)|t_1 z^*|^{\alpha+1}|t_2 w^*|^{\beta+1}\,dx &= t_1^{\alpha+1} t_2^{\beta+1} \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx \\ &= t_1^{\alpha+1} t_2^{\beta+1} M_{\lambda,\mu} \\ &> M_{\lambda,\mu} = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0\right\},\end{aligned}$$

a contradiction. Thus $E_1(z^*) = 1$ or $E_2(w^*) = 1$. If $E_1(z^*) = 1,\ E_2(w^*) < 1$ or $E_1(z^*) < 1,\ E_2(w^*) = 1$, we can obtain another contradiction. Hence $(z^*, w^*) \in X_\lambda \times X_\mu$ is a solution of (6.3.3). By Lemma 6.2.4 it follows that (z^*, w^*) is a critical point of I. By Remark 6.2.2 we may assume $z^* \geqslant 0$ and $w^* \geqslant 0$. Thus, by Lemma 6.2.3, $(u_1 = r_1 z^*, v_1 = \rho_1 w^*)$ is a critical point of J. Therefore $(u_1, v_1) \in Y$ is a nonnegative weak solution of (6.0.1), (6.0.2). Using the same arguments as in [23] we deduce that $u_1 > 0,\ v_1 > 0$. This completes the proof. □

REMARK 6.3.3. In the scalar case it is known that weak solutions of

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + b(x)|u|^{q-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

belong to $C^{1,\nu}_{\mathrm{loc}}(\Omega)$ for some $\nu \in (0,1)$ (see [23]). Since our system is subcritical (see (6.1.9)), we expect that a similar result holds for (6.0.1). The regularity problem for weak solutions of quasilinear variational elliptic systems of type (6.0.1) will be studied elsewhere.

PROPOSITION 6.3.4. *Suppose that* (6.1.5)–(6.1.16) *hold and that, in addition,* $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$. *Then problem* (6.0.1), (6.0.2) *has another positive weak solution* $(u_2, v_2) \in Y$.

PROOF. Consider the following:

$$0 < \hat{M}_{\lambda,\mu} := \sup\left\{ \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \;\middle|\; E_1(z) + E_2(w) = 1 \right\}. \tag{6.3.4}$$

Then the set

$$X_{\lambda,\mu} = \left\{(z,w) \in Y \;\middle|\; E_1(z) + E_2(w) = 1\right\}$$

is not empty. By $E_1(z) + E_2(w) = 1$ and Lemma 6.1.1, for any $(z,w) \in X_{\lambda,\mu}$ we have

$$\|z\|_p^p + \|w\|_q^q \leqslant 1 + \frac{\lambda}{\lambda_1}\|z\|_p^p + \frac{\mu}{\mu_1}\|w\|_q^q,$$

that is,

$$\frac{\lambda_1 - \lambda}{\lambda_1}\|z\|_p^p + \frac{\mu_1 - \mu}{\mu_1}\|w\|_q^q \leqslant 1.$$

Since each of the summands above is strictly positive (recall that $\lambda < \lambda_1$, $\mu < \mu_1$), the latter inequality implies

$$\|z\|_p^p \leqslant \frac{\lambda_1}{\lambda_1 - \lambda}$$

and

$$\|w\|_q^q \leqslant \frac{\mu_1}{\mu_1 - \mu}.$$

Therefore $\|(z,w)\|$ is bounded. Hence, we may suppose that a maximizing sequence (z_n, w_n) for (6.3.4) is bounded in Y. Thus we can assume that (z_n, w_n) converges weakly in Y to some (z^*, w^*). By (6.1.15) it follows that

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}\,dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx = \hat{M}_{\lambda,\mu} > 0.$$

In particular $z^* \not\equiv 0$ and $w^* \not\equiv 0$.

The weakly lower semicontinuity of the corresponding norms, (6.1.5), (6.1.6), and $E_1(z_n) + E_2(w_n) = 1$ imply that

$$E_1(z^*) + E_2(w^*) \leqslant 1,$$

that is,

$$\left(\|z^*\|_p^p - \lambda \int_\Omega a(x)|z^*|^p\,dx\right) + \left(\|w^*\|_q^q - \lambda \int_\Omega b(x)|w^*|^q\,dx\right) \leqslant 1.$$

Since $\lambda < \lambda_1,\ \mu < \mu_1$ both summands above are positive. Hence

$$0 < E_1(z^*) + E_2(w^*) \leqslant 1.$$

We claim that actually

$$E_1(z^*) + E_2(w^*) = 1.$$

Indeed, if $E_1(z^*) + E_2(w^*) < 1$, then there exists $t > 1$ such that

$$t\big(E_1(z^*) + E_2(w^*)\big) = 1.$$

Then $(t^{1/p} z^*, t^{1/q} w^*) \in X_{\lambda,\mu} := X_\lambda \times X_\mu$ and

$$\begin{aligned}
&\int_\Omega c(x)\big|t^{1/p} z^*\big|^{\alpha+1}\big|t^{1/q} w^*\big|^{\beta+1}\,dx \\
&\quad = t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx \\
&\quad = t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \hat{M}_{\lambda,\mu} \\
&\quad > \hat{M}_{\lambda,\mu} = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \;\middle|\; E_1(z) + E_2(w) = 1\right\},
\end{aligned}$$

a contradiction (note that we have used (6.1.7)). Therefore we have proved the claim. Hence $(z^*, w^*) \in X_{\lambda,\mu}$ is a solution of (6.3.4). By an analogue of Lemma 6.2.4 for one constraint of type $E(z,w) = \text{const}$, (z^*, w^*) is a critical point of I. Indeed, since in our case $E(z,w) = E_1(z) + E_2(w) = 1$, the condition $E'(z,w)(z,w) \neq 0$ if $E(z,w) = 1$ is easily verified. The rest of the proof is the same as that of Proposition 6.3.2. □

PROOF OF THEOREM 6.3.1. It remains to show that the solutions found in Propositions 6.3.2 and 6.3.4 are distinct. The proof is by contradiction. Suppose that $(u_1, v_1) = (u_2, v_2)$. By the proofs of Propositions 6.3.2 and 6.3.4 it follows that

$$\frac{E_1(u_1)}{r_1^p} = \frac{E_2(v_1)}{\rho_1^q} = 1$$

and

$$\frac{E_1(u_2)}{r_2^p} + \frac{E_2(v_2)}{\rho_2^q} = 1,$$

where $r_i, \rho_i, i = 1, 2$, are determined by (6.2.12) and (6.2.13), with $z_i^*,\ w_i^*,\ i = 1, 2$. These relations imply that if the solutions are not distinct then there exists a number $m > 1$ such that

$$r_1^p = \frac{r_2^p}{m}, \qquad \rho_1^q = \frac{\rho_2^q}{m'}, \qquad \frac{1}{m} + \frac{1}{m'} = 1.$$

By (6.2.12) and (6.2.13) we have

$$r_1 = \left(c_1 C^{-q}\right)^{1/d}, \qquad \rho_1 = \left(c_2 C^{-p}\right)^{1/d},$$
$$r_2 = \left(c_1 C^{-q} \frac{(1-s)^{\beta+1}}{s^{\beta+1-q}}\right)^{1/d}, \qquad \rho_2 = \left(c_2 C^{-p} \frac{s^{\alpha+1}}{(1-s)^{\alpha+1-p}}\right)^{1/d},$$

where we have introduced the parameter $s = E_1(z_2^*)$. We note that the exact values of c_1 and c_2 are not important for the proof. Since $s \in (0,1)$, it is easy to show that the conditions $m > 1$ and $m' > 1$ are equivalent to

$$s^{\beta+1-q} < (1-s)^{\beta+1}$$

and

$$s^{\alpha+1} > (1-s)^{\alpha+1-p}.$$

From the last two inequalities we have that

$$s^d > 1,$$

where $d > 0$ is given by (6.1.8). This is impossible for $s \in (0,1)$. Thus we have reached a contradiction. This concludes the proof. □

6.3.2. *The eigenvalue case $\lambda = \lambda_1$, $\mu = \mu_1$.* We consider problem (6.3.4) with $\lambda = \lambda_1$ and $\mu = \mu_1$. In this case the corresponding set $X_{\lambda,\mu}$ is not bounded in Y. Therefore, we need to impose an additional condition on our data. Henceforth we shall suppose that condition (6.1.17) is fulfilled.

THEOREM 6.3.5. *Suppose that* (6.1.5)–(6.1.17) *hold and* $\lambda = \lambda_1$, $\mu = \mu_1$. *Then problem* (6.0.1), (6.0.2) *has at least one positive weak solution* $(u, v) \in Y$.

PROOF. The arguments of the proof of this theorem would be the same as those of Proposition 6.3.2 if we could prove that problem (6.3.4) with $\lambda = \lambda_1$, $\mu = \mu_1$ has a solution.

Let (z_n, w_n) be a maximizing sequence such that

$$E_1(z_n) + E_2(w_n) = 1, \qquad \int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}\,dx = \hat{m}_n \to \hat{M}_{\lambda_1,\mu_1} > 0.$$

Suppose that $\|(z_n, w_n)\| \to \infty$ and put

$$s_n = \frac{z_n}{\|(z_n, w_n)\|^{1/p}}, \qquad t_n = \frac{w_n}{\|(z_n, w_n)\|^{1/q}}, \qquad \|(s_n, t_n)\| = 1.$$

Then

$$\|(z_n, w_n)\|\left[\left(\|s_n\|_p^p - \lambda_1 \int_\Omega a(x)|s_n|^p\,dx\right) + \left(\|t_n\|_q^q - \mu_1 \int_\Omega b(x)|t_n|^q\,dx\right)\right] = 1.$$

Therefore

$$\|s_n\|_p^p - \lambda_1 \int_\Omega a(x)|s_n|^p\,dx + \|t_n\|_q^q - \mu_1 \int_\Omega b(x)|t_n|^q\,dx = \frac{1}{\|(z_n, w_n)\|} \to 0, \quad n \to \infty.$$

Hence

$$\|(s_n, t_n)\| - \lambda_1 \int_\Omega a(x)|s_n|^p\,dx - \mu_1 \int_\Omega b(x)|t_n|^q\,dx = \frac{1}{\|(z_n, w_n)\|} \to 0, \tag{6.3.5}$$

and thus

$$\lim_{n\to\infty}\left[\lambda_1 \int_\Omega a(x)|s_n|^p\,dx + \mu_1 \int_\Omega b(x)|t_n|^q\,dx\right] = 1,$$

since $\|(s_n, t_n)\| = 1$. We may assume that (s_n, t_n) converges weakly in Y to some (s^*, t^*). Thus

$$\lambda_1 \int_\Omega a(x)|s^*|^p\,dx + \mu_1 \int_\Omega b(x)|t^*|^q\,dx = 1,$$

which implies that $(s^*, t^*) \not\equiv (0, 0)$. Furthermore,

$$\|(s^*, t^*)\| \leqslant \lim_{n\to\infty} \inf\|(s_n, t_n)\| = 1.$$

Now from (6.3.5) we deduce that

$$\left(\|s^*\|_p^p - \lambda_1 \int_\Omega a(x)|s^*|^p\,dx\right) + \left(\|t^*\|_q^q - \mu_1 \int_\Omega b(x)|t^*|^q\,dx\right) = 0.$$

The variational properties of the first eigenvalue of the p- and q-Laplacian imply that both summands in the above relation are nonnegative. Hence both are zero, which means, by Lemma 6.1.1, that

$$s^* = c_1\varphi, \qquad t^* = c_2\psi.$$

Since

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}\,dx = \big\|(z_n, w_n)\big\|^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_\Omega c(x)|s_n|^{\alpha+1}|t_n|^{\beta+1}\,dx$$
$$= \hat{m}_n \to \hat{M}_{\lambda_1,\mu_1} > 0,$$

we conclude that

$$\int_\Omega c(x)|s^*|^{\alpha+1}|t^*|^{\beta+1}\,dx \geqslant 0$$

and therefore

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx \geqslant 0,$$

which contradicts (6.1.17). Thus we can assume that (z_n, w_n) is bounded and

$$\lim_{n\to\infty}(z_n, w_n) = (z^*, w^*)$$

weakly in Y. Then

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}\,dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx = M_{\lambda_1,\mu_1} > 0.$$

This means that $z^* \neq 0$ and $w^* \neq 0$. Furthermore,

$$0 \leqslant E_1(z^*) + E_2(w^*) \leqslant 1.$$

We claim that

$$0 < E_1(z^*) + E_2(w^*) \leqslant 1.$$

Indeed, first suppose that

$$0 = E_1(z^*) + E_2(w^*),$$

that is,

$$\left(\|z^*\|_p^p - \lambda_1 \int_\Omega a(x)|z^*|^p\,dx\right) + \left(\|w^*\|_q^q - \mu_1 \int_\Omega b(x)|w^*|^q\,dx\right) = 0.$$

Therefore, by Lemma 6.1.1 we know that

$$z^* = k_1\varphi, \qquad w^* = k_2\psi$$

for some $k_1, k_2 \neq 0$, and then

$$\int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx = |k_1|^{\alpha+1}|k_2|^{\beta+1}\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx$$
$$= \hat{M}_{\lambda_1,\mu_1} > 0,$$

which is a contradiction since (6.1.17) holds.

Next, suppose that

$$0 < E_1(z^*) + E_2(w^*) < 1.$$

Then we can find $t > 1$ such that

$$t\big(E_1(z^*) + E_2(w^*)\big) = 1.$$

Further

$$\begin{aligned}
&\int_\Omega c(x)\big|t^{1/p}z^*\big|^{\alpha+1}\big|t^{1/q}w^*\big|^{\beta+1}\,dx \\
&\quad = t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}\,dx \\
&\quad = t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\hat{M}_{\lambda_1,\mu_1} > \hat{M}_{\lambda_1,\mu_1} \\
&\quad = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \,\middle|\, E_1(z) + E_2(w) = 1\right\},
\end{aligned}$$

another contradiction.

In this way, we have proved that

$$E_1(z^*) + E_2(w^*) = 1,$$

and therefore (z^*, w^*) is a maximizer of problem (6.3.4) with $\lambda = \lambda_1$, $\mu = \mu_1$. The rest of the proof is the same as that of Proposition 6.3.1. This completes the proof. □

6.3.3. *Existence of three distinct solutions for $\lambda > \lambda_1$, $\mu > \mu_1$.*

THEOREM 6.3.6. *Suppose that* (6.1.5)–(6.1.17) *hold*, $\lambda > \lambda_1$, *and* $\mu > \mu_1$. *Then there exist* $\delta > 0$ *and* $\varepsilon > 0$ *such that for* $\lambda \in (\lambda_1, \lambda_1 + \delta)$, $\mu \in (\mu_1, \mu_1 + \varepsilon)$ *problem* (6.0.1), (6.0.2) *has at least three positive weak solutions in* Y.

The proof of the above theorem will be a consequence of several lemmas. To begin with, we define

$$M_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \,\middle|\, E_1(z) = 1 \text{ and } E_2(w) = 1\right\} \tag{6.3.6}$$

and

$$\tilde{M}_{\lambda,\mu} := \sup\left\{ \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx > 0 \;\middle|\; E_1(z) \leqslant 1 \text{ and } E_2(w) \leqslant 1 \right\}. \tag{6.3.7}$$

LEMMA 6.3.7. *Problems* (6.3.6) *and* (6.3.7) *are equivalent.*

PROOF. Since $c^+ \not\equiv 0$ (see (6.1.16)), any maximizer of (6.3.6) is a maximizer of (6.3.7). Suppose for a moment that $(z, w) \in Y$ is a maximizer of (6.3.7) and $E_1(z) < 1$ or $E_2(w) < 1$. For instance, let $E_1(z) < 1$. Therefore there exists $k > 1$ such that $E_1(kz) = 1$. Then

$$\begin{aligned} \int_\Omega c(x)|kz|^{\alpha+1}|w|^{\beta+1}\,dx &= k^{\alpha+1}\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx \\ &= k^{\alpha+1}\tilde{M}_{\lambda,\mu} > \tilde{M}_{\lambda,\mu}, \end{aligned} \tag{6.3.8}$$

which is a contradiction. Thus $E_1(z) = E_2(w) = 1$. Hence any maximizer of (6.3.7) is a maximizer of (6.3.6). □

LEMMA 6.3.8. *Let* (6.1.5)–(6.1.17) *hold. Then there exist* $\delta_1 > 0$ *and* $\varepsilon_1 > 0$ *such that for* $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$, $\mu \in (\mu_1, \mu_1 + \varepsilon_1)$ *problem* (6.3.6) *has a nontrivial solution* $(z, w) \in Y$.

PROOF. Due to Lemma 6.3.7, it suffices to deduce the existence of $\delta > 0$ and $\varepsilon > 0$ corresponding to problem (6.3.7). Suppose that the claim is not true, that is, there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that problem (6.3.7) with $\lambda = \lambda^s = \lambda_1 + \delta_s$ and $\mu = \mu^s = \mu_1 + \varepsilon_s$ does not have a solution. Fix an integer s and consider (6.3.7) with λ^s and μ^s. Denoting by (z_n^s, w_n^s) the corresponding maximizing sequence, we have

$$\lim_{n\to\infty}\int_\Omega c(x)\big|z_n^s\big|^{\alpha+1}\big|w_n^s\big|^{\beta+1}\,dx = \tilde{M}_{\lambda^s,\mu^s} > 0,$$

$$E_1\big(z_n^s\big) \leqslant 1$$

and

$$E_2\big(w_n^s\big) \leqslant 1.$$

If (z_n^s, w_n^s) is bounded, we may assume that it converges weakly in Y to some (z_0^s, w_0^s) as $n \to \infty$. Then

$$\int_\Omega c(x)\big|z_n^s\big|^{\alpha+1}\big|w_n^s\big|^{\beta+1}\,dx \to \int_\Omega c(x)\big|z_0^s\big|^{\alpha+1}\big|w_0^s\big|^{\beta+1}\,dx = \tilde{M}_{\lambda^s,\mu^s} > 0,$$

$$\int_\Omega \big|\nabla z_0^s\big|^p\,dx - \lambda^s\int_\Omega a(x)\big|z_0^s\big|^p\,dx \leqslant 1,$$

$$\int_\Omega \left|\nabla w_0^s\right|^q dx - \mu^s \int_\Omega b(x)\left|w_0^s\right|^q dx \leqslant 1.$$

Therefore (z_0^s, w_0^s) is a solution of (6.3.7)—a contradiction. Thus we may consider (z_n^s, w_n^s) to be unbounded. Let

$$\left(h_n^s, t_n^s\right) = \frac{(z_n^s, w_n^s)}{\|(z_n^s, w_n^s)\|}.$$

Since $\|(h_n^s, t_n^s)\| = 1$ we may assume that

$$\lim_{n\to\infty}\left(h_n^s, t_n^s\right) = \left(h_0^s, t_0^s\right)$$

weakly in Y. Then

$$\begin{aligned}&\int_\Omega c(x)\left|z_n^s\right|^{\alpha+1}\left|w_n^s\right|^{\beta+1} dx\\ &\quad = \left\|\left(z_n^s, w_n^s\right)\right\|^{\alpha+\beta+2}\int_\Omega c(x)\left|h_n^s\right|^{\alpha+1}\left|t_n^s\right|^{\beta+1} dx \to \tilde{M}_{\lambda^s,\mu^s} > 0,\end{aligned}$$

therefore

$$\int_\Omega c(x)\left|h_0^s\right|^{\alpha+1}\left|t_0^s\right|^{\beta+1} dx \geqslant 0. \tag{6.3.9}$$

From the inequality $E_1(z_n^s) \leqslant 1$, that is,

$$\left\|\left(z_n^s, w_n^s\right)\right\|^p\left(\left\|h_n^s\right\|_p^p - \lambda^s \int_\Omega a(x)\left|h_0^s\right|^p dx\right) \leqslant 1,$$

it follows that

$$\left\|h_n^s\right\|_p^p - \lambda^s \int_\Omega a(x)\left|h_n^s\right|^p dx \leqslant \frac{1}{\|(z_n^s, w_n^s)\|^p}.$$

By letting $n \to \infty$ we get

$$\left\|h_0^s\right\|_p^p - \lambda^s \int_\Omega a(x)\left|h_0^s\right|^p dx \leqslant 0. \tag{6.3.10}$$

On the other hand, summing up

$$\lambda^s \int_\Omega a(x)\left|h_n^s\right|^p dx \geqslant \left\|h_n^s\right\|_p^p - \frac{1}{\|(z_n^s, w_n^s)\|^p}$$

and

$$\mu^s \int_\Omega b(x)\left|t_n^s\right|^q dx \geqslant \left\|t_n^s\right\|_q^q - \frac{1}{\|(z_n^s, w_n^s)\|^q}$$

and letting $n \to \infty$, we obtain

$$\lambda^s \int_\Omega a(x)\left|h_0^s\right|^p dx + \mu^s \int_\Omega b(x)\left|t_0^s\right|^q dx \geqslant 1. \tag{6.3.11}$$

Clearly $\|(h_0^s, t_0^s)\| \leqslant 1$. This allows us to suppose that (h_0^s, t_0^s) converges weakly in Y to some (h_0, t_0). Letting $s \to \infty$ in (6.3.11), we get that

$$\lambda_1 \int_\Omega a(x)|h_0|^p dx + \mu_1 \int_\Omega b(x)|t_0|^q dx \geqslant 1.$$

Hence $(h_0, t_0) \not\equiv (0, 0)$. Next, from inequality (6.3.10) we obtain

$$0 \leqslant \|h_0\|_p^p - \lambda^s \int_\Omega a(x)|h_0|^p dx \leqslant 0.$$

The latter and Lemma 6.1.1 imply that $h_0 = l\varphi$, $l \neq 0$. Starting with $E_2(w_n^s) \leqslant 1$ we can obtain $t_0 = k\psi$, $k \neq 0$. Then by (6.3.9) we get that

$$\int_\Omega c(x)|h_0|^{\alpha+1}|t_0|^{\beta+1} dx \geqslant 0,$$

and thus

$$|l|^{\alpha+1}|k|^{\beta+1} \int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1} dx \geqslant 0.$$

This contradicts our assumption (6.1.17).

Therefore there exist $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ and $\mu \in (\mu_1, \mu_1 + \varepsilon_1)$ problem (6.3.7) has a solution $(z_1, w_1) \in Y$. By Lemma 6.3.7 $(z_1, w_1) \in Y$ is a solution of (6.3.6). □

LEMMA 6.3.9. *The set*

$$W_- = \left\{(z, w) \in Y \;\middle|\; \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} dx = -1\right\}$$

is not empty and $m_{\lambda,\mu} < 0$, $\lambda > \lambda_1$, $\mu > \mu_1$, *where*

$$m_{\lambda,\mu} = \inf\left\{E_1(z) + E_2(w) \;\middle|\; \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} dx = -1\right\}. \tag{6.3.12}$$

PROOF. Set $z = \varphi$ and $w = \psi$. Then by (6.1.17) we have

$$\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}\,dx = \int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx < 0.$$

Therefore there exist $k, l \in \mathbb{R}$ such that

$$\int_\Omega c(x)|k\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx = -1$$

and

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|l\psi|^{\beta+1}\,dx = -1.$$

Since $\lambda > \lambda_1$ and $\mu > \mu_1$, we have

$$E_1(k\varphi) = |k|^p(\lambda_1 - \lambda)\int_\Omega a(x)|\varphi|^p\,dx < 0$$

and

$$E_2(l\psi) = |l|^q(\lambda_1 - \lambda)\int_\Omega b(x)|\psi|^q\,dx < 0.$$

These inequalities imply that $m_{\lambda,\mu} < 0$. □

LEMMA 6.3.10. *Assume that* (6.1.5)–(6.1.17) *hold. Then there exist* $\delta_2 > 0$ *and* $\varepsilon_2 > 0$ *such that for any* $\lambda \in (\lambda_1, \lambda_1 + \delta_2)$ *and* $\mu \in (\mu_1, \mu_1 + \varepsilon_2)$ *problem* (6.3.11) *has a nontrivial solution* $(z_2, w_2) \in Y$ *satisfying* $E_1(z_2) + E_2(w_2) < 0$.

PROOF. The proof is by contradiction and it is analogous to that of Lemma 6.3.8.

Assume that the opposite assertion holds. Then there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that problem (6.3.12) with $\lambda = \lambda^s = \lambda_1 + \delta_s$ and $\mu = \mu^s = \mu_1 + \varepsilon_s$ does not have a solution. Fix an integer s and consider (6.3.12) with λ^s and μ^s. Denote by (z_n^s, w_n^s) the corresponding maximizing sequence:

$$\int_\Omega c(x)\left|z_n^s\right|^{\alpha+1}\left|w_n^s\right|^{\beta+1}\,dx = -1,$$

$$\int_\Omega \left|\nabla z_n^s\right|^p dx - \lambda^s\int_\Omega a(x)\left|z_n^s\right|^p dx + \int_\Omega \left|\nabla w_n^s\right|^q dx - \mu^s\int_\Omega b(x)\left|w_n^s\right|^q dx$$
$$\to m_{\lambda^s,\mu^s} < 0.$$

If (z_n^s, w_n^s) is bounded, we can obtain as before that there exists a solution (z_0^s, w_0^s) of (6.3.12):

$$\int_\Omega c(x)\left|z_0^s\right|^{\alpha+1}\left|w_0^s\right|^{\beta+1}dx = -1$$

and

$$\int_\Omega \left|\nabla z_0^s\right|^p dx - \lambda^s \int_\Omega a(x)\left|z_0^s\right|^p dx + \int_\Omega \left|\nabla w_0^s\right|^q dx - \mu^s \int_\Omega b(x)\left|w_0^s\right|^q dx$$
$$= m_{\lambda^s,\mu^s} < 0,$$

which is a contradiction. Thus we may assume that (z_n^s, w_n^s) is unbounded. With the same notation as in Lemma 6.3.8, it follows that

$$\int_\Omega c(x)\left|h_n^s\right|^{\alpha+1}\left|t_n^s\right|^{\beta+1}dx = -\frac{1}{\|(z_n^s, w_n^s)\|^{\alpha+\beta+2}} \to 0.$$

Since the functional f_3 (see (6.1.15)) is lower weakly semicontinuous, we obtain

$$\int_\Omega c(x)\left|h_0^s\right|^{\alpha+1}\left|t_0^s\right|^{\beta+1}dx = 0. \tag{6.3.13}$$

Analogously to previous proofs, (6.3.13) enables us to conclude that

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx = 0.$$

This contradicts (6.1.17).

The fact that $E_1(z_2) + E_2(w_2) < 0$ follows from Lemma 6.3.9. This completes the proof. □

LEMMA 6.3.11. *Let* (6.1.5)–(6.1.17) *hold. Then there exist* $\delta_3 > 0$ *and* $\varepsilon_3 > 0$ *such that for* $\lambda \in (\lambda_1, \lambda_1 + \delta_3)$ *and* $\mu \in (\mu_1, \mu_1 + \varepsilon_3)$ *problem* (6.3.5) *has another nontrivial solution* $(z_3, w_3) \in Y$.

PROOF. Set

$$N_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \;\middle|\; E_1(z) + E_2(w) = 1\right\} \tag{6.3.14}$$

and

$$\hat{N}_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \;\middle|\; E_1(z) + E_2(w) \leqslant 1\right\}. \tag{6.3.15}$$

Following the argument of Lemma 6.3.7, it is easy to prove that problems (6.3.14) and (6.3.15) are equivalent (see the end of the proof of Proposition 6.3.4). Therefore, we shall deduce the existence of $\delta_3 > 0$ and $\varepsilon_3 > 0$ corresponding to problem (6.3.15). Suppose that this is not true, that is, there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that problem (6.3.15) with $\lambda = \lambda^s = \lambda_1 + \delta_s$ and $\mu = \mu^s = \mu_1 + \varepsilon_s$ does not have a solution. Fix an integer s and consider (6.3.15) with λ^s and μ^s. Denoting by (z_n^s, w_n^s) the corresponding maximizing sequence, we have

$$\lim_{n\to\infty} \int_\Omega c(x)\left|z_n^s\right|^{\alpha+1}\left|w_n^s\right|^{\beta+1} dx = \hat{N}_{\lambda^s,\mu^s} > 0,$$

$$E_1\left(z_n^s\right) + E_2\left(w_n^s\right) \leqslant 1.$$

If (z_n^s, w_n^s) is bounded, we may assume that it converges weakly in Y to some (z_0^s, w_0^s) as $n \to \infty$. Then

$$\int_\Omega c(x)\left|z_n^s\right|^{\alpha+1}\left|w_n^s\right|^{\beta+1} dx \to \int_\Omega c(x)\left|z_0^s\right|^{\alpha+1}\left|w_0^s\right|^{\beta+1} dx = \hat{N}_{\lambda^s,\mu^s} > 0,$$
$$E_1\left(z_0^s\right) + E_2\left(w_0^s\right) \leqslant 1.$$

Therefore (z_0^s, w_0^s) is a solution of (6.3.15)—a contradiction. Thus we may consider (z_n^s, w_n^s) to be unbounded. Let

$$h_n^s = \frac{z_n^s}{\|(z_n^s, w_n^s)\|^{1/p}}, \qquad t_n^s = \frac{w_n^s}{\|(z_n^s, w_n^s)\|^{1/q}}, \qquad \left\|\left(h_n^s, t_n^s\right)\right\| = 1.$$

Thus we may assume that

$$\lim_{n\to\infty}\left(h_n^s, t_n^s\right) = \left(h_0^s, t_0^s\right)$$

weakly in Y. Then

$$\int_\Omega c(x)\left|z_n^s\right|^{\alpha+1}\left|w_n^s\right|^{\beta+1} dx = \left\|\left(z_n^s, w_n^s\right)\right\|^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_\Omega c(x)\left|h_n^s\right|^{\alpha+1}\left|t_n^s\right|^{\beta+1} dx$$
$$\to \hat{N}_{\lambda^s,\mu^s} > 0,$$

therefore

$$\int_\Omega c(x)\left|h_0^s\right|^{\alpha+1}\left|t_0^s\right|^{\beta+1} dx \geqslant 0. \tag{6.3.16}$$

From the inequality $E_1(z_n^s) + E_2(w_n^s) \leqslant 1$, that is,

$$\|(z_n^s, w_n^s)\| \biggl[\biggl(\|h_n^s\|_p^p - \lambda^s \int_\Omega a(x)|h_n^s|^p\,dx\biggr) + \biggl(\|t_n^s\|_q^q - \mu^s \int_\Omega b(x)|t_n^s|^q\,dx\biggr)\biggr] \leqslant 1,$$

it follows that

$$\|h_n^s\|_p^p - \lambda^s \int_\Omega a(x)|h_n^s|^p\,dx + \|t_n^s\|_q^q - \mu^s \int_\Omega b(x)|t_n^s|^q\,dx \leqslant \frac{1}{\|(z_n^s, w_n^s)\|}. \tag{6.3.17}$$

By letting $n \to \infty$ we get

$$\biggl(\|h_0^s\|_p^p - \lambda^s \int_\Omega a(x)|h_0^s|^p\,dx\biggr) + \biggl(\|t_0^s\|_q^q - \mu^s \int_\Omega b(x)|t_0^s|^q\,dx\biggr) \leqslant 0. \tag{6.3.18}$$

On the other hand, from (6.3.17) we can obtain that

$$\lambda^s \int_\Omega a(x)|h_0^s|^p\,dx + \mu^s \int_\Omega b(x)|t_0^s|^q\,dx \geqslant 1. \tag{6.3.19}$$

Clearly $\|(h_0^s, t_0^s)\| \leqslant 1$. This allows us to suppose that (h_0^s, t_0^s) converges weakly in Y to some (h_0, t_0). Letting $s \to \infty$ in (6.3.19), it follows that

$$\lambda_1 \int_\Omega a(x)|h_0|^p\,dx + \mu_1 \int_\Omega b(x)|t_0|^q\,dx \geqslant 1.$$

Hence $(h_0, t_0) \not\equiv (0, 0)$.

Now, from (6.3.18), by letting $s \to \infty$, we infer

$$\biggl(\|h_0\|_p^p - \lambda_1 \int_\Omega a(x)|h_0|^p\,dx\biggr) + \biggl(\|t_0\|_q^q - \mu_1 \int_\Omega b(x)|t_0|^q\,dx\biggr) \leqslant 0.$$

By the definitions of λ_1 and μ_1, both summands above are nonnegative. Therefore,

$$\|h_0\|_p^p - \lambda_1 \int_\Omega a(x)|h_0|^p\,dx = 0$$

and

$$\|t_0\|_q^q - \mu_1 \int_\Omega b(x)|t_0|^q\,dx = 0.$$

The last two equalities and Lemma 6.1.1 imply that $h_0 = l\varphi$, $l \neq 0$, and $t_0 = k\psi$, $k \neq 0$. Then by (6.3.16), letting $s \to \infty$, we get that

$$\int_\Omega c(x)|h_0|^{\alpha+1}|t_0|^{\beta+1}\,dx \geqslant 0,$$

and thus

$$|l|^{\alpha+1}|k|^{\beta+1}\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}\,dx \geqslant 0,$$

a contradiction to (6.1.17). This completes the proof. □

PROOF OF THEOREM 6.3.6. Let δ_1, ε_1, $(z_1, w_1) \in Y$, δ_2, ε_2, $(z_2, w_2) \in Y$, and δ_3, ε_3, $(z_3, w_3) \in Y$ be as in Lemmas 6.3.8, 6.3.10, and 6.3.11 respectively. Denote $\delta = \min(\delta_1, \delta_2, \delta_3)$ and $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Now we substitute (z_i, w_i), $i = 1, 2, 3$, into (6.2.12) and (6.2.13). In this way we obtain three pairs of positive numbers: (r_i, ρ_i), $i = 1, 2, 3$. Set

$$u_i = r_i z_i, \quad v_i = \rho_i w_i, \quad i = 1, 2, 3.$$

By Lemma 6.3.4, (u_1, v_1), (u_2, v_2), and (u_3, v_3) are weak solutions of (6.0.1) and (6.0.2). By Lemma 6.3.8 it follows that

$$\frac{E_1(u_1)}{r_1^p} = E_1(z_1) = 1$$

and

$$\frac{E_2(v_1)}{\rho_1^q} = E_2(w_1) = 1.$$

Thus

$$(u_1, v_1) \in S = \left\{(u, v) \;\middle|\; \frac{E_1(u_1)}{r_1^p} = 1 \text{ and } \frac{E_2(v_1)}{\rho_1^q} = 1\right\}.$$

On the other hand, by Lemma 6.3.10 we have

$$\frac{E_1(u_2)}{|r_2|^p} + \frac{E_2(v_2)}{|\rho_2|^q} = E_1(z_2) + E_2(w_2) < 0.$$

Hence at least one of $E_1(u_2)$ and $E_2(v_2)$ is negative. Therefore (u_2, v_2) does not belong to S. We conclude that (u_1, v_1) and (u_2, v_2) are distinct. Similarly (u_2, v_2) and (u_3, v_3) are distinct. An argument analogous to that in the proof of Theorem 6.3.1 shows that (u_1, v_1) and (u_3, v_3) are distinct too. The rest of the proof is the same as that of Theorem 6.3.1. This completes the proof of Theorem 6.3.6. □

6.4. *Nonexistence results for classical solutions*

In this subsection we shall comment on nonexistence results for classical solutions of a potential system associated to (p,q)-Laplacian operators. However, as in Subsection 5.5, it is clear that the assumption "the considered solutions are classical" does not seem to be a natural hypothesis for this kind of problem. Indeed, the context of this book suggests that the natural class to consider should be the class of *weak* solutions.

Our argument, which is based on a variational identity [41] (see also [26,52]), enables only to consider classical solutions. We should mention that Guedda and Véron [26] proved a variational identity for *weak* solutions of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

under some suitable growth assumptions on f. We are confident that a variational identity for weak solutions of potential systems associated to p-Laplacian operators still holds if the potential does not grow very fast. However, in the present chapter we shall not consider this kind of generalization.

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Consider the following quasilinear potential system:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q-2}\nabla v) = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.4.1}$$

where $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R})$. Let $(u,v) \in (C^2(\Omega) \cap C^0(\overline{\Omega}))^2$ be a classical solution of (6.4.1). Then the variational identity [41] for (6.4.1) can be written in the form

$$\begin{aligned} &\frac{N-p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{N-q}{q}\int_\Omega |\nabla v|^q\,dx \\ &\quad - N\int_\Omega F(x,u,v)\,dx - \int_\Omega D_x F(x,u,v)\,dx \\ &\quad = -\left(1-\frac{1}{p}\right)\int_{\partial\Omega} |\nabla u|^p(x,\nu)\,d\sigma - \left(1-\frac{1}{q}\right)\int_{\partial\Omega} |\nabla v|^q(x,\nu)\,d\sigma. \end{aligned} \tag{6.4.2}$$

Now we are ready to prove the next

THEOREM 6.4.1. *Suppose that Ω is strictly star-shaped with respect to the origin. Let $a,b,c \in C^1(\overline{\Omega})$ and $(u,v) \in (C^2(\Omega) \cap C^0(\overline{\Omega}))^2$ be a solution of* (6.0.1) *and* (6.0.2). *Suppose that the assumptions of Subsection* 6.1 *hold. In addition, assume that for any $\gamma, \sigma \in \mathbb{R}$ the following inequalities are valid*:

$$\frac{N-p}{p} + \gamma \geqslant 0,$$

$$\frac{N-q}{q}+\sigma \geqslant 0,$$

and for $x \in \Omega$ we have

$$\left(\frac{\lambda N}{p}-\gamma\lambda\right)a(x)-\frac{\lambda}{p}\big(\nabla a(x),x\big)\geqslant 0,$$
$$\left(\frac{\mu N}{q}-\sigma\lambda\right)b(x)-\frac{\mu}{q}\big(\nabla b(x),x\big)\geqslant 0,$$
$$-Nc(x)-N\big(\nabla c(x),x\big)-\big((\alpha+1)\gamma+(\beta+1)\sigma\big)c(x)\geqslant 0.$$

Then $u=v=0$ in Ω.

PROOF. Multiplying the first equation of (6.0.1) by γu and integrating by parts, we get

$$\gamma\int_\Omega |\nabla u|^p\,dx=\gamma\lambda\int_\Omega a(x)|u|^p\,dx+\gamma(\alpha+1)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx. \tag{6.4.3}$$

Similarly

$$\sigma\int_\Omega |\nabla v|^q\,dx=\sigma\mu\int_\Omega b(x)|v|^q\,dx+\sigma(\beta+1)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx. \tag{6.4.4}$$

Now we recall that the potential F is given by (6.2.1). Then substitute (6.2.1) into (6.4.1). Further, sum the obtained identity up with (6.4.3) and (6.4.4). Then the resulting identity, the inequalities given in the theorem, and the fact that Ω is strictly star-shaped imply that $u=v=0$ in Ω. □

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CHAPTER 3

Superlinear Elliptic Equations and Systems

Bernhard Ruf

Dipartimento de Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano, Italy
E-mail: *ruf@mat.unimi.it*

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 5
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1. Introduction

In this article we survey some recent results on superlinear elliptic equations and systems. A particular focus will be the borderline situations of so-called *critical growth*. In the existence theorems, we will use mostly variational methods, that is we look for critical points of functionals associated to the equations and systems. In dealing with variational problems, the choice of the function spaces in which the functionals are defined is of essential importance. There are two competing factors which determine this choice: on the one hand, the space should be "sufficiently small" so that the functional under consideration has the desired regularity; and, on the other hand, the space should "not be too small" since otherwise the required compactness properties may get lost.

When working with scalar equations one is used that the above smallness and largeness requirements usually lead to a unique choice of (Sobolev) space, in which the problem is well posed and hence solvable. Indeed, the borderline situation (critical growth) may be defined as the limiting situation in which this space setup works. We will discuss the various phenomena connected with critical growth; for more details on this, cf. [26,42,46].

We then treat recent results on systems of superlinear equations. We will see that for the functionals associated to systems there is more freedom in the choice of the space; in fact, one may choose among a whole continuum of products of Sobolev spaces. Each choice yields different maximal growths for the respective nonlinearities, but again we find that for a fixed pair of such critical growth nonlinearities there exists a unique choice of a product Sobolev space. The pairs of critical growth nonlinearities form together the so-called "critical hyperbola".

We will then concentrate on some limiting cases of elliptic systems. Contrary to the situation in scalar equations and in (nonlimiting case) systems, we will find a wide range of (Sobolev) spaces available in which the corresponding functionals may be defined, and the question of the "right" functional setup becomes quite delicate. Indeed, we will see that in some limiting cases the various possible choices of Sobolev spaces yield, for the same functional, *different maximal growths*. We will then see that the more refined Sobolev–Lorentz spaces provide an "optimal" functional setup.

Much space will be devoted to the less widely known situation in dimension $N = 2$, where critical growth is of exponential type, given by the so-called Trudinger–Moser inequality. Working with systems in dimension $N = 2$, we will see that also here the Sobolev–Lorentz spaces yield the suitable functional setup in which the analogue of the critical hyperbola (involving nonlinearities of different exponential growths) can be found.

2. Elliptic equations

2.1. *Some history*

For studying the model elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is an bounded open domain and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, one may try to use the well-known *Dirichlet principle* which consists in minimizing an associated functional over a suitable set of functions; then, the corresponding *critical points* correspond by the *Euler–Lagrange principle* to (weak) solutions of problem (2.1). The functional associated to equation (2.1) takes the form

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx,$$

where $F(t) = \int_0^t f(s)\,dx$ is the primitive of $f(t)$.

If we for instance assume that

$$\bigl|f(s)\bigr| \leqslant M, \quad \forall s \in \mathbb{R},$$

and hence

$$\bigl|F(s)\bigr| \leqslant c + M|s|, \quad \forall s \in \mathbb{R},$$

we can estimate

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - M\int_\Omega |u|\,dx \\ &\geqslant \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - M|\Omega|^{1/2}\left(\int_\Omega |u|^2\,dx\right)^{1/2}. \end{aligned}$$

By the Poincaré inequality

$$\int_\Omega |\nabla u|^2 \geqslant d\int_\Omega |u|^2$$

we find that $I(u)$ is bounded from below, and hence is makes sense to look for the global minimum of this functional. It is clear that there exist functionals which are bounded below but which do *not attain* their minimum: consider, e.g., $j : \mathbb{R} \to \mathbb{R} : j(s) = e^s$; clearly, $\inf_{s\in\mathbb{R}} j(s) = 0$, and any *minimizing sequence* (s_n) satisfies $s_n \to -\infty$. The above functional $I(u)$ seems better behaved since we see easily that any minimizing sequence $(u_n) \subset H_0^1(\Omega)$ is in fact bounded: Setting $m = \inf_{u\in H_0^1(\Omega)} I(u_n)$, we have

$$m + 1 \geqslant I(u_n) \geqslant \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - M|\Omega|^{1/2} d\left(\int_\Omega |\nabla u_n|^2\,dx\right)^{1/2}$$

and hence

$$c \geqslant \left(\int_\Omega |\nabla u_n|^2\,dx\right)^{1/2} = \|u_n\|_{H_0^1}.$$

In finite dimensions this would lead immediately to a convergent subsequence and to the conclusion that the minimum is attained. However, in the infinite-dimensional space $H_0^1(\Omega)$ the situation is more delicate. Indeed, in Dirichlet's time, and also much later, this question was not rigorously posed, and it was often tacitly assumed that the minima for functionals of the form $I(u)$ are attained, without questioning by what kind of function. It took the famous Weierstrass example, namely

$$J(u) = \int_0^1 x^2 |u'|^2 \, dx, \qquad u : [0,1] \to \mathbb{R}, \qquad u(0) = 0, \qquad u(1) = 1$$

to change things. Minimizing $J(u)$ for instance over the (natural) class $E = \{u \in C^1([0,1], \mathbb{R});\ u(0) = 0, u(1) = 1\}$, one sees easily that $\inf J(u) = 0$, but that 0 cannot be attained by a C^1-function. This threw the field of *Calculus of Variations* (and in fact all of *Analysis*) into a crisis; but the crisis was overcome by the efforts of Weierstrass himself, by Arzelà, Fréchet, Hilbert, Lebesgue and others, leading to the foundation of modern analysis.

2.2. *My space or yours?*

In today's words, the upshot from this crisis regarding the *Dirichlet principle* is precisely the question over what class (or space) of functions the minimization should be taken. Indeed, there is a large variety of spaces available, the spaces of continuous and differentiable functions, the more refined Hölder spaces, the more general Lebesgue and Sobolev spaces, and (as we will see later on) generalizations of these, the Orlicz spaces and Lorentz spaces. So the actual choice of the space to work in seems somewhat arbitrary—and only restricted by the expectation that the "outcome" should be (essentially) the same, and not really depend on the choice of the space. This apparent ambiguity in the choice of the space may be hard to understand for people working in other fields—and led even to the somewhat derogatory saying: "if the space matters, then it does not matter ..."

There are two competing requirements which intervene in the choice of the space: the functional must be continuous and differentiable, and the functional must possess a suitable compactness; for the first requirement, the space should be "small", i.e. the topology must be sufficiently fine. Indeed, if we take the "small" space of differentiable C^1-functions, then the functional of the form

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(u) \, dx$$

is certainly defined and continuous in u; however, due to the incompleteness of this space with respect to the *Dirichlet-norm* $\|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$, compactness fails to hold, and it is in general impossible to prove directly that $\inf_{C^1(\Omega)} I(u)$ is attained. On the other hand, for the compactness requirement, the space should be sufficiently "large", i.e. the coarser the topology the better.

Indeed, by Sobolev's work we know that a "good" space to work with is the Sobolev space $H_0^1(\Omega)$, i.e. the space of functions whose (weak) derivative belongs to the space $L^2(\Omega)$. By the very definition of this space, the Dirichlet integral is well defined and continuous on this space; in fact, it is the largest space on which this is the case. Now, we need to check that also the second part of the functional $I(u)$, i.e. $\int_\Omega F(u)\,dx$, is well defined on $H_0^1(\Omega)$. To obtain this, one needs to impose a *growth condition* on $f(s)$, namely

$$\left|f(s)\right| \leqslant c + c|s|^{\frac{N+2}{N-2}}, \quad s \in \mathbb{R}.$$

This implies that $F(s) = \int_0^s f(t)\,dt$ satisfies the restriction

$$\left|F(s)\right| \leqslant c + c|s|^{2^*}, \quad s \in \mathbb{R},\ 2^* := \frac{2N}{N-2},\ N \geqslant 3.$$

And then, by the famous Sobolev embedding theorem $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, one concludes that indeed the second term of the functional $I(u)$ is well defined on $H_0^1(\Omega)$, and that the functional is continuous and differentiable.

2.3. *Compactness*

Let us now return to the question raised above, namely whether it is true that "bounded minimizing sequences contain a convergent subsequence". In the context of general critical point theory, this is known today as the *Palais–Smale property*, and goes back to a famous work [34] of these two authors where they study a generalized Morse theory.

Indeed, if we impose the stronger growth condition (so-called subcritical growth):

$$\left|f(s)\right| \leqslant c_1 + c_2|s|^p, \quad \text{for some } 1 < p < \frac{N+2}{N-2},$$

then we have by Rellich's theorem a compact embedding $H_0^1(\Omega \subset L^{p+1}(\Omega)$, and consequently the Palais–Smale property is often (that is, under suitable "technical" conditions) satisfied.

On the other hand, if we consider the model problem[1] with "critical growth"

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u(0) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

then interesting phenomena appear: first of all, by the famous *Pohozaev identity* [35], one proves that equation (2.2) has *no nontrivial solution* if Ω is starshaped. Thus, one might argue that the problem is completely solved:

– the natural choice of the space $H_0^1(\Omega)$ yields compactness and hence solvability for subcritical growth (defined precisely by this choice of the space);
– on the other hand, the Palais–Smale property fails and solvability is lost at the limiting "critical growth".

[1]To simplify notation, we will write throughout the text: $s^p := |s|^{p-1}s$.

But due to the continued interest in equations of form (2.2), mainly because of their importance in geometry, the studies continued, and many surprising and fascinating phenomena were discovered.

2.4. *Critical growth*

In this section we discuss some of the main phenomena of critical growth:

2.4.1. *Loss of compactness.* The "loss of compactness" derives from the noncompactness of the limiting case of the Sobolev embedding

$$H_0^1(\Omega) \subset L^{2^*}(\Omega), \quad 2^* = \frac{2N}{N-2}, \ N \geqslant 3.$$

To see that this embedding is not compact, it suffices to find a bounded sequence $(u_n) \subset H_0^1(\Omega)$ which does not admit a convergent subsequence in $L^{2^*}(\Omega)$. Such a sequence can be easily constructed: Choose a ball of radius a such that $B_a(x_0) \subset \Omega$. Clearly we may assume that $x_0 = 0$. Let

$$u_n(x) = u_n\big(|x|\big) = u_n(r) = \begin{cases} \frac{1}{n^{\frac{N-2}{2}}} \frac{1}{r^{N-2}} - d_n, & \frac{1}{n} \leqslant r \leqslant a, \\ n^{\frac{N-2}{2}} - d_n, & 0 \leqslant r \leqslant \frac{1}{n}, \end{cases}$$

where $d_n = \frac{1}{n^{\frac{N-2}{2}} a^{N-2}}$, and $u_n(x) = 0$ for $x \in \Omega \setminus B_a(0)$.

A direct calculation shows that the Dirichlet norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ is bounded: indeed

$$\int_\Omega |\nabla u_n|^2\,dx = \omega_{N-1} \frac{(N-2)^2}{n^{N-2}} \int_{1/n}^a r^{2-2N} r^{N-1}\,dr = C_N\left(1 - \frac{c(a)}{n^{N-2}}\right) \leqslant C_N,$$

where ω_{N-1} denotes the surface area of the unit sphere in $\mathbb{R}^N$.

Furthermore, we observe that pointwise $u_n(r) \to 0$, for all $0 < r < a$, as $n \to \infty$. But (u_n) cannot have a subsequence which converges to 0 in $L^{2^*}(\Omega)$, since

$$\int_\Omega |u_n|^{2^*}\,dx \geqslant \omega_{N-1} \int_0^{1/n} \big(n^{\frac{N-2}{2}} - d_n\big)^{\frac{2N}{N-2}} r^{N-1}\,dr \geqslant c_1 > 0, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

From the simple sequence above, we may make another very important observation. Looking at the "principal term" of u_n, namely,

$$\tilde{u}_n(r) := \frac{1}{n^{\frac{N-2}{2}}} \frac{1}{r^{N-2}} = n^{\frac{N-2}{2}} \frac{1}{(nr)^{N-2}}$$

we observe that

$$\tilde{u}_n(r) = n^{\frac{N-2}{2}} \tilde{u}_1(nr),$$

and from the simple calculations above we see that $\int_\Omega |\nabla \tilde{u}_1|^2\,dx$ as well as $\int_\Omega |\tilde{u}_1|^{2^*}\,dx$ remain *invariant* under this "group action". This is in fact true in general, i.e. in the limiting Sobolev embedding we have a

2.4.2. *Group invariance.* For $u \in H_0^1(\Omega)$, define the continuous group action or "rescaling"

$$u_\lambda(x) := \lambda^{\frac{N-2}{2}} u(\lambda x);$$

this is defined for all $\lambda > 0$ if $\Omega = \mathbb{R}^N$, and for $0 < \lambda \leqslant 1$ if Ω is starshaped (w.r.t. the origin). One then shows by direct calculation that

$$\int_\Omega |\nabla u_\lambda|^2\,dx = c, \qquad \int_\Omega |u_\lambda|^{2^*}\,dx = d, \quad \forall \lambda > 0.$$

It is in fact the appearance of this invariance under rescaling which is the deeper reason for the loss of compactness.

2.4.3. *Nonexistence of solutions in bounded starshaped domains.* With the loss of compactness, one loses the main instrument to prove existence of a solution to equation (2.2). And in fact, one may show that if Ω is a bounded starshaped domain, then indeed there does not exist a (nontrivial) solution.

This is due to the famous *Pohozaev identity*, see [35]. This identity is obtained from equation (2.2) by multiplication by $x \cdot \nabla u$ and integration, and it says that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (2.2), then the following relation holds:

$$\frac{N-2}{2} \int_\Omega |\nabla u|^2\,dx - \frac{N}{2^*} \int_\Omega |u|^{2^*}\,dx + \frac{1}{2} \int_{\partial\Omega} |\partial_\nu u|^2\, x \cdot \nu\,d\sigma = 0,$$

where ν is the exterior normal to $\partial\Omega$. On the other hand, multiplying (2.2) by u and integration yields

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega |u|^{2^*}\,dx.$$

From these two identities follows that

$$\int_{\partial\Omega} |\partial_\nu u|^2 x \cdot \nu\,d\sigma = 0.$$

Since Ω is starshaped, we have $x \cdot \nu > 0$, for all $x \in \partial\Omega$, and hence $\partial_\nu u = 0$ on $\partial\Omega$. But this implies that $u \equiv 0$ by the principle of unique continuation.

The situation changes if we consider the problem on the whole of $\mathbb{R}^N$. In the case with critical growth we have:

2.4.4. *Existence of explicit solutions on $\mathbb{R}^n$: instantons.* The explicit solutions of the equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u(0) \to 0 & \text{for } |x| \to 0 \end{cases} \tag{2.4}$$

are

$$u_\lambda(x) = \big(N(N-2)\big)^{\frac{N-2}{4}} \frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x|^2)^{\frac{N-2}{2}}}, \quad \lambda > 0. \tag{2.5}$$

They were found independently by G. Talenti [43] and Th. Aubin [6]. Note that (2.5) represent a *family of solutions*, parametrized by $\lambda > 0$. This reflects again the group (or scaling) invariance of the equation. In addition, there is also (an obvious) invariance of the equation by translation. One knows that, up to the rescaling and translations, the above solutions are the only solutions of equation (2.4). The solutions are characterized by

$$S_N\big(\mathbb{R}^N\big) = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2^*} \, dx)^{2/2^*}}. \tag{2.6}$$

This expression characterizes the *best Sobolev embedding constant* for the embedding $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, and thus this constant is *attained* by the function (2.5), and it can be explicitly calculated. On the other hand, by the Pohozaev identity we infer that the best Sobolev embedding constant $S_N(\Omega)$ cannot be attained if Ω is starshaped: otherwise, we would obtain a nontrivial solution to equation (2.4) which is impossible. In fact, by the unique continuation property, one shows that this constant is never attained if $\Omega \neq \mathbb{R}^N$.

We have already seen that there is a loss of compactness for the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, for bounded domains $\Omega \subset \mathbb{R}^n$, by giving an explicit bounded sequence in $H_0^1(\Omega)$ which does not have a convergent subsequence in $L^{2^*}(\Omega)$. Using the instantons (2.5), it is easy to obtain a minimizing sequence for

$$S_N(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^{2^*} \, dx)^{2/2^*}}.$$

First note that evidently

$$S_N(\Omega) \geqslant S_N(\mathbb{R}^n) =: S_N,$$

since every function in $H_0^1(\Omega)$ may be extended by 0 to an H^1-function on $\mathbb{R}^n$. On the other hand, taking

$$\tilde{u}_\lambda(x) = \eta(x) u_\lambda(x), \tag{2.7}$$

where $\eta \in C_0^\infty(\Omega)$ is a cut-off function, i.e. $\eta \equiv 1$ on a neighborhood $B_\rho(0) \subset \Omega$. One then estimates that for $\lambda \to 0^+$

$$\int_\Omega |\nabla \tilde{u}_\lambda|^2\,dx = \int_\Omega |\nabla u_\lambda|^2 + O\big(\lambda^{N-2}\big) = S_N^{N/2} + O\big(\lambda^{N-2}\big),$$
$$\int_\Omega |\tilde{u}_\lambda|^{2^*}\,dx = \int_\Omega |u_\lambda|^{2^*}\,dx + O\big(\lambda^N\big) = S_N^{N/2} + O\big(\lambda^N\big) \tag{2.8}$$

and thus

$$S_N(\Omega) \leqslant \frac{\int_\Omega |\nabla \tilde{u}_\lambda|^2\,dx}{(\int_\Omega |\tilde{u}_\lambda|^{2^*}\,dx)^{\frac{2}{2^*}}} = \frac{S_N^{N/2}}{S_N^{N/N^*}} + O\big(\lambda^{N-2}\big) = S_N + O\big(\lambda^{N-2}\big).$$

From this we conclude that for any domain $\Omega \subset \mathbb{R}^N$

$$S_N(\Omega) = S_N, \tag{2.9}$$

and that $(\tilde{u}_\lambda)$ is a minimizing sequence for S_N in $H_0^1(\Omega)$ which is noncompact in $L^{2^*}(\Omega)$.

As already mentioned, the above stated results on the equations with subcritical and critical growth seem to imply that we have a complete result: that is

- compactness and existence of nontrivial solutions in the subcritical case in bounded domains;
- loss of compactness due to the appearance of a group invariance, and loss of solutions (in starshaped domains) in the critical case.

Indeed, as pointed out by H. Brezis, this seemingly complete result blocked further research for many years—until H. Brezis and L. Nirenberg published their groundbreaking result, see [9]:

2.4.5. *The Brezis–Nirenberg result.* As mentioned above, the equation with "pure critical growth" (2.2) has no nontrivial solution if Ω is bounded and starshaped. From the variational point of view, this is due to the lack of compactness, caused by the concentration phenomenon. The crucial observation by H. Brezis and L. Nirenberg is that this concentration is the only way in which compactness can be lost. And if compactness is lost due to concentration, then this happens at precise energy levels (the energy of the concentrating instantons). Brezis and Nirenberg consider in [9] an equation with critical growth and with a lower order perturbation, and then search for solutions by minimizing a suitable constrained energy functional. They then calculate the lowest "level of noncompactness", i.e. the limit value of the functional along the concentrating instantons. Finally, they show that the actual minimum value of the functional is below this "value of noncompactness", and conclude that hence the minimum is attained.

More precisely, they consider, for $0 < \lambda < \lambda_1$, the equation

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{n+2}{n-2}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ u > 0, & \text{on } \Omega, \end{cases} \tag{2.10}$$

and look for solutions by considering the minimization problem

$$m := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 - \lambda |u|^2)\, dx}{(\int_\Omega |u|^{2^*}\, dx)^{2/2^*}}.$$

The only noncompactness level for this minimization is again the level S_N, since along the concentrating instantons $\int_\Omega |\tilde{u}_\lambda|^2 \to 0$, for $\lambda \to 0$. We now show that $m < S_N$. For this we do a more precise estimate: indeed, one can estimate explicitly that (for $N \geqslant 5$):

$$\int_\Omega |\tilde{u}_\lambda|^2\, dx \geqslant c\lambda^2, \quad \text{for some } c > 0, \text{ and } \lambda > 0 \text{ small.}$$

Thus we get, using the sequence $(\tilde{u}_\lambda)$ and the estimates (2.8)

$$m \leqslant S_N + O\big(\lambda^{N-2}\big) - c\lambda^2 < S_N \quad (N \geqslant 5),$$

for $\lambda > 0$ sufficiently small. Thus we have confirmed that the infimum m lies below the noncompactness level S_N, and hence it is attained! (Similar estimates hold for $N = 4$, and with some restrictions for $N = 3$.)

This result is by now classical, and has had an enormous influence on the research of the last 25 years.

3. The case of dimension $N = 2$

The case of dimension $N = 2$ is special, since the corresponding Sobolev space $H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^2$, is a borderline case for Sobolev embeddings: one has

$$H_0^1(\Omega) \subset L^p(\Omega), \quad \text{for all } \ p \geqslant 1,$$

but

$$H_0^1(\Omega) \nsubseteq L^\infty(\Omega);$$

indeed, the function $\lg(x/a)$ belongs to $H_0^1(B_a(0))$, $B_a \subset \Omega$, but not to $L^\infty(\Omega)$.

So, one is lead to ask if there is another kind of *maximal growth* in this situation. And indeed, this is the result of Pohozaev [36] and Trudinger [44], and which is now called the *Trudinger inequality*: it says that if $\Omega \subset \mathbb{R}^2$ is a bounded domain, then

$$u \in H_0^1(\Omega) \quad \Longrightarrow \quad \int_\Omega e^{|u|^2}\, dx \leqslant c. \tag{3.1}$$

We can express this fact alternatively in terms of an embedding. For this we need to introduce the notion of Orlicz space which generalize the L^p-spaces. Let $\varphi(t) = e^{t^2} - 1$. This is a so-called N-function (see Section 5 below, where Orlicz spaces will be discussed in more detail). Let $K_M = \{u : \Omega \to \mathbb{R},\ \int_\Omega \varphi(u) < +\infty\}$. The Orlicz space L_φ is the linear vector space generated by K_M. For more details we refer to Section 5. The result of Pohozaev and Trudinger now says that one has a continuous embedding

$$H_0^1(\Omega) \subset L_\varphi(\Omega), \quad \text{for } \Omega \subset \mathbb{R}^2 \text{ bounded}$$

and

$$H_0^1(\Omega) \subset L_\psi(\Omega) \quad \text{compact, for any } \psi \prec\prec \varphi, \tag{3.2}$$

where $\psi \prec\prec \varphi$ means that ψ increases essentially more slowly than φ, see Section 5.

Inequality (3.1) was improved and made precise by J. Moser [33] who proved that:

$$\sup_{\|\nabla u\|_{L^2} \leqslant 1} \int_\Omega e^{\alpha u^2}\,dx \begin{cases} \leqslant c, & \text{if } \alpha \leqslant 4\pi, \\ = +\infty, & \text{if } \alpha > 4\pi. \end{cases} \tag{3.3}$$

One can now ask if the “critical growth” (3.1) produces similar phenomena for equation (2.1) as the case $N \geqslant 3$; indeed, one has many similarities, but also remarkable differences.

3.0.6. *Loss of compactness.* Similarly as in the case $N \geqslant 3$, we can give an explicit sequence (u_n) which is bounded in $H_0^1(\Omega)$, and such that (u_n) has no convergent subsequence in L_φ. For simplicity, assume that $\Omega = B_1(0)$, the unit ball. Let

$$w_n = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if } 0 \leqslant |x| \leqslant \frac{1}{n}, \\ \frac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \text{if } \frac{1}{n} \leqslant |x| \leqslant 1. \end{cases}$$

One checks easily that $\int_{B_1} |\nabla w_n|^2\,dx = 1$, and hence $w_n \rightharpoonup w$. Furthermore, one checks that $\int_{B_1} (e^{w_n^2} - 1)\,dx \to \pi$. On the other hand, we have pointwise $w_n(x) \to 0$, for any $x \neq 0$, and hence $w = 0$. From this one concludes that there cannot exist a subsequence with $\|w_{n_k} - w\|_{L_\varphi} \to 0$.

3.0.7. *Group invariance.* A fundamental difference to the case $N \geqslant 3$ is that no analogue of the group invariance in $N \geqslant 3$ is known for the case $N = 2$. Connected with this, also no identity of Pohozaev type is known for dimension $N = 2$, which could be important for obtaining nonexistence results.

Before discussing further the issue of existence and nonexistence, let us give a more precise notion of critical growth: We say that $f \in C(\mathbb{R})$ has *subcritical growth* if $f(t) \prec\prec e^{t^2} - 1$ (see (3.2) and Section 5), i.e. if

$$\lim_{|t| \to \infty} \frac{f(t)}{e^{\alpha t^2}} = 0, \quad \text{for every } \alpha > 0, \tag{3.4}$$

and $f(t)$ has *critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{e^{\alpha t^2}} = 0 \quad \text{if } \alpha > \alpha_0, \quad \text{and} \quad \lim_{|t|\to\infty} \frac{f(t)}{e^{\alpha t^2}} = +\infty \quad \text{if } 0 < \alpha < \alpha_0. \tag{3.5}$$

We first consider

3.1. *Subcritical growth*

Concerning subcritical growth, one has the following existence result for equation (2.1):

THEOREM 3.1. (See [3,14].) *Assume that* $f \in C(\mathbb{R})$ *satisfies*

(H1) *there exist constants* $t_0 > 0$ *and* $M > 0$ *such that*

$$0 < F(t) = \int_0^t f(s)\,ds \leqslant M\big|f(t)\big|, \quad \forall |t| \geqslant t_0,$$

(H2) $0 < F(t) \leqslant \frac{1}{2} f(t)t, \quad \forall t \in \mathbb{R} \setminus \{0\}.$

Then equation (2.1), *with* $\Omega \subset \mathbb{R}^2$ *a bounded domain, has a nontrivial solution. Moreover, if* $f(t)$ *is an odd function in* t, *then equation* (2.1) *has infinitely many solutions.*

PROOF. The proof follows (by now) standard lines: the assumptions guarantee that the functional has a mountain pass structure around the origin, cf. [4,38]. The subcritical growth yields compactness, cf. (3.2), and hence the critical level is attained. □

3.2. *Critical growth*

We consider now equation (2.1) with critical growth in the sense specified in (3.5) above. For the case $N = 2$ the situation is more complicated than for dimensions $N \geqslant 3$, and the known results are less complete. The difficulties start already with the fact that there is no natural "model problem" for the critical case. Thus, let us write the "critical" equation (with $\alpha_0 = 4\pi$, see (3.5)) in the form

$$-\Delta u = h(u)e^{4\pi u^2} = e^{4\pi u^2 + \log h(u)}, \quad \Omega; \qquad u = 0, \quad \partial\Omega,$$

where $h \in C(\mathbb{R})$ is subcritical, i.e. satisfying condition (3.4). Thus, $\log h(u)$ is a "lower order perturbation" of the principal growth term $4\pi u^2$, and in analogy to the Brezis–Nirenberg case we can ask for conditions on $h(t)$ such that we have again the situation of nonexistence or existence of solutions.

Related to the study of this question is the behavior of the supremum in (3.3). Indeed, it came as a surprise when L. Carleson and A. Chang [11] proved in 1986 that the supremum in (3.3) is attained on the unit ball in $\mathbb{R}^2$. In fact, consider for comparison the following maximization problem, for $N \geqslant 3$

$$\sup_{\|\nabla u\|_{L^2} \leqslant 1} \int_\Omega |u|^{2^*}\, dx = M. \tag{3.6}$$

This problem corresponds to the minimization problem (2.6), and by (2.9) it is clear that

$$M = \frac{1}{S_N^{\frac{N}{N-2}}}.$$

Furthermore, by the remarks made following (2.6), the supremum in (3.6) is never attained if $\Omega \neq \mathbb{R}^N$.

Thus, the result of Carleson and Chang (for $\Omega = B_1(0)$) is in striking contrast to the case $N \geqslant 3$. We remark that the result of Carleson and Chang was extended to arbitrary domains in $\mathbb{R}^2$ by M. Flucher [20].

Carleson and Chang prove their result by the following steps: Let $\{u_n\}$ be a maximizing sequence:

- by radial symmetrization, one sees that $\{u_n\}$ may be assumed radial, and thus characterized by an ODE (the radial equation);
- if the supremum is not attained, then the maximizing sequence is a "normalized concentrating sequence,", i.e. it tends weakly to 0 and concentrates in the origin;
- determine an explicit upper bound for any such normalized concentrating sequence $(u_n) \in H_0^1(B_1)$, namely

$$\int_{B_1(0)} e^{4\pi u_n^2} \leqslant (1+e)\pi;$$

- provide an explicit normalized function $w \in H_0^1(B_1)$ with

$$\int_{B_1} e^{4\pi w^2} > (1+e)\pi.$$

Clearly, $(1+e)\pi$ takes the rôle of the (highest) noncompactness level, analogous to the situation in $\mathbb{R} \geqslant 3$ described above, and since the supremum lies above this noncompactness level, it is attained.

In a recent paper by de Figueiredo, do Ó and the author [15] the following explicit normalized concentrating and *maximizing* sequence for $(1+e)\pi$ was constructed:

For $n \in \mathbb{N}$ set $\delta_n = \frac{2\log n}{n}$ and $A_n = \frac{1}{en^2} + O(\frac{1}{n^4})$; then define

$$y_n(|x|) = \begin{cases} (\frac{1-\delta_n}{n})^{1/2} \log \frac{1}{|x|}, & 1/n \leqslant |x| \leqslant 1, \\ \frac{1}{(n(1-\delta_n))^{1/2}} \log \frac{A_n+1}{A_n+n|x|} + (n(1-\delta_n))^{1/2}, & 0 \leqslant |x| \leqslant 1/n. \end{cases} \tag{3.7}$$

The constants A_n are chosen such that $\int_{B_1} |\nabla u_n|^2 \, dx = 1$. This sequence allows to give a *new proof* of the last step in the argument of Carleson–Chang (and also a generalization of their result): one shows that this sequence approaches the value $(1+e)\pi$ from above, i.e.

$$\int_{B_1} e^{4\pi u_n^2} \, dx > (1+e)\pi, \quad \text{for } n \text{ large.}$$

This is in complete analogy to the case of Brezis–Nirenberg, whereby the sequence (3.7) takes the rôle of the sequence $\tilde{u}_\lambda$, see (2.7).

PROBLEM. In view of this and the above remarks, it is of interest to consider

$$\sup_{u \in H_0^1(\Omega), \|u\|=1} \int_\Omega h(u) e^{4\pi u^2} = S$$

and give optimal (= sharp) conditions on the subcritical function $h(t)$ such that the supremum S is *attained*, respectively *not attained*.

3.3. *Critical growth: existence*

For the corresponding equation

$$\begin{cases} -\Delta u = h(u) \, e^{4\pi u^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

some progress has been made recently concerning the determination of an optimal subcritical function $h(t)$. We remark that concerning nonexistence results, a fundamental difference to the case $N \geqslant 3$ is that (up to now) no suitable identity of Pohozaev type is known for the case $N = 2$.

In [14] the following theorem was proved by de Figueiredo, Miyagaki and the author (see also Adimurthi [2]):

THEOREM 3.2. *Assume that* $h \in C(\mathbb{R})$ *and let* $f(s) = h(s)e^{4\pi s^2}$. *Assume that*

(H1) $f(0) = 0$.

(H2) *There exist constants* $s_0 > 0$ *and* $M > 0$ *such that*

$$0 < F(s) = \int_0^s f(r) \, dr \leqslant M \big| f(s) \big|, \quad \forall |s| \geqslant s_0.$$

(H3) $\quad 0 < F(s) \leqslant \dfrac{1}{2} f(s)s, \quad \forall s \in \mathbb{R} \setminus \{0\}.$

Furthermore, let d *denote the inner radius of* Ω, *i.e.*

$d :=$ *radius of the largest ball* $\subset \Omega$.

Then equation (3.8) *has a nontrivial solution provided that*

(H4) $$\lim_{|s|\to\infty} h(s)s = \beta > \frac{1}{2\pi d^2}.$$

The *proof* of this theorem follows closely the scheme by Brezis–Nirenberg mentioned above, that is

– determine (explicitly) the level of noncompactness;
– use an explicit concentrating sequence and the hypothesis on $h(t)$ to show that the min–max level is below this noncompactness level;
– thus, compactness is recovered and the existence of a solution follows.

The concentrating sequence used in the proof of this theorem is the so-called *Moser sequence* given by

$$w_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{1}{(\log n)^{1/2}} \log \frac{1}{|x|}, & \frac{1}{n} \leqslant |x| \leqslant 1, \\ (\log n)^{1/2}, & 0 \leqslant |x| \leqslant \frac{1}{n}. \end{cases}$$

We remark that this sequence is *not* an optimal concentrating sequence; in fact, one easily calculates that

$$\lim_{n\to\infty} \int_{B_1} e^{4\pi w_n^2} = 2\pi < (1+e)\pi.$$

We remark that the condition (H4) in Theorem 3.2 may be slightly improved to

$$\beta > \frac{1}{e\pi d^2}$$

by using the optimal maximizing sequence (3.7) mentioned above instead of the Moser sequence.

3.4. *Critical growth: nonexistence*

Concerning nonexistence, only a partial result is known; in the following theorem, the nonexistence of a positive radial solution on $\Omega = B_1(0)$ is proved under conditions comparable to those of Theorem 3.2.

THEOREM 3.3. (See de Figueiredo and Ruf [18].) *Let* $\Omega = B_1(0)$. *Suppose that* $h \in C^2(\mathbb{R})$, *and that there exist constants* $r_1 > 0$ *and* $\sigma > 0$ *such that for some constants* $K > 0$, $c > 0$:

(1) $h(r) = \frac{K}{r}$, *for* $r \geqslant r_1$;
(2) $0 \leqslant h(r) \leqslant cKr^{1+\sigma}$, *for* $0 \leqslant r \leqslant r_1$.

Then there exists a constant $K_0 > 0$ *such that for* $K < K_0$ *the equation*

$$\begin{cases} -\Delta u = h(u)e^{4\pi u^2} & \textit{in } B_1(0) \subset \mathbb{R}^2, \\ u = 0 & \textit{on } \partial B_1(0) \end{cases} \tag{3.9}$$

has no nontrivial radial solution.

We remark that by Gidas, Ni and Nirenberg [21] any positive solution of equation (3.9) is *radial*, and hence Theorem 3.3 says that equation (3.9) has *no positive solution*.

Comparing Theorems 3.2 and 3.3 one notes that Theorem 3.3 assures (under some "technical" conditions) the existence of a positive solution for $\Omega = B_1(0)$ provided that asymptotically

$$h(s) \sim \frac{\beta}{s}, \quad \text{with } \beta > \frac{1}{2\pi},$$

while Theorem 3.3 gives nonexistence for

$$h(s) = \frac{\beta}{s} \quad \text{for } s \text{ large, and } \beta > 0 \text{ sufficiently small.}$$

The *proof* of Theorem 3.3 uses techniques of the theory of ordinary differential equations, in particular the *shooting method*. More precisely, considering only the radial solutions on $\Omega = B_1(0)$, one can reduce equation (3.9) to the radial equation

$$\begin{cases} u_r r + \frac{1}{r} u_r + h(u)e^{4\pi u^2} = 0 & \text{in } (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{3.10}$$

Using the transformation $t = -2\log\frac{r}{2}$ and setting $y(t) = u(r)$ we obtain

$$\begin{cases} -y'' = h(y)e^{4\pi y^2 - t}, & \text{for } t > 2\log 2, \\ y(2\log 2) = 0, & y'(+\infty) = 0. \end{cases} \tag{3.11}$$

That is, we have transformed equation (3.10), which has a singularity in 0, to equation (3.11) on $(2\log 2, +\infty)$, thus transporting the singularity in 0 to $+\infty$. The shooting method consists now in considering solutions $y(t)$ of (3.11) with $y'(+\infty) = \gamma$, i.e. one shoots horizontally from infinity and tries to land at the point $2\log 2$. The estimates to achieve this are delicate and lengthy, and are a refinement of the work of Atkinson and Peletier [5].

We summarize: if we assume that the asymptotic condition in the existence Theorem 3.2 is optimal (at least on the unit ball B_1), then the major *open problem* may be stated as follows:

> *Find a good model equation* (*i.e. properties of* $h(u)$) *under which one may prove*: *existence of a nontrivial solution for* $\lim_{t\to\infty} h(t)t > \frac{1}{e\pi}$, *and nonexistence of a solution for* $\lim_{t\to\infty} h(t)t < \frac{1}{e\pi}$.

As already mentioned, what seems to be missing is a kind of *Pohozaev identity* to obtain a sharp nonexistence result.

4. Elliptic systems, $N \geqslant 3$

4.1. *Strongly indefinite functionals*

In this section we begin the discussion of elliptic systems of the following simple form

$$\begin{cases} -\Delta u = g(v), & \\ -\Delta v = f(u) & \text{in } \Omega \subset \mathbb{R}^N,\ N \geqslant 3, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and superlinear functions, i.e.

$$\frac{f(s)}{s} \to +\infty, \qquad \frac{g(s)}{s} \to +\infty, \quad \text{as } |s| \to \infty.$$

These systems are of so-called *Hamiltonian form*; indeed, we can define the Hamiltonian $H(u, v) = F(u) + G(v)$, where F and G are the primitives of f and g, respectively. Then we get the system: $-\Delta u = H_v(u, v)$ and $-\Delta v = H_u(u, v)$.

As in the scalar case, our first interest is to find the *maximal* or “critical” growth for the nonlinearities f and g.

We can employ the same procedure as for the scalar equation: Write down a functional for the system (such that critical points yield weak solutions), and then find the appropriate function space on which the functional is well defined. The functional we choose is the following:

$$I(u, v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega F(u)\, dx - \int_\Omega G(v)\, dx. \tag{4.2}$$

As a first attempt, we may define the functional on the space

$$E := H_0^1(\Omega) \times H_0^1(\Omega)$$

by estimating

$$\left| \int_\Omega \nabla u \nabla v \, dx \right| \leqslant \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}.$$

In order to have the functional well defined and continuous, we then obtain as in the scalar case the following growth conditions for the primitives $F(s) = \int_0^s g(t)\, dt$ and $G(s) = \int_0^s g(t)\, dt$:

$$F(s) \leqslant c_1 + d_1 |s|^{2^*}, \qquad G(s) \leqslant c_2 + d_2 |s|^{2^*}.$$

Suppose now that $(\bar{u}, \bar{v}) \in E$ is a critical point of $I(u, v)$; then we have

$$\int_\Omega \nabla\varphi\nabla\bar{v} + \int_\Omega \nabla\bar{u}\nabla\psi = \int_\Omega f(\bar{u})\varphi + \int_\Omega g(\bar{v})\psi, \quad \forall(\varphi, \psi) \in E.$$

Choosing in particular the directions $(\varphi, 0)$ and $(0, \psi)$, we obtain that $(\bar{u}, \bar{v})$ is a weak solution of system (4.1).

Note that the functional $I(u, v)$ has a more complicated structure than the functionals considered up to now: the quadratic term $\int_\Omega \nabla u\nabla v\,dx$ is *strongly indefinite* near the origin; indeed, it is positive respectively negative definite on infinite-dimensional subspaces of E. In recent years much research has been devoted to the study of such situations, we refer to [30], [28] and [7]. We will describe below, in a more general situation, a detailed approach for such problems.

4.2. *The critical hyperbola*

We have seen above that the "natural choice" of space $E = H_0^1(\Omega) \times H_0^1(\Omega)$ leads to the known Sobolev growth restriction for both nonlinearities $F(s)$ and $G(s)$.

However, by *a different choice* of the space E, this result can be considerably generalized. In independent works by Hulshof and van der Vorst [23] and de Figueiredo and Felmer [13] it was shown that one may have different maximal growths for F and G; more precisely, the condition for the two nonlinearities is given by

$$F(s) \leqslant c_1 + d_1|s|^{p+1}, \qquad G(s) \leqslant c_2 + d_2|s|^{q+1}$$

with $p + 1$ and $q + 1$ satisfying the condition

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}.$$

This is the so-called *critical hyperbola*. We will show in the next sections in some detail how the critical hyperbola arises. Indeed, we will see that it can be obtained in two quite different ways, first in a Hilbert space setting, working with *fractional Sobolev spaces* H^s, and then in a Banach space setting, working with $W^{1,\alpha}$*-spaces.*

It is interesting that this critical hyperbola has many of the features of the critical exponents, namely there is

- compactness below the critical hyperbola, i.e. for nonlinearities with exponential growths p and q with

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N},$$

  and in consequence, existence of solutions for such equations; see Section 5.4 below;
- loss of compactness and concentration phenomena for systems with critical growth, i.e. when the exponents lie on the critical hyperbola, see Section 8.2;

- nonexistence of solutions for the pure power case and on starshaped domains, due to a Pohozaev (or Rellich) type inequality; see Section 8.1;
- existence of instantons; however, in contrast to the scalar equation, these are not explicitly known, but it is possible to derive their asymptotic behavior, see Section 8.3;
- group invariance.

4.3. *The H^s-approach*

In this section we will use *fractional Sobolev spaces* H^s on which the functional $I(u, v)$ will be defined.

4.3.1. *Fractional Sobolev spaces and the functional setting.* To describe the idea of de Figueiredo and Felmer [13] and Hulshof and van der Vorst [23], we begin by defining *fractional* Sobolev spaces.

Consider the Laplacian as the operator

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

and $\{e_i\}_{i=1}^{\infty}$ a corresponding system of orthogonal and L^2-normalized eigenfunctions, with eigenvalues $\{\lambda_i\}$. Then, writing

$$u = \sum_{n=1}^{\infty} a_n e_n, \quad \text{with } a_n = \int_{\Omega} u e_n \, dx,$$

we set

$$E^s = \left\{ u \in L^2(\Omega) \colon \sum_{n=1}^{\infty} \lambda_n^s |a_n|^2 < \infty \right\}$$

and

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n e_n, \quad \forall u \in D(A^s) := E^s.$$

The spaces E^s are *fractional Sobolev spaces* with the inner product

$$(u, v)_s = \int_{\Omega} A^s u A^s v \, dx,$$

see Lions and Magenes [32].

In the next lemma we collect a few properties of the operators A^s and the spaces E^s.

LEMMA 4.1. *Let $s > 0$ and $t > 0$.*
(1) *$z \in E^s$ iff $A^s z \in L^2$, and $\|z\|_{E^s} = \|A^s z\|_{L^2}$.*
(2) *Let $z \in E^{s+t} = E^2 = H^2$; then $A^{s+t} z = A^s A^t z = A^t A^s z$.*

PROOF. (1) follows immediately from the definitions.

(2) We have

$$A^{s+t}z = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{(s+t)/2} e_i = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{s/2} \lambda_i^{t/2} e_i = A^s \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{t/2} e_i = A^s A^t z. \qquad \square$$

With these definitions, we now define the functional

$$\begin{aligned} &I : E^s \times E^t \to \mathbb{R}, \\ &I(u, v) = \int_\Omega A^s u A^t v - \int_\Omega \big(F(u) + G(v)\big)\, dx \end{aligned} \tag{4.3}$$

with s and t such that $s + t = 2$; loosely speaking, this means that we distribute the two derivatives given in the first term of the functional I differently on the variables u and v.

The first term of $I(u, v)$ is well defined on the space $E^s \times E^t$ by the estimate

$$\left| \int_\Omega A^s u\, A^t v\, dx \right| \leqslant \|A^s u\|_{L^2} \left\|A^t v\right\|_{L^2} = \|u\|_{E^s} \|v\|_{E^t}.$$

By the Sobolev embedding theorem we have continuous embeddings

$$E^s \subset L^{p+1}(\Omega), \quad \text{if } \frac{1}{p+1} \geqslant \frac{1}{2} - \frac{s}{N},$$

and these embeddings are compact if $\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}$, and similarly for the embedding

$$E^t \subset L^{q+1}(\Omega), \quad \text{if } \frac{1}{q+1} \geqslant \frac{1}{2} - \frac{s}{N}.$$

Summing the two conditions above we now obtain the growth restrictions

$$\frac{1}{p+1} + \frac{1}{q+1} \geqslant 1 - \frac{s+t}{N} = 1 - \frac{2}{N},$$

i.e. we have found the critical hyperbola.

Of course, it is crucial to recuperate from critical points (u, v) of this functional (weak) solutions of system (4.1). We state this in the following

PROPOSITION 4.2. *Suppose that $(u, v) \in E^s \times E^t$ is a critical point of the functional I, i.e. u and v are weak solutions of the system*

$$\begin{cases} \int_\Omega A^s u A^t \phi = \int_\Omega g(v)\phi, & \forall \phi \in E^s, \\ \int_\Omega A^s \psi A^t v = \int_\Omega f(u)\psi, & \forall \psi \in E^t. \end{cases} \tag{4.4}$$

Then $v \in W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$ *and* $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \forall q \geqslant 1$, *and hence* u *and* v *are "strong" solutions of* (4.4), *i.e.*

$$\begin{cases} \int_\Omega (-\Delta u)\phi = \int_\Omega g(v)\phi, & \forall \phi \in C_0^\infty(\Omega), \\ \int_\Omega (-\Delta v)\psi = \int_\Omega f(u)\psi, & \forall \psi \in C_0^\infty(\Omega). \end{cases} \tag{4.5}$$

From this proposition follows by standard bootstrap arguments that u and v are classical solutions of (4.1) if f, g and Ω are smooth.

For the proof of the proposition, see de Figueiredo and Felmer [13].

4.3.2. *Compactness and existence of solutions for systems with subcritical growth.* Subcritical growth is given for nonlinearities whose growth restrictions satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{1}{N}. \tag{4.6}$$

REMARK 4.3. To simplify the exposition, we state the theorems and give the proof for the so-called "model problem", i.e. system (4.1) with polynomial-type nonlinearities, i.e. $f(u) = u^p$ and $g(v) = v^q$. For the more general versions, the reader is referred to the literature.

Thus, we consider the system

$$\begin{cases} -\Delta u = v^q, \\ -\Delta v = u^p & \text{in } \Omega \subset \mathbb{R}^N,\ N \geqslant 3, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

We then have

THEOREM 4.4. *For subcritical growth, i.e. if* $p+1$ *and* $q+1$ *satisfy* (4.6), *system* (4.7) *has a nontrivial solution.*

PROOF. We have defined the functional

$$I : E^s(\Omega) \times E^t(\Omega) \to \mathbb{R},$$
$$I(u,v) = \int_\Omega A^s u A^t v\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx - \frac{1}{q+1}\int_\Omega |v|^{q+1}\, dx.$$

An inherent difficulty to systems of type (4.1) is that the associated functional is strongly indefinite, in the sense that near the origin it is positive respectively negative definite on infinite-dimensional subspaces. This precludes a direct application of the (by now classical) "linking theorems" of Critical Point Theory. In recent years, several approaches have been devised to overcome this problem:

– approximation by finite-dimensional problems;
– introduction of a new (weak-strong) topology: W. Kryszewski and A. Szulkin [28], Th. Bartsch and Y. Ding [7];
– infinite linking with compactness: S. Li and M. Willem [30].

We will use the last method to prove the above theorem. In later chapters, we will also show an application of the first method.

In [30], S. Li and M. Willem prove the following theorem:

THEOREM 4.5. *Let $\Phi : E \to \mathbb{R}$ be a strongly indefinite C^1-functional satisfying*

(A1) *Φ has a local linking at the origin, i.e. for some $r > 0$:*

$$\Phi(z) \geqslant 0 \quad \text{for } z \in E^+,\ \|z\|_E \leqslant r, \qquad \Phi(z) \leqslant 0, \quad \text{for } z \in E^-,\ \|z\|_E \leqslant r.$$

(A2) *Φ maps bounded sets into bounded sets.*

(A3) *Let E_n^+ be any n-dimensional subspace of E^+; then $\phi(z) \to -\infty$ as $\|z\| \to \infty$, $z \in E_n^+ \oplus E^-$.*

(A4) *Φ satisfies the Palais–Smale condition* (*PS*) (*Li and Willem* [30] *require a weaker "*(PS^*)*-condition", however, in our case the classical* (*PS*) *condition will be satisfied*).

Then Φ has a nontrivial critical point.

We now verify that our functional $I(u, v)$ satisfies the assumptions of this theorem. We assume, without restricting generality, that

$$1 < q \leqslant p, \quad \text{and hence} \quad s \geqslant t.$$

Also, we may assume that, e.g., the embedding $E^t \subset L^{q+1}$ is compact.

First, it is clear, with the choices of s and t made above, that $I(u, v)$ is a C^1-functional on $E^s \times E^t$.

(A1) Following de Figueiredo and Felmer [13] we can define the spaces

$$E^+ = \big\{ \big(y, A^{s-t} y\big) \bigm| y \in E^s \big\} \subset E^s \times E^t,$$
$$E^- = \big\{ \big(y, -A^{s-t} y\big) \bigm| y \in E^s \big\} \subset E^s \times E^t,$$

which give a natural splitting $E^+ \oplus E^- = E$. It is easy to see that $I(u, v)$ has a local linking with respect to E^+ and E^- at the origin.

(A2) Let $B \subset E^s \times E^t$ be a bounded set, i.e. $\|u\|_{E^s} \leqslant c$, $\|v\|_{E^t} \leqslant c$, for all $(u, v) \in B$. Then, by the embeddings $E^s \subset L^{p+1}$ and $E^t \subset L^{q+1}$

$$\begin{aligned} \big|I(u, v)\big| &\leqslant \big\|A^s u\big\|_{L^2} \big\|A^t v\big\|_{L^2} + \frac{1}{p+1} \int_\Omega |u|^{p+1} + \frac{1}{q+1} \int_\Omega |v|^{q+1} \\ &\leqslant \|u\|_{E^s} \|v\|_{E^t} + c\|u\|_{E^s}^{p+1} + c\|v\|_{E^t}^{q+1} \leqslant C. \end{aligned}$$

(A3) Let $z_k = z_k^+ + z_k^- \in E = E_n^+ \oplus E^-$ denote a sequence with $\|z_k\| \to \infty$. By the above, z_k may be written as

$$z_k = \left(u_k, A^{s-t} u_k\right) + \left(v_k, -A^{s-t} v_k\right), \quad \text{with } u_k \in E_n^s,\ v_k \in E^s,$$

where E_n^s denotes an n-dimensional subspace of E^s. Thus, the functional $I(z_k)$ takes the form

$$\begin{aligned} I(z_k) &= \int_\Omega A^s u_k A^t A^{s-t} u_k - \int_\Omega A^s v_k A^t A^{s-t} v_k \\ &\quad - \frac{1}{p+1} \int_\Omega |u_k + v_k|^{p+1} - \frac{1}{q+1} \int_\Omega \left|A^{s-t}(u_k - v_k)\right|^{q+1} \\ &= \int_\Omega |A^s u_k|^2 - \int_\Omega |A^s v_k|^2 - \frac{1}{p+1} \int_\Omega |u_k + v_k|^{p+1} \\ &\quad - \frac{1}{q+1} \int_\Omega \left|A^{s-t}(u_k - v_k)\right|^{q+1}. \end{aligned}$$

Note that $\|z_k\| \to \infty \Leftrightarrow \int |A^s u_k|^2 + \int |A^t v_k|^2 = \|u_k\|_{E^s}^2 + \|v_k\|_{E^s}^2 \to \infty$.

Now, if

(1) $\|u_k\|_{E^s} \leqslant c$, then $\|v_k\|_{E^s} \to \infty$, and then $J(z_k) \to -\infty$;

(2) $\|u_k\|_{E^s} \to \infty$, then we estimate (c, c_1, c_2 and d are positive constants)

$$\begin{aligned} \int_\Omega |u_k + v_k|^{p+1} &\geqslant c \left(\int_\Omega |u_k + v_k|^2 \right)^{\frac{p+1}{2}} - c \\ &\geqslant c_1 \|u_k + v_k\|_{L^2}^{p+1} - c \geqslant c_2 \|u_k + v_k\|_{L^2}^{q+1} - \bar{c}, \end{aligned}$$

and, by Poincaré's inequality, since $s \geqslant t$

$$\int_\Omega \left|A^{s-t}(u_k - v_k)\right|^{q+1} \geqslant c_1 \left\|A^{s-t}(u_k - v_k)\right\|_{L^2}^{q+1} - c \geqslant c_2 \|u_k - v_k\|_{L^2}^{q+1} - \bar{c}$$

and hence we obtain the estimate

$$I(z_k) \leqslant \|u_k\|_{E^s}^2 - c\left(\|u_k + v_k\|_{L^2}^{q+1} + \|u_k - v_k\|_{L^2}^{q+1}\right) + d.$$

Since $\phi(t) = t^{q+1}$ is convex, we have $\frac{1}{2}(\phi(t) + \phi(s)) \geqslant \phi(\frac{1}{2}(s+t))$, and hence

$$\begin{aligned} I(z_k) &\leqslant \frac{1}{2} \|u_k\|_{E^s}^2 - c \frac{1}{2^q} \left(\|u_k - v_k\|_{L^2} + \|u_k + v_k\|_{L^2}\right)^{q+1} + d \\ &\leqslant \frac{1}{2} \|u_k\|_{E^s}^2 - c \frac{1}{2^q} \|u_k\|_{L^2}^{q+1} + d. \end{aligned}$$

Since on E_n^s the norms $\|u_k\|_{E^s}$ and $\|u_k\|_{L^2}$ are equivalent, we conclude that also in this case $J(z_k) \to -\infty$.

(A4) Let $\{z_n\} \subset E$ denote a (PS)-sequence, i.e. such that

$$\big|I(z_n)\big| \to c, \quad \text{and} \quad \big|\big(I'(z_n), \eta\big)\big| \leqslant \epsilon_n \|\eta\|_E, \quad \forall \eta \in E, \text{ and } \epsilon_n \to 0. \tag{4.8}$$

We first show:

LEMMA 4.6. *The (PS)-sequence $\{z_n\}$ is bounded in E.*

PROOF. By (4.8) we have for $z_n = (u_n, v_n) \in E$

$$I(u_n, v_n) = \int_\Omega A^s u_n A^t v_n - \frac{1}{p+1}\int_\Omega |u_n|^{p+1} - \frac{1}{q+1}\int_\Omega |v_n|^{q+1} \to c, \tag{4.9}$$

$$I'(u_n, v_n)(\phi, \psi) = \int_\Omega A^s u_n A^t \psi + \int_\Omega A^t v_n A^s \phi - \int_\Omega u_n^p \psi - \int_\Omega v_n^q \phi = \epsilon_n \big\|(\phi, \psi)\big\|_E. \tag{4.10}$$

Choosing $(\phi, \psi) = (u_n, v_n) \in E^s \times E^t$ we get by (4.10)

$$2\int_\Omega A^s u_n A^t v_n - \int |u_n|^{p+1} - \int_\Omega |v_n|^{q+1} = \epsilon_n\big(\|u_n\|_{E^t} + \|v_n\|_{E^s}\big) \tag{4.11}$$

and subtracting this from $2I(u_n, v_n)$ we obtain

$$\left(1 - \frac{2}{p+1}\right)\int_\Omega |u_n|^{p+1} + \left(1 - \frac{2}{q+1}\right)\int_\Omega |v_n|^{q+1} \leqslant C + \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big) \tag{4.12}$$

and thus

$$\int_\Omega |u_n|^{p+1} \leqslant C + \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big), \tag{4.13}$$

$$\int_\Omega |v_n|^{q+1} \leqslant C + \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big). \tag{4.14}$$

Next, note that $A^{s-t} u_n \in E^t$; indeed, $u_n \in E^s$ implies that $A^t(A^{s-t} u_n) = A^s u_n \in L^2 \Leftrightarrow A^{s-t} u_n \in E^t$.

Thus, choosing $(\phi, \psi) = (0, A^{s-t} u_n) \in E^s \times E^t$ in (4.10) we get

$$\int_\Omega |A^s u_n|^2 = \int_\Omega v_n^q A^{s-t} u_n + \epsilon_n \big\|A^{s-t} u_n\big\|_{E^t}$$

and hence by Hölder

$$\|u_n\|_{E^s}^2 = \|A^t u_n\|_{L^2}^2 \leqslant \left(\int_\Omega |v_n|^{q+1}\right)^{\frac{q}{q+1}} \left(\int_\Omega |A^{s-t} u_n|^{p+1}\right)^{\frac{1}{p+1}} + \epsilon_n \|u_n\|_{E^s}.$$

Noting that

$$\|A^{s-t} u_n\|_{q+1} \leqslant c\|A^{s-t} u_n\|_{E^s} = c\|A^s u_n\|_{L^2} = c\|u_n\|_{E^s}$$

we obtain, using (4.14)

$$\|u_n\|_{E^s}^2 \leqslant \left[C + \epsilon_n\left(\|u_n\|_{E^s} + \|v_n\|_{E^t}\right)\right]^{q/(q+1)} \cdot c\|u_n\|_{E^s} + \epsilon_n\|u_n\|_{E^s}$$

and thus

$$\|u_n\|_{E^s} \leqslant C + \epsilon_n\left(\|u_n\|_{E^s} + \|v_n\|_{E^t}\right). \tag{4.15}$$

Similarly as above we note that $A^{t-s} v_n \in E^s$, and thus, choosing $(\phi, \psi) = (A^{t-s} v_n, 0) \in E^s \times E^t$ in (4.10) we obtain as above

$$\|v_n\|_{E^t} \leqslant C + \epsilon_n\left(\|v_n\|_{E^t} + \|u_n\|_{E^s}\right). \tag{4.16}$$

Joining (4.15) and (4.16) we finally get

$$\|u_n\|_{E^s} + \|v_n\|_{E^t} \leqslant C + 2\epsilon_n\left(\|u_n\|_{E^s} + \|v_n\|_{E^t}\right).$$

Thus, $\|u_n\|_{E^s} + \|v_n\|_{E^t}$ is bounded. □

With this it is now possible to complete the proof of the (PS)-condition:

Since $\|u_n\|_{E^s}$ is bounded, we find a weakly convergent subsequence $u_n \rightharpoonup u$ in E^s. Since the mappings $A^s : E^s \to L^2$ and $A^{-t} : L^2 \to E^t$ are continuous isomorphisms, we get $A^s(u_n - u) \rightharpoonup 0$ in L^2 and $A^{s-t}(u_n - u) \rightharpoonup 0$ in E^t. Furthermore, since $E^t \subset L^{q+1}$ compactly, we conclude that $A^{s-t}(u_n - u) \to 0$ strongly in L^{q+1}.

Similarly, we find a subsequence of $\{v_n\}$ which is weakly convergent in E^t and such that v_n^q is strongly convergent in $L^{\frac{q+1}{q}}$.

Choosing $(\phi, \psi) = (0, A^{s-t}(u_n - u) \in E^s \times E^t$ in (4.10) we thus conclude

$$\int_\Omega A^s u_n A^s(u_n - u) = \int_\Omega v_n^q A^{s-t}(u_n - u) + \epsilon_n\|A^{s-t}(u_n - u)\|. \tag{4.17}$$

By the above considerations, the right-hand side converges to 0, and thus

$$\int_\Omega |A^s u_n|^2 \to \int_\Omega |A^s u|^2.$$

Thus, $u_n \to u$ strongly in E^s.

To obtain the strong convergence of $\{v_n\}$ in E^t, one proceeds similarly: as above, one finds a subsequence $\{v_n\}$ converging weakly in E^t to v, and then $A^{t-s}v_n \rightharpoonup A^{t-s}v$ weakly in E^s and hence also in L^{p+1}, while by the above $u_n \to u$ strongly in E^s and hence in L^{p+1}, and then $u_n^p \to u_n^p$ in $L^{\frac{p+1}{p}}$. Choosing in (4.8) $(\phi, \psi) = (A^{t-s}(v_n - v), 0)$, we get

$$\int_\Omega A^t(v_n - v)A^t v_n = \int_\Omega |u_n|^{p-1}u_n A^{t-s}(v_n - v) + \epsilon_n\big(\|A^{t-s}(v_n - v)\|\big) \tag{4.18}$$

and thus one concludes again that

$$\int_\Omega \left|A^t v_n\right|^2 \to \int_\Omega \left|A^t v\right|^2$$

and hence also $v_n \to v$ strongly in E^t.

Thus, the conditions of Theorem 4.5 are satisfied; hence, we find a (positive) critical point (u, v) for the functional I, which yields a weak solution to system (4.7). □

4.4. *The $W^{1,\alpha}$-approach*

In this section we use that, alternatively, the functional $I(u, v)$ can be defined on a product of $W^{1,\alpha}$ spaces.

4.4.1. *The functional framework.* We define the functional $I(u, v)$ on a product space of Sobolev spaces. The term $\int_\Omega \nabla u \nabla v\, dx$ can be defined on the product space

$$W_0^{1,\alpha}(\Omega) \times W_0^{1,\beta}(\Omega), \quad \text{with } \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

by using the Hölder inequality

$$\left|\int_\Omega \nabla u \nabla v\, dx\right| \leqslant \|\nabla u\|_{L^\alpha} \|\nabla v\|_{L^\beta} = \|u\|_{W_0^{1,\alpha}} \|v\|_{W_0^{1,\beta}}.$$

Thus, to have the terms $F(s) = \frac{1}{p+1}|s|^{p+1}$ and $G(s) = \frac{1}{q+1}|s|^{q+1}$ in $I(u, v)$ well defined, we need to impose, using the Sobolev embedding theorem:

$$p + 1 \leqslant \alpha^* \frac{\alpha N}{N - \alpha N}, \qquad q + 1 \leqslant \beta^* = \frac{\beta N}{N - \beta N}.$$

From this we obtain the condition

$$\frac{1}{p+1} + \frac{1}{q+1} \geqslant \frac{N - \alpha}{\alpha N} + \frac{N - \beta}{\beta N} = 1 - \frac{2}{N}.$$

Thus, we have found again the critical hyperbola.

5. Generalization to Orlicz spaces

The $W^{1,\alpha}$-approach has the advantage that it can be generalized to more general settings by using *Orlicz spaces*. These spaces were already mentioned in Section 4 when we discussed the Trudinger inequality. We discuss now some details about Orlicz spaces.

5.1. *Orlicz spaces*

We recall here some basic facts about Orlicz spaces, for more details see for instance [1,27,39,22].

DEFINITION 5.1. A continuous function $M : \mathbb{R} \to [0, +\infty)$ is called an N-function, if it is convex, even, $M(t) = 0$ if and only if $t = 0$, and

$$M(t)/t \to 0 \quad \text{as } t \to 0 \quad \text{and} \quad M(t)/t \to +\infty \quad \text{as } t \to +\infty.$$

DEFINITION 5.2. Let A and B be N-functions. We say that

(1) A *dominates* B (near infinity) if, for some positive constant k,

$$B(x) \leqslant A(kx) \quad \text{for } x \geqslant x_0, \quad \text{and write} \quad B \prec A.$$

(2) A and B are *equivalent* if A dominates B and B dominates A; then we write $A \sim B$.

(3) B *increases essentially more slowly* than A if

$$\lim_{t\to\infty} \frac{B(kt)}{A(t)} = 0, \quad \text{for all } k > 0;$$

in this case we write $B \prec\prec A$.

Associated to the N-function M we introduce the following class of functions.

DEFINITION 5.3 (*Orlicz class*). The *Orlicz class* is the set of functions defined by

$$K_M(\Omega) := \left\{ u : \Omega \to \mathbb{R}\colon\ u \text{ measurable and } \int_\Omega M\big(u(x)\big)\,dx < \infty \right\}.$$

Orlicz classes are convex sets, but in general not linear spaces. One then defines

DEFINITION 5.4 (*Orlicz space*). The vector space $L_M(\Omega)$ generated by $K_M(\Omega)$ is called Orlicz space.

PROPOSITION 5.1. *The Orlicz class $K_M(\Omega)$ is a vector space, and hence equal to $L_M(\Omega)$ if and only if M satisfies the following Δ_2-condition*:

DEFINITION 5.5 (Δ_2-*condition*). There exist numbers $k > 1$ and $t_0 \geqslant 0$ such that

$$M(2t) \leqslant kM(t), \quad \text{for } t \geqslant t_0.$$

Furthermore, we define

DEFINITION 5.6 (∇_2-*condition*). There exist numbers $h > 1$ and $t_1 \geqslant 0$ such that

$$M(t) \leqslant \frac{1}{2h} M(ht), \quad \text{for } t \geqslant t_1.$$

We call a function satisfying the Δ_2- and the ∇_2-condition Δ-*regular*.

We remark that the Orlicz class depends only on the asymptotic growth of the function M; therefore, also the Δ_2-condition and the ∇_2-condition need to be satisfied only near infinity.

We define the following norm on $L_M(\Omega)$:

DEFINITION 5.7 (*Luxemburg norm*).

$$\|u\|_{(M)} = \inf\left\{\lambda > 0\colon \int_\Omega M\left(\frac{|u|}{\lambda}\right) \leqslant 1\right\}.$$

PROPOSITION 5.2. $(L_M, \|\cdot\|_{(M)})$ *is a Banach space*.

DEFINITION 5.8 (*Conjugate function*). Let

$$\widetilde{M}(x) = \sup_{y>0}\left\{xy - M(y)\right\}.$$

$\widetilde{M}$ is called the conjugate function of M.

It is clear that $\widetilde{\widetilde{M}} = M$, and M and $\widetilde{M}$ satisfy the *Young inequality*:

$$st \leqslant M(t) + \widetilde{M}(s), \quad \forall s, t \in \mathbb{R},$$

with *equality* when $s = M'(t)$ or $t = \widetilde{M}'(s)$.

PROPOSITION 5.3. *In the spaces L_M and $L_{\widetilde{M}}$ the* Hölder inequality *holds*:

$$\left|\int_\Omega u(x)v(x)\,dx\right| \leqslant 2\|u\|_{(M)}\|v\|_{(\widetilde{M})}.$$

Hence, for every $\tilde{u} \in L_{\widetilde{M}}$ we can define a continuous linear functional $l_{\tilde{u}}\, v := \int_\Omega \tilde{u} v \, dx$ and $l_{\tilde{u}} \in (L_M)^*$. Then we can define

$$\|\tilde{u}\|_{\widetilde{M}} := \|l_{\tilde{u}}\| = \sup_{\|v\|_{(M)} \leqslant 1} \int_\Omega \tilde{u}(x) v(x)\, dx.$$

DEFINITION 5.9. $\|\tilde{u}\|_{\widetilde{M}}$ is called the *Orlicz norm* on the space $L_{\widetilde{M}}$, and analogously one defines the Orlicz norm $\|u\|_M$ on L_M.

Thus, we have two different norms on L_M, the Luxemburg (or gauge) norm $\|\cdot\|_{(M)}$ and the Orlicz norm $\|\cdot\|_M$; they are equivalent, and satisfy

$$\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}. \tag{5.1}$$

In order to be precise about which norm is considered in the spaces, we are going to use from now on the following notations:

$$\big(L_M, \|\cdot\|_M\big) := L_M \quad \text{and} \quad \big(L_M, \|\cdot\|_{(M)}\big) := L_{(M)},$$

and similarly for $\widetilde{M}$.

It follows from the definition of Orlicz norm that

PROPOSITION 5.4. *If $u \in L_M$ and $\tilde{w} \in L_{\widetilde{M}}$, then one has the following Hölder inequality*

$$\left| \int_\Omega u\, \tilde{w}\, dx \right| \leqslant \|u\|_M \|\tilde{w}\|_{(\widetilde{M})}. \tag{5.2}$$

PROPOSITION 5.5. (See J.P. Gossez [22], Rao and Ren [39, p. 111].) L_M *is* reflexive *if and only if M and $\widetilde{M}$ satisfy the Δ_2-condition, and then*

$$(L_{(M)})^* = L_{\widetilde{M}} \quad \textit{and} \quad (L_{(\widetilde{M})})^* = L_M.$$

PROPOSITION 5.6. (See Rao and Ren [39, Theorem 2, p. 297].) *If Φ is Δ-regular, then there exists a $\Phi_1 \sim \Phi$ such that $L_\Phi = L_{\Phi_1}$ as sets, and their Luxemburg norms* (*respectively Orlicz norms*) *are equivalent, with the following additional structure*:

(a) *L_Φ and L_{Φ_1} are isomorphic, and both are reflexive spaces,*

(b) *L_{Φ_1} is uniformly convex and uniformly smooth.*

Next, we define the *Orlicz–Sobolev* spaces: Let A be a N-function. Then set

DEFINITION.

$$W^1 L_A = \left\{ u : \Omega \to \mathbb{R};\ \max_{|\alpha| \in \{0,1\}} \int_\Omega A\big(|D^\alpha u|\big) < +\infty \right\}$$

with Luxemburg norm

$$\|u\|_{W^1 L_A} := \max\{\|D^\alpha u\|_{(A)}\colon |\alpha| \in \{0, 1\}\}.$$

On the space $W_0^1 L_{(A)}$, i.e. the space of functions in $W^1 L_A$ which vanish on the boundary, an equivalent Luxemburg norm is given by

$$\|u\|_{1,(A)} = \|\nabla u\|_{(A)} = \inf\left\{\lambda > 0\colon \int_\Omega A\left(\frac{|\nabla u|}{\lambda}\right) \leqslant 1\right\}.$$

The equivalence of these two norms is a consequence of the Poincaré inequality,

$$\|u\|_{(M)} \leqslant C \sum_{i=1}^n \|D_i u\|_{(M)}, \quad \forall u \in W_0^1 L_M(\Omega)$$

(see [22]). In analogy with the above definition of the Orlicz norm in L_M we can define an Orlicz norm in $W_0^1 L_{(A)}$ by

$$\|u\|_{1,A} := \sup\left\{\int_\Omega \nabla u \nabla \tilde{w}\, dx\colon \tilde{w} \in W_0^1 L_{(\tilde{A})},\ \|\tilde{w}\|_{1,(\tilde{A})} \leqslant 1\right\}.$$

The space $W_0^1 L_{(A)}$ endowed with this new norm is denoted by $W_0^1 L_A$.

DEFINITION 5.10 (*Sobolev conjugate*). (See Adams [1, p. 248].) Suppose that

$$\int_1^\infty \frac{A^{-1}(t)}{t^{1+1/n}} dt = +\infty.$$

Then the Sobolev conjugate function $\Phi(t)$ is given by

$$\Phi^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{1+1/n}} d\tau, \quad t \geqslant 0. \tag{5.3}$$

PROPOSITION 5.7. *Let Ω be bounded, and satisfying the cone property. Then the embedding*

$$W^1 L_A(\Omega) \hookrightarrow L_\Phi(\Omega)$$

is continuous, and compact into $L_G(\Omega)$ where G is any N-function increasing essentially more slowly than Φ, i.e. $\lim_{t\to\infty} \frac{G(kt)}{\Phi(t)} = 0$, *for all* $k > 0$.

EXAMPLE 5.1. One easily checks that for $\Phi(s) = s^{p+1}$ the above formula (5.3) yields $A(s) = cs^\alpha$, with α satisfying $\frac{1}{\alpha} = \frac{1}{p+1} + \frac{1}{N}$; indeed, $\Phi^{-1}(t) = t^{1/(p+1)}$, and hence

$$\frac{A^{-1}(t)}{t^{1+\frac{1}{N}}} = \frac{d}{dt}\Phi^{-1}(t) = \frac{1}{p+1} t^{\frac{1}{p+1}-1},$$

i.e.

$$A^{-1}(t) = \frac{1}{p+1} t^{\frac{1}{p+1}+\frac{1}{N}}.$$

It follows that $p+1 = \frac{\alpha N}{N-\alpha}$, and thus we recover the classical Sobolev embedding theorem $W^{1,\alpha}(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

Next, we make the following

DEFINITION 5.11. Let $g \in C(\mathbb{R})$ be a N-function, and G its primitive. Then we say that G is θ-regular, if there exists a constant $\theta_G > 1$ such that

$$\lim_{s\to\infty} \frac{sg(s)}{G(s)} = \theta_G. \tag{5.4}$$

Let $F(t) = G^{-1}(t)$, and $f(t) = F'(t)$. Then the above condition is equivalent to

$$\lim_{t\to\infty} \frac{tf(t)}{F(t)} = \frac{1}{\theta_G}. \tag{5.5}$$

Indeed, we have $G(s) = t \Leftrightarrow F(t) = s$, and $f(t) = \frac{d}{dt}[G^{-1}(t)] = \frac{1}{g(s)}$.

We have

PROPOSITION 5.8. (See Rao and Ren [39, p. 26].) *If G is θ-regular, then G is Δ-regular, i.e. $G \in \Delta_2 \cap \nabla_2$.*

5.2. *Orlicz space criticality*

In the last section we have seen that Orlicz spaces are a generalization of L^p-spaces (which correspond to the N-functions $|s|^p$) to more general N-functions. It is natural to look for an analogue of the critical hyperbola for N-functions.

DEFINITION (*Critical Orlicz pair*). Let Φ and Ψ be Δ-regular N-functions. Then (Φ, Ψ) are a *critical Orlicz pair* if there exist Δ-regular and conjugate N-functions A and $\widetilde{A}$ such that L_Φ and L_Ψ are the smallest Orlicz spaces with

$$W^1 L_A \hookrightarrow L_\Phi, \qquad W^1 L_{\widetilde{A}} \hookrightarrow L_\Psi.$$

We have the following:

THEOREM 5.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain. Let $\Phi \in C^1$ be a given N-function, and set $\varphi(t) = \Phi'(t)$. Assume that*

$$\lim_{s\to\infty} \frac{\varphi(s)s}{\Phi(s)} = \theta_\Phi > \frac{N}{N-2}. \tag{5.6}$$

Then there exists an associated N-function Ψ such that (Φ, Ψ) form a critical Orlicz pair. Furthermore, the limit

$$\lim_{s\to\infty} \frac{s\Psi(s)}{\Psi(s)} = \theta_\Psi$$

exists, and

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = 1 - \frac{2}{N}. \tag{5.7}$$

Consider the following

EXAMPLE 5.2. In Example 5.1 we saw that to $\Phi(s) = s^{p+1}$ corresponds the inverse Sobolev conjugate $A(s) = cs^\alpha$, with

$$\frac{1}{p+1} + \frac{1}{N} = \frac{1}{\alpha}.$$

The conjugate function $\widetilde{A}$ to A is given by $\widetilde{A}(s) = cs^\beta$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, which in turn has as Sobolev conjugate $\Psi(s) = s^{q+1}$, with

$$\frac{1}{q+1} + \frac{1}{N} = \frac{1}{\beta}.$$

Adding the two equations yields

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{5.8}$$

This is the *critical hyperbola*, see de Figueiredo and Felmer [13] and Hulshof and van der Vorst [23]. Thus, $(|s|^{p+1}, |s|^{q+1})$ are a critical Orlicz pair, and so the above theorem contains the critical hyperbola as a special case. We remark that the proof given here is also new in the polynomial case; in [13] and [23] fractional Sobolev spaces H^s were used in order to conserve the Hilbert space structure.

We now give some examples of critical Orlicz pairs:

EXAMPLE 5.3. Let

$$\Phi(s) \sim |s|^{p+1}\big(\log|s|\big)^\alpha \quad \text{and} \quad \Psi(s) \sim |s|^{q+1}\big(\log|s|\big)^{-\alpha\frac{q+1}{p+1}},$$

with $\alpha > 0$ and $p,\ q \in (1, +\infty)$ satisfying (5.8). Then Φ and Ψ satisfy (5.6), with $\theta_\Phi = p+1$ and $\theta_\Psi = q+1$, respectively, and (Φ, Ψ) form a critical Orlicz-pair; for the proof, see [17].

REMARK 5.12. The restriction $\theta_\Phi > N/(N-2)$ in Theorem 5.9 is necessary in order to obtain a Ψ which is θ-regular, in the sense of Definition 5.11. Also in the polynomial case such a restriction, which here is $p+1 > N/(N-2)$, is necessary in order to obtain $q > 1$.

5.3. *Critical Orlicz-pairs: proof of Theorem 5.9*

(1) Hypothesis (5.6) expresses the fact that the function Φ is θ-regular with $\theta_\Phi > N/(N-2)$. Let A be the inverse Sobolev conjugate of Φ, see Definition 5.10. Note that W^1L_A is the largest Orlicz–Sobolev space that embeds into L_Φ.

CLAIM 1. A is θ-regular, with $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

Indeed, let $F(s) = \Phi^{-1}(s)$ and $B(t) = A^{-1}(t)$. Then $F(s) = \int_0^s \frac{B(t)}{t^{1+1/N}}\,dt$, and hence

$$f(s) = \frac{B(s)}{s^{1+1/N}}.$$

Then we have by (5.5)

$$\frac{1}{\theta_\Phi} = \lim_{s\to\infty} \frac{f(s)s}{F(s)} = \lim_{s\to\infty} \frac{B(s)s^{-1/N}}{F(s)}.$$

Then, by l’Hôpital’s rule

$$\lim_{s\to\infty} \frac{B(s)s^{-1/N}}{F(s)} = \lim_{s\to\infty} \frac{b(s)s^{-1/N} - \frac{1}{N}s^{-\frac{1}{N}-1}B(s)}{\frac{B(s)}{s^{1+1/N}}} = \lim_{s\to\infty} \frac{b(s)s}{B(s)} - \frac{1}{N}.$$

We conclude that

$$\frac{1}{\theta_\Phi} = \lim_{s\to\infty} \frac{b(s)s}{B(s)} - \frac{1}{N}$$

and thus

$$\lim_{s\to\infty} \frac{b(s)s}{B(s)} = \frac{1}{\theta_\Phi} + \frac{1}{N} < 1.$$

This implies that A is θ-regular, with $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

(2) Next, let $\widetilde{A}$ be the conjugate function of A, given by Definition 5.8. $\widetilde{A}$ is a N-function, and Δ-regular, see Rao and Ren [39, Corollary 4, p. 26].

In the following, suppose that $s = A'(t)$ (iff $t = \widetilde{A}'(s)$); note that $t \to \infty$ iff $s \to \infty$. Then

$$\frac{1}{\theta_A} = \lim_{t\to\infty} \frac{A(t)}{tA'(t)} = \lim_{t\to\infty} \frac{A(t)}{ts} = \lim_{s\to\infty} \frac{st - \widetilde{A}(s)}{st} = 1 - \lim_{s\to\infty} \frac{\widetilde{A}(s)}{s\widetilde{A}'(s)}$$

$$= 1 - \frac{1}{\theta_{\widetilde{A}}}.$$

Thus, $\widetilde{A}$ is θ-regular, with $\theta_{\widetilde{A}} > 1$.

We can now define the corresponding Orlicz–Sobolev space $W^1 L_{\widetilde{A}}$.

(3) Next, use Definition 5.10 again to define the function Ψ; by Adams and Fournier [1, p. 248], Ψ is an N-function.

CLAIM 2. Ψ is θ-regular, with $\theta_\Psi = \frac{N\theta_{\widetilde{A}}}{N-\theta_{\widetilde{A}}}$.

This follows similarly as in Claim 1, reversing the direction in the arguments.

Finally, L_Ψ is the smallest Orlicz space into which $W^1 L_{\widetilde{A}}$ embeds continuously.

Thus, we have shown that (Φ, Ψ) is a critical Orlicz pair.

Finally, we have

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = \frac{N - \theta_A}{N\theta_A} + \frac{N - \theta_{\widetilde{A}}}{N\widetilde{A}} = \frac{1}{\theta_A} - \frac{1}{N} + \frac{1}{\theta_{\widetilde{A}}} - \frac{1}{N} = 1 - \frac{2}{N}.$$

5.4. *Orlicz space subcritical: an existence theorem*

We have the following existence theorem for nonlinearities which have *subcritical growth* with respect to a given *critical Orlicz pair* (Φ, Ψ).

THEOREM 5.13. *Suppose that (Φ, Ψ) is a critical Orlicz-pair. Suppose that f and g are continuous functions, and let F and G denote their primitives. Assume that*

(H1) *there exists $\theta > 2$ and $t_0 > 0$ such that for all $t \geqslant t_0$*

$$0 < \theta F(t) \leqslant tf(t) \quad \textit{and} \quad 0 < \theta G(t) \leqslant tg(t).$$

(H2) *F and G are uniformly superquadratic near zero, i.e. there exist numbers $\sigma > 2$ and $c \geqslant 1$ such that*

$$F(st) \leqslant cs^\sigma F(t), \qquad G(st) \leqslant c\, s^\sigma G(t), \quad \forall t > 0,\ \forall s \in [0, 1].$$

(H3) *F and G have an essentially lower growth than Φ and Ψ, respectively* (*see Definition* 5.2).

(H4) *F and G satisfy*

$$\lim_{t \to 0} \frac{F(t)}{\Phi(t)} = C_F < \infty \quad \textit{and} \quad \lim_{t \to 0} \frac{G(t)}{\Psi(t)} = C_G < \infty.$$

Then system (4.1) *has a nontrivial solution.*

EXAMPLE 5.4. Let (Φ, Ψ) denote the critical Orlicz pair given in Example 5.3. Suppose that $F(s) \sim s^{p+1}(\log s)^{\beta}$ and $G(s) \sim s^{q+1}(\log s)^{-\gamma}$, for s positive and large with $\beta < \alpha$ and $\gamma > \alpha\frac{q+1}{p+1}$. Then F and G have essentially slower growth than Φ and Ψ, respectively.

PROOF. (Outline; for more details, see [17].)

We now define the functional

$$I(u, \tilde{v}) = \int_{\Omega} \nabla u \nabla \tilde{v}\, dx - \int_{\Omega} F(u)\, dx - \int_{\Omega} G(\tilde{v})\, dx$$

on the space

$$I : W_0^1 L_A(\Omega) \times W_0^1 L_{(\widetilde{A})}(\Omega) \to \mathbb{R}. \tag{5.9}$$

Here $\tilde{v} \in W_0^1 L_{(\widetilde{A})}(\Omega)$ is an independent variable; we write $\tilde{v}$ to emphasize that $\tilde{v}$ belongs to the space $W_0^1 L_{(\widetilde{A})}(\Omega)$.

We recall that the functional (5.9) is strongly indefinite, being positive definite, respectively negative definite (near the origin) on an infinite-dimensional subspace. We now make this more precise, by showing that there is the geometry of (local) infinite-dimensional linking. For this, we introduce the

TILDE MAP. For $u \in W_0^1 L_A(\Omega)$ consider

$$S_u := \sup\left\{ \int_{\Omega} \nabla u(x) \nabla \tilde{w}(x)\, dx \colon \tilde{w} \in W_0^1 L_{(\widetilde{A})},\ \|\tilde{w}\|_{1,(\widetilde{A})} = \|u\|_{1,A} \right\}. \tag{5.10}$$

Then we have

LEMMA 5.14. *There exists a unique* $\tilde{u} \in W_0^1 L_{(\widetilde{A})}$ *such that*

$$\|\tilde{u}\|_{1,(\widetilde{A})} = \|u\|_{1,A} \quad \textit{and} \quad S_u = \int_{\Omega} \nabla u(x) \nabla \tilde{u}(x)\, dx = \|u\|_{1,A} \|\tilde{u}\|_{1,(\widetilde{A})}.$$

Furthermore, $\tilde{u}$ *depends continuously* (*but nonlinearly*) *on* u.

We now define two submanifolds of $E := W_0^1 L_A(\Omega) \times W_0^1 L_{(\widetilde{A})}$:

$$E^+ = \left\{(u, \tilde{u}) \colon u \in W_0^1 L_A\right\} \quad \text{and} \quad E^- = \left\{(u, -\tilde{u}) \colon u \in W_0^1 L_A\right\}.$$

We note that E^+ and E^- are nonlinear submanifolds of E when regarded with respect to the standard vector space structure of E, but they are *linear* with respect to the following notion of

TILDE SUM. Given $(u, \tilde{v}), (y, \tilde{z}) \in E$, set

$$(u, \tilde{v}) \mathbin{\tilde{+}} (y, \tilde{z}) := (u + y, \widetilde{v + z}).$$

One proves easily that with this notion one has

$$E = E^+ \widetilde{\oplus} E^-.$$

One then proves the following *local linking* structure:

LEMMA 5.15.
(1) *There exist* $\rho_0, \sigma_0 > 0$ *such that* $I(z) \geqslant \sigma_0$, *for all* $z \in \partial B_{\rho_0} \cap E^+$.
(2) *There exist positive constants* R_0, R_1 *such that* $I(z) \leqslant 0$ *for all* $z \in \partial Q$, *where* $Q = \{r(e_1, \tilde{e}_1) \widetilde{+} w\colon\ w \in E^-,\ \|w\| \leqslant R_0$ *and* $0 \leqslant r \leqslant R_1\}$ (e_1 *denotes the first eigenfunction of the Laplacian, with* $\|(e_1, \tilde{e}_1)\| = 1$).

We emphasize that this linking is only formal, since it involves two *infinite*-dimensional manifolds. We now proceed by a *finite-dimensional approximation*; this will lead to an actual linking structure. The proof is then completed by a limiting argument.

Define

$$E_n^+ := \{(z, \tilde{z});\ z \in E_n\} \quad \text{and} \quad E_n^- := \{(z, -\tilde{z});\ z \in E_n\},$$

where E_n is the space spanned by the first n eigenfunctions of the Laplacian.

We now restrict the functional I to $E_n^+ \widetilde{\oplus} E_n^- = E_n \times \widetilde{E}_n$, and consider the set

$$Q_n = \{w \widetilde{+} r(e_1, \tilde{e}_1);\ w \in E_n^1,\ \|w\| \leqslant R_0,\ 0 \leqslant r \leqslant R_1\} \subset E_n^+ \widetilde{\oplus} E_n^-.$$

Furthermore, we define the family of maps

$$H_n = \{h \in C(Q_n, E_n^+ \widetilde{E}_n^-)\colon\ h(z) = z \text{ on } \partial Q_n\}.$$

One shows, using the topological degree on oriented manifolds, that

LEMMA 5.16. *The sets* Q_n *and* $\partial B_{\rho_0} \cap E_n^+$ *link, i.e.*

$$H(Q_n) \cap (\partial B_{\rho_0} \cap E_n^+) \neq 0, \quad \forall h \in H_n.$$

Finally, we set

$$c_n = \inf_{h \in H_n} \max_{z \in Q_n} I(h(z)).$$

Using the linking property and Lemma 5.15 one obtains that the values c_n satisfy $c_n \in [\sigma_0, cR_1^2]$, $\forall n \in \mathbb{N}$. Furthermore, by the “Linking Theorem” of P.H. Rabinowitz [38] one obtains a PS-sequence $(u_m, \tilde{v}_m) \in E_n^+ \widetilde{\oplus} E_n^-$, i.e. satisfying $I(u_m, \tilde{v}_m) \to c_n$, as $m \to \infty$, and $|I'(u_m, \tilde{v}_m)[(\xi, \tilde{\eta})]| \leqslant \epsilon_m \|(\xi, \tilde{\eta})\|$, with $\epsilon_m \to 0$ as $m \to \infty$.

One now shows

LEMMA 5.17. *Let $(u_m, \tilde{v}_m)$ be a PS-sequence. Then the sequence $(u_m, \tilde{v}_m)$ is bounded in E, i.e. there exists a constant C such that $\|u_m\| \leqslant C$ and $\|\tilde{v}_m\| \leqslant C$, for all $m \in \mathbb{N}$.*

The *proof* of this lemma uses standard methods, which are however rendered complicated by the fact that L^p estimates must be replaced by more delicate estimates in Orlicz spaces.

With this, one now obtains that c_n is a critical level of $I|_{E_n^+ \widetilde{\oplus} E_n^-}$, for each $n \in \mathbb{N}$, with a corresponding sequence of critical points $z_n \in E_n^+ \widetilde{\oplus} E_n^-$, with $\|z_n\| \leqslant c$, where c does not depend on n.

To complete the proof, we take the limit $n \to \infty$: by the uniform bounds on c_n and on the finite-dimensional critical points z_n, it is easy to conclude that $z_n = (u_n, \tilde{v}_n) \rightharpoonup z = (u, \tilde{v})$ in $E = W_0^1 L_A \times W_0^1 L_{\widetilde{A}}$, and that z is a weak solution of system (4.1).

It remains to show that $z = (u, \tilde{v})$ is nontrivial; assume by contradiction that $u = 0$, then by equations (4.1) also $\tilde{v} = 0$. Note that we can find a suitable Δ-regular N-function F_1 with $F_1 \prec\prec \Phi$ and the properties $F(x) \leqslant F_1(x)$, $f(x) \leqslant f_1(x)$, $\forall x \in \mathbb{R}^+$. Thus, by the compact embedding $W_0^1 L_A \hookrightarrow L_{F_1}$, we get

$$\|u_n\|_{(F_1)} \to 0, \quad \text{i.e.} \quad \inf\left\{\lambda > 0;\ \int_\Omega F_1\left(\frac{u_n}{\lambda}\right) \leqslant 1\right\} =: \lambda_n \to 0.$$

Since, for $\lambda_n < 1$ holds $\frac{1}{\lambda_n} \int_\Omega F_1(u_n) \leqslant \int_\Omega F_1(\frac{u_n}{\lambda_n}) \leqslant 1$, we conclude that

$$\int_\Omega F(u_n) \leqslant \int_\Omega F_1(u_n) \leqslant \lambda_n \to 0.$$

Since F_1 is Δ-regular, we have $x f_1(x) \leqslant c F_1(x)$, for some $c > 1$, and hence

$$0 \leqslant \int_\Omega f(u_n) u_n \leqslant \int_\Omega f_1(u_n) u_n \leqslant c \int_\Omega F_1(u_n)\, dx \to 0. \tag{5.11}$$

It is easily seen that this implies that also $\int_\Omega \nabla u_n \nabla \tilde{v}_n\, dx \to 0$, and thus also $I(u_n, \tilde{v}_n) \to 0$. But this contradicts that $I(u_n, \tilde{v}_n) \geqslant \sigma_0 > 0$, for all $n \in \mathbb{N}$.

This concludes the proof of Theorem 5.13. □

In Theorem 5.13 we have obtained solutions for system (4.1) in the case of Orlicz-subcriticality; in particular, for the polynomial "model" system

$$\begin{cases} -\Delta u = v^q, \\ -\Delta v = u^p \quad \text{in } \Omega \subset \mathbb{R}^N,\ N \geqslant 3, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.12}$$

this means that we have existence of solutions if the pair of nonlinearities $(|s|^{p+1}, |s|^{q+1})$ are Orlicz subcritical which, as we have seen, is in this case equivalent to say that the pair (p, q) is below the critical hyperbola, i.e.

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

6. A system with no growth restriction in one nonlinearity

We now turn our attention to some other aspects of the critical hyperbola. Note that the asymptotes of the critical hyperbola are given by

$$\left(\frac{N}{N-2}, s\right), \quad \text{and} \quad \left(s, \frac{N}{N-2}\right), \quad s \in \mathbb{R}, \quad \text{see Figure 1.}$$

We show next that if one of the nonlinearities in system (4.1), say $f(s)$, has a polynomial growth restriction with exponent lying to the left of the asymptote $(\frac{N}{N-2}, s)$ of the hyperbola, then *no growth restriction* on the other nonlinearity is needed. More precisely, restricting attention to the following model situation:

$$\begin{cases} -\Delta u = g(v), & \\ -\Delta v = u^p & \text{in } \Omega \subset \mathbb{R}^N, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

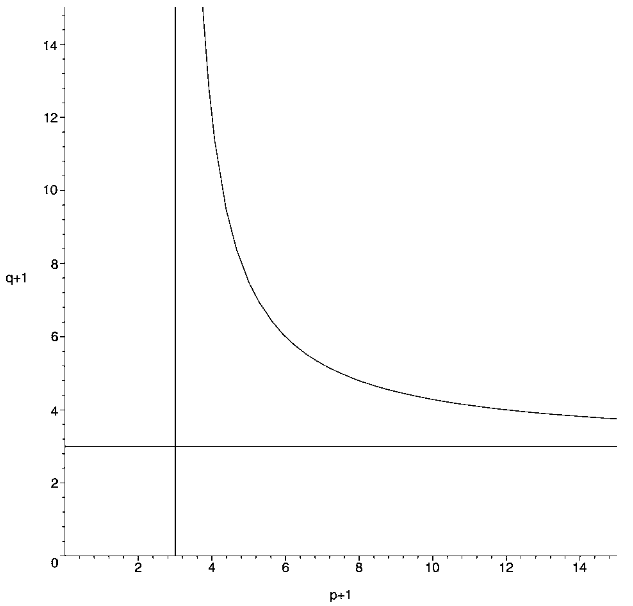

Fig. 1. The critical hyperbola, $N = 3$.

we have the following theorem, cf. [19]:

THEOREM 6.1. *Suppose that*

$$\begin{cases} 0 < p, & \text{if } N = 2, \\ 0 < p < \frac{2}{N-2}, & \text{if } N \geqslant 3 \end{cases}$$

and assume that $g \in C(\mathbb{R})$, *with* $G(s) = \int_0^s g(t)\,dt$ *its primitive. Then the functional*

$$I(u,v) = \int_\Omega \nabla u \nabla v\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx - \int_\Omega G(v)\,dx$$

is well defined and of class C^1 *on the space*

$$E = E^s(\Omega) \times E^t(\Omega), \quad \text{with } s \text{ satisfying } p+1 = \frac{2N}{N-2s}.$$

PROOF. We proceed as in Section 4.3 and define the equivalent functional (cf. (4.3))

$$I(u,v) = \int_\Omega A^s u A^t v - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx - \int_\Omega G(v)\,dx.$$

The first term of the functional is naturally defined on the space E, and the second term is well defined, continuous and differentiable, by the continuous embedding $E^s \subset L^{p+1}$, for $p+1 = \frac{2N}{N-2s}$.

Next, note that

$$t = 2 - s = 2 - \frac{N}{2} + \frac{N}{p+1} > 2 - \frac{N}{2} + N - 2 = \frac{N}{2},$$

where we used the assumption $p+1 < \frac{N}{N-2}$. Thus, we have the continuous embedding, cf. [1]

$$E^t(\Omega) \subset C^{0,\frac{1}{2}}(\Omega),$$

and hence we have for $v \in E^t$ that v is continuous, and hence also $G(v)$, and then $\int_\Omega G(v)\,dx$ is well defined, continuous and differentiable. □

Next, we give an existence theorem for system (6.1). Since in this survey we concentrate on *superlinear* problems, i.e. with $p > 1$, we obtain from $2 < p+1 < \frac{N}{N-2}$ the restriction $N \leqslant 3$. For corresponding results on "superlinear-sublinear" systems, we refer to [19]. Concerning system (6.1), with $N = 2, 3$, we have the following existence theorem

THEOREM 6.2. *Suppose that*

$$\begin{cases} 0 < p, & \text{if } N = 2, \\ 0 < p < \frac{2}{N-2}, & \text{if } N \geqslant 3 \end{cases}$$

and assume that $g \in C(\mathbb{R})$, *with* $G(s) = \int_0^s g(t)\,dt$ *its primitive, and that there exist constants* $\theta > 2$ *and* $s_0 \geqslant 0$ *such that*

$$\theta G(s) \leqslant g(s)s, \quad \forall |s| \geqslant s_0.$$

Finally, for s *near* 0 *we assume* $g(s) = o(s)$.

Then system (6.1) *has a nontrivial* (*strong*) *solution* (u, v).

PROOF. The proof follows closely the proof of Theorem 4.4, and we omit it. For details, we refer to [19]. □

7. A borderline case

In this section we will consider the *borderline case* where one of the nonlinearities has a growth corresponding to the asymptote $(\frac{N}{N-2}, s)$ of the critical hyperbola (5.8): we assume that

$$F(s) \sim |s|^{\frac{N}{N-2}},$$

and consider the system

$$\begin{cases} -\Delta u = g(v), \\ -\Delta v = u^{\frac{2}{N-2}} & \text{in } \Omega \subset \mathbb{R}^N, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

We will see that then the nonlinearity $g(s)$ is governed by a *limiting Sobolev space embedding*, i.e. we find a maximal growth defined by a Trudinger-type embedding. However, there is a *surprise*!

We may again apply the H^s-approach or the $W^{1,\alpha}$-approach; and we will also consider a "mixed approach", defining the functional on $W^{s,\alpha}$-spaces, i.e. the space of functions whose fractional derivative of order s lies in L^α. We have seen that the H^s-approach and the $W^{1,\alpha}$-approach give the same critical hyperbola, and we will show that this is also the case for the "mixed approach". Thus, all these approaches seem equivalent. We will however show the surprising result that for the borderline case $F(s) \sim |s|^{\frac{N}{N-2}}$, these methods yield *different maximal growth conditions* for the nonlinearity $G(s)$. We will then use *Lorentz spaces* to obtain a truly maximal growth for $G(s)$.

7.1. *The* $W^{s,\alpha}$*-approach*

This is a "mixed" approach between the H^s and the $W^{1,\alpha}$-approach. The space $W_0^{s,\alpha}$ consists of the functions whose fractional derivative of order s lie in L^α; a precise definition may be given by interpolation, see Adams and Fournier [1]. For these spaces, the following Sobolev embedding theorems hold (see [1])

$$W_0^{s,\alpha} \subset L^{p+1}, \quad \text{with } p+1 = \frac{\alpha N}{N - s\alpha}. \tag{7.2}$$

We define the functional $I(u, v)$ on the space

$$W_0^{s,\alpha}(\Omega) \times W_0^{t,\beta}(\Omega), \quad \text{with } s+t=2,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

that is, as in Section 4.3.1 we consider the functional

$$I(u, v) = \int_\Omega A^s u A^t v - \frac{1}{p+1} \int_\Omega |u|^{p+1} - \frac{1}{q+1} \int_\Omega |v|^{q+1}.$$

The first term of the functional, $\int_\Omega A^s u A^t v\, dx$, is well defined on $W_0^{s,\alpha} \times W_0^{t,\beta}$ by estimating

$$\left| \int_\Omega A^s u A^t v\, dx \right| \leqslant \|u\|_{W^{s,\alpha}} \|v\|_{W^{t,\beta}}, \quad \text{with } s+t=2 \text{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

From the embeddings (7.2) we then get the following maximal growth conditions for $F(u) = \frac{1}{p+1}|u|^{p+1}$ and $G(v) = \frac{1}{q+1}|v|^{q+1}$:

$$p+1 = \frac{\alpha N}{N - s\alpha}, \qquad q+1 = \frac{\beta N}{N - s\beta}$$

with the conditions:

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N - s\alpha}{\alpha N} + \frac{N - t\beta}{\beta N} = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{s+t}{N} = 1 - \frac{2}{N};$$

that is, we have found again the critical hyperbola!

7.1.1. *Different results using the H^s-method and the $W^{s,\alpha}$-method.* We now consider the border line case mentioned above: we assume that one of the nonlinearities has an exponent on the asymptote of the critical hyperbola, i.e.

$$F(u) = \frac{1}{p+1}|u|^{p+1} = \frac{N-2}{N}|u|^{\frac{N}{N-2}}.$$

We will see that the maximal growth for the other nonlinearity, $g(v)$, is given by a Trudinger-type inequality, i.e. it is of exponential growth. We will consider both, the H^s-approach and the $W^{s,\alpha}$-approach, and surprisingly, we will get *different maximal growths*!

As seen in Section 4.4, the corresponding functional

$$I(u,v)=\int_\Omega \nabla u\nabla v-\frac{N-2}{N}\int_\Omega |u|^{\frac{N}{N-2}}-\int G(v)$$

can be defined on the space

$$W_0^{1,\alpha}(\Omega)\times W_0^{1,\beta}(\Omega),\quad \text{with } \frac{1}{\alpha}+\frac{1}{\beta}=1. \tag{7.3}$$

The requirement

$$W^{1,\alpha}(\Omega)\subset L^{\frac{N}{N-2}}(\Omega)$$

yields the condition

$$\frac{N}{N-2}=\alpha^*=\frac{N\alpha}{N-\alpha}\quad\Longleftrightarrow\quad \frac{N}{N-1}=\alpha.$$

By (7.3) this gives $\beta=N$, i.e. we are in the limiting Sobolev case $W^{1,\beta}(\Omega)=W^{1,N}(\Omega)$. We look now for the largest possible growth function $\phi(s)$ such that $\int_\Omega \phi(u)\,dx$ is finite. Indeed, by Trudinger [44] and Pohozaev [36] we have

$$W^{1,N}(\Omega)\subset L_\phi(\Omega),$$

where L_ϕ is the Orlicz space with the corresponding N-function

$$\phi(s)=e^{|s|^{\frac{N}{N-1}}}-1.$$

Thus, with the $W^{1,\alpha}$-method we obtain as "critical growth" for the primitive $G(s)$:

$$G(s)\sim e^{|s|^{\frac{N}{N-1}}}, \tag{7.4}$$

i.e. we have the maximal growths

$$F(s)=|s|^{p+1}=|s|^{\frac{N}{N-2}},\qquad G(s)\sim e^{|s|^{\frac{N}{N-1}}}.$$

We now use the fractional Sobolev space approach: we work directly with the mixed approach, i.e. the $W^{s,\alpha}$-method. Given $s\in(0,2)$, the optimal fractional Sobolev space $W_0^{s,\alpha}$ for having a continuous embedding $W_0^{s,\alpha}\subset L^{\frac{N}{N-2}}$ is given by the condition:

$$\frac{N}{N-2}=p+1=\frac{\alpha N}{N-s\alpha}$$

which implies

$$\alpha = \frac{N}{N-(2-s)},$$

and hence by $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we get for β the condition:

$$\beta = \frac{N}{2-s} = \frac{N}{t}.$$

The space $W_0^{t,\beta}(\Omega) = W_0^{t,\frac{N}{t}}(\Omega)$ is again a *limiting Sobolev case*, and by R.S. Strichartz [41] we have the optimal embedding

$$W_0^{t,\beta}(\Omega) = W_0^{t,\frac{N}{t}}(\Omega) \subset L_\phi(\Omega)$$

with

$$\phi(r) = e^{|r|^{\frac{N}{N-t}}} - 1, \quad 0 < t < 2.$$

Thus, the maximal growths by the $W^{s,\alpha}$-method are

$$F(s) = |s|^{\frac{N}{N-2}}, \quad \text{and} \quad G(s) \sim e^{|s|^{\frac{N}{N-t}}},\ 0 < t < 2,$$

i.e. we have found a *variable* "critical growth" which depends on the choice of the value of t!

We emphasize once more that we have found for system (7.1), by using the *same functional* $I(u,v)$ in changing function space settings, different maximal growths for the nonlinearity $G(s)$. This is of course *somewhat disturbing*!

7.2. *Lorentz spaces*

We now change from Sobolev space settings to *Lorentz spaces*.

We recall the definition of a Lorentz space: For $\phi : \Omega \to \mathbb{R}$ a measurable function, we denote by

$$\mu_\phi(t) = \left|\left\{x \in \Omega:\ \phi(x) > t\right\}\right|, \quad t \geqslant 0,$$

its *distribution function*. The *decreasing rearrangement* $\phi^*(s)$ of ϕ is defined by

$$\phi^*(s) = \sup\left\{t > 0;\ \mu_\phi(t) > s\right\}, \quad 0 \leqslant s \leqslant |\Omega|.$$

The Lorentz space $L(p,q)$ is defined as follows:

$$\phi \in L(p,q), \quad 1 < p < \infty,\ 1 \leqslant q < \infty,$$

if

$$\|\phi\|_{p,q} = \left(\int_0^\infty \left[\phi^*(t) t^{1/p}\right]^q \frac{dt}{t} \right)^{1/q} < +\infty.$$

Lorentz spaces have the following main properties (see Adams and Fournier [1]):

(1) $L(p,p) = L^p,\ 1 < p < +\infty$.

(2) The following inclusions hold for $1 < q < p < r < \infty$:

$$L^r \subset L(p,1) \subset L(p,q) \subset L(p,p) = L^p \subset L(p,r) \subset L^q.$$

(3) Hölder inequality:

$$\left| \int_\Omega fg\,dx \right| \leqslant \|f\|_{p,q} \|g\|_{p',q'}, \quad \text{where } p' = \frac{p}{p-1},\ q' = \frac{q}{q-1}.$$

Furthermore, the following embedding theorems hold:

THEOREM A. *Suppose that* $1 \leqslant p < N$, *and that* $\nabla u \in L(p,q)$; *then* $u \in L(p^*, q)$, *where* $p^* = \frac{Np}{N-p}$ *and* $1 \leqslant q < \infty$.

Note that this theorem improves slightly Sobolev's embedding theorem, which gives $u \in L^{p^*} = L(p^*, p^*)$, which is a larger space than $L(p^*, p)$.

For the next theorem, see H. Brezis [8]:

THEOREM B. *Suppose that* $u \in W^{j,p}$, *with* $p < \frac{N}{j}$; *then* $u \in L(p^*, p)$ *with* $\frac{1}{p^*} = \frac{1}{p} - \frac{j}{N}$.

The following theorem deals with the limiting Sobolev case, and is a generalization of Trudinger's result (see H. Brezis and S. Wainger [10], H. Brezis [8]). It is of particular importance for our considerations:

THEOREM C. *Assume* $\nabla u \in L(N,q)$ *for some* $1 < q < \infty$. *Then* $e^{|u|^{\frac{q}{q-1}}} \in L^1$.

This generalizes the Trudinger embedding, which gives for $\nabla u \in L(N,N)$ that $e^{|u|^{\frac{N}{N-1}}} \in L^1$, i.e. the maximal growth $e^{|u|^{\frac{N}{N-1}}}$; we point out that in the Brezis–Wainger embedding the maximal growth depends only on the second Lorentz exponent q, but not on N.

We make the following

DEFINITION. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that $1 < p < \infty$, $1 < q < \infty$, and set

$$W_0^1 L(p,q)(\Omega) = cl\left\{u \in C_0^\infty(\Omega) \colon\ \|\nabla u\|_{p,q} < \infty\right\}.$$

On $W_0^1 L(p,q)$ we have the following norm

$$\|u\|_{1;p,q} := \|\nabla u\|_{p,q}$$

with which $W_0^1 L(p,q)$ becomes a reflexive Banach space.

One now has the following sharpening of Theorem C, in analogy to Moser's sharpening of the Trudinger inequality.

THEOREM D. *There exists* $\alpha_0 = \alpha_0(N, p, \Omega) > 0$ *such that*

$$\sup_{\|\nabla u\|_{L(N,q)}=1} \int_\Omega e^{\alpha |u|^{\frac{q}{q-1}}} < +\infty, \quad \textit{for } \alpha \leqslant \alpha_0.$$

7.3. *Lorentz spaces and the asymptotic borderline case*

We consider again the functional

$$I(u,v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega G(v) - \frac{N-2}{N} \int_\Omega |u|^{\frac{N}{N-2}} \, dx. \tag{7.5}$$

We want to consider the term $\int_\Omega \nabla u \nabla v \, dx$ on a product of Lorentz spaces, i.e. we want to estimate

$$\left| \int_\Omega \nabla u \nabla v \, dx \right| \leqslant \|\nabla u\|_{L(p,q)} \|\nabla v\|_{L(p',q')},$$

where we determine p, q and p', q' such that the last term in $I(u,v)$ is well defined, i.e.

$$u \in L^{\frac{N}{N-2}}(\Omega) = L\left(\frac{N}{N-2}, \frac{N}{N-2}\right)(\Omega).$$

Thus, $q = \frac{N}{N-2}$, and by Theorem A we obtain the following condition for p

$$\frac{N}{N-2} = p^* = \frac{Np}{N-p},$$

and hence

$$p = \frac{N}{N-1}.$$

Thus, we have to impose

$$\nabla u \in L\left(\frac{N}{N-1}, \frac{N}{N-2}\right).$$

Next, we calculate

$$p' = \frac{p}{p-1} = N \quad \text{and} \quad q' = \frac{q}{q-1} = \frac{N}{2},$$

and hence we get the condition

$$\nabla v \in L(p', q') = L\left(N, \frac{N}{2}\right).$$

By Theorem C above we now find for $\nabla v \in L(N, \frac{N}{2})$ that $e^{|v|^{\frac{N}{N-2}}} \in L^1(\Omega)$.

Thus we have

THEOREM 7.1. *The functional*

$$I(u, v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega G(v) - \frac{N-2}{N} \int_\Omega |u|^{\frac{N}{N-2}} \, dx$$

is well defined on the space

$$W_0^1 L\left(\frac{N}{N-1}, \frac{N}{N-2}\right)(\Omega) \times W_0^1 L\left(N, \frac{N}{2}\right)(\Omega),$$

and the maximal growth for $G(s)$ is given by

$$G(s) \sim e^{|v|^{\frac{N}{N-2}}}. \tag{7.6}$$

We remark that the growth (7.6) is considerably larger than the growth $G(s) \sim e^{|s|^{\frac{N}{N-1}}}$ found by the H^s-method in (7.4), and it corresponds to the limiting case $t \to 2$ in the $W^{s,\alpha}$-method (which however cannot be reached in that framework).

7.4. *Subcritical with respect to the asymptotic borderline case: existence of solution*

In this section we prove the existence of solutions for systems (7.1) in the “subcritical case”. Again, since we are interested in the superlinear case, we restrict to dimension $N = 3$, that is, we consider the system

$$\begin{cases} -\Delta u = g(v), & \\ -\Delta v = u^2 & \text{in } \Omega \subset \mathbb{R}^3, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.7}$$

As mentioned, the corresponding functional is

$$I(u, v) = \int_\Omega \nabla u \nabla v - \int_\Omega \frac{1}{3} |u|^3 - \int_\Omega G(v) \tag{7.8}$$

on the space

$$E := W_0^1 L\left(\frac{3}{2}, 3\right) \times W_0^1 L\left(3, \frac{3}{2}\right)$$

and we assume that $g(s)$ has a subcritical growth, i.e. an essentially lower growth than $e^{|s|^3}$. More precisely, we assume that

$$\lim_{s\to+\infty} \frac{g(s)}{e^{\alpha|s|^3}} = 0, \quad \forall \alpha > 0. \tag{7.9}$$

Furthermore, we make the following assumptions on g (our aim is to give simple assumptions, at the expense of greater generality):

(A1) g is a continuous function, with $g(s) = o(s)$ near the origin.

(A2) There exist constants $\nu > 2$ and $s_0 > 0$ such that

$$0 < \nu G(s) \leqslant s g(s), \quad \forall |s| > 0.$$

(A3) There exist constants $s_1 > 0$ and $M > 0$ such that

$$0 < G(s) \leqslant M g(s) \quad \text{for all } |s| \geqslant s_1.$$

EXAMPLE.

$$G(s) = e^{|s|^\beta} - 1 - |s|^\beta, \quad \text{with } 0 < \beta < 3.$$

THEOREM 7.2. (See [40].) *Under assumptions* (A1)–(A3) *and* (7.9), *system* (7.1) *has a nontrivial positive* (*weak*) *solution* $(u, v) \in E$.

PROOF. The proof follows ideas from Section 5.3. We remark that the proof is similar to the one in Section 9.2 below, and therefore we limit ourselves to give an outline of the main idea.

First, note that the functional I given in (7.8) is strongly indefinite. In order to overcome this difficulty, we introduce, as in Section 5.4, a suitable "tilde map":

(a) *The tilde map.*

Consider the bilinear map $\int_\Omega \nabla u \nabla v\, dx$ on the space $W_0^1 L(\frac{3}{2}, 3) \times W_0^1 L(3, \frac{3}{2})$. For $u \in W_0^1 L(\frac{3}{2}, 3)$ denote with $\tilde{u} \in W_0^1 L(3, \frac{3}{2})$ the unique element such that

$$\sup_{\{v\in W_0^1 L(3,\frac{3}{2}):\ \|\nabla v\|_{3,\frac{3}{2}} = \|\nabla u\|_{\frac{3}{2},3}\}} \int_\Omega \nabla u \nabla v\, dx = \int_\Omega \nabla u \nabla \tilde{u}\, dx = \|\nabla u\|_{\frac{3}{2},3} \|\nabla \tilde{u}\|_{3,\frac{3}{2}}$$

and hence

$$\int_\Omega \nabla u \nabla \tilde{u}\, dx = \|\nabla u\|^2_{\frac{3}{2},3} = \|\nabla \tilde{u}\|^2_{3,\frac{3}{2}}. \tag{7.10}$$

The existence and uniqueness of $\tilde{u}$ follows from the reflexivity and convexity of $W_0^1 L(\frac{3}{2}, 3)$, see [17].

We can thus define the "tilde-map": $W_0^1 L(\frac{3}{2}, 3) \to W_0^1 L(3, \frac{3}{2})$, $u \mapsto \tilde{u}$, which is continuous and positively homogeneous.

On the product space $E = W_0^1 L(\frac{3}{2}, 3) \times W_0^1(3, \frac{3}{2})$ we now define two continuous submanifolds

$$E^+ = \left\{(u, \tilde{u})\colon\ u \in W_0^1 L\left(\frac{3}{2}, 3\right)\right\}, \qquad E^- = \left\{(u, -\tilde{u})\colon\ u \in W_0^1 L\left(\frac{3}{2}, 3\right)\right\}.$$

As remarked in [17], the nonlinear manifolds E^+ and E^- have a *linear* structure with respect to the following notion of *tilde-sum*:

$$(u, \tilde{v}) \mathbin{\tilde{+}} (y, \tilde{z}) := (u + y, \widetilde{v + z}),$$

and one has

$$E = E^+ \mathbin{\tilde{\oplus}} E^-, \quad \text{with norm } \|w\|_E^2 = \big\|(u, \tilde{v})\big\|_E^2 = \|\nabla u\|_{3,\frac{3}{2}}^2 + \|\nabla \tilde{v}\|_{\frac{3}{2},3}^2.$$

(b) *Linking structure.*

Next, one verifies that the functional I has a "linking structure" in the origin. This is proved similarly as in the corresponding proof in Section 9.2 below, and we thus omit it here. For details, we refer to [40].

(c) *Palais–Smale sequences are bounded.*

Let $(u_n, \tilde{v}_n) \in E$ be a Palais–Smale sequence for the functional I, i.e. such that with $|I(u_n, \tilde{v}_n)| \leqslant d$, and $|I'(u_n, \tilde{v}_n)[(\phi, \tilde{\psi})]| \leqslant \varepsilon_n \|(\phi, \tilde{\psi})\|_E$, $\varepsilon_n \to 0$, $\forall (\phi, \tilde{\psi}) \in E$. Then $\|(u_n, \tilde{v}_n)\|_E \leqslant c$.

Again, the proof is similar to Section 9.2 below.

(d) *Finite-dimensional approximation.*

Since the functional I is strongly indefinite on the space E (i.e. positive and negative definite on infinite-dimensional manifolds), the standard linking theorems cannot be applied. We therefore proceed by finite-dimensional approximations (Galerkin procedure):

Denote by $(e_i)_{i\in\mathbb{N}}$ an orthonormal set of eigenfunctions corresponding to the eigenvalues (λ_i), $i \in \mathbb{N}$, of the Laplacian, with Dirichlet boundary conditions. Set

$$E_n^+ = \operatorname{span}\big\{(e_i, \tilde{e}_i) \bigm| i = 1, \ldots, n\big\}, \qquad E_n^- = \operatorname{span}\big\{(e_i, -\tilde{e}_i) \bigm| i = 1, \ldots, n\big\}$$

and

$$E_n = E_n^- \mathbin{\tilde{\oplus}} E_n^-.$$

Exploiting the local linking structure given in (b), it is now standard to conclude that for each $n \in \mathbb{N}$ the functional $I_n := I|_{E_n}$ has a critical point $z_n = (u_n, \tilde{v}_n) \in E_n$ at level c_n, with

$$I(z_n) = c_n \in [\sigma_0, \sigma_1], \quad \sigma_i > 0,\ i = 1, 2, \tag{7.11}$$

and $I'(z_n)[(\phi, \tilde{\psi})] = 0$, for all $(\phi, \tilde{\psi}) \in E_n$. Hence we have

$$\begin{cases} \int_\Omega \nabla u_n \nabla \tilde{\psi} = \int_\Omega g(\tilde{v}_n)\tilde{\psi}, \\ \int_\Omega \nabla \tilde{v}_n \nabla \phi = \int_\Omega u_n^2 \phi, \end{cases} \quad \forall (\phi, \tilde{\psi}) \in E_n. \tag{7.12}$$

(e) *Limit* $n \to \infty$.

It remains to pass to the limit $n \to \infty$. By (c) we have that $\|(u_n, \tilde{v}_n)\|_E \leqslant c$, and hence $(u_n, \tilde{v}_n) \rightharpoonup (u, \tilde{v})$ in E. Furthermore, we may assume that

$$\tilde{v}_n \to \tilde{v} \quad \text{in } L^\alpha, \text{ for all } \alpha \geqslant 1; \tag{7.13}$$

indeed, by the properties (1) and (2) of Lorentz spaces, see Section 7.2, we have the following continuous embeddings

$$W_0^1 L\left(3, \frac{3}{2}\right) \subset W_0^1 L(3-\delta, 3-\delta) = W_0^{1,3-\delta} \subset L^{\frac{(3-\delta)N}{N-(3-\delta)}} = L^{\frac{(3-\delta)3}{\delta}}, \quad \text{for } \delta > 0,$$

and hence we have a compact embedding into L^α, for all $1 \leqslant \alpha < \frac{(3-\delta)3}{\delta}$.

Proceeding as in Section 9.2 below, one concludes that by taking the limit $n \to \infty$

$$\begin{cases} \int_\Omega \nabla u \nabla \tilde{\psi} = \int_\Omega g(\tilde{v})\tilde{\psi}, \\ \int_\Omega \nabla \tilde{v} \nabla \phi = \int_\Omega u^2 \phi, \end{cases} \quad \forall (\phi, \tilde{\psi}) \in \bigcup E_n = E, \tag{7.14}$$

i.e. $(u, \tilde{v}) \in E$ is a (weak) solution of (7.14).

Finally, we prove that $(u, \tilde{v}) \in E$ is nontrivial. If we assume to the contrary that $u = 0$, then by (7.14) also $\tilde{v} = 0$. Since g is subcritical, we obtain by (7.9) that for all $\delta > 0$

$$\left|g(t)\right| \leqslant c_\delta e^{\delta |t|^3}, \quad \forall t \in \mathbb{R}.$$

Now we choose $\tilde{\psi} = \tilde{v}_n$ in the first equation of (7.14), and estimate by Hölder

$$\left| \int_\Omega g(\tilde{v}_n)\tilde{v}_n \right| \leqslant c_\delta \left\| e^{\delta |\tilde{v}_n|^3} \right\|_{L^\beta} \|\tilde{v}_n\|_{L^\alpha} \leqslant d_\delta \|\tilde{v}_n\|_{L^\alpha}, \tag{7.15}$$

where we have used that $\|\nabla \tilde{v}_n\|_{3,\frac{3}{2}} \leqslant c$, and hence by Theorem C above, for $\beta > 1$ sufficiently small:

$$\left\| e^{\delta |\tilde{v}_n|^3} \right\|_{L^\beta} = \int_\Omega e^{\delta\beta |\tilde{v}_n|^3} \leqslant c.$$

Since by (7.13) $\|\tilde{v}_n\|_{L^\alpha} \to 0$, and hence by (7.15) $|\int_\Omega g(\tilde{v}_n)\tilde{v}_n| \to 0$, we conclude by the first equation in (7.12), by multiplication by $\tilde{v}_n$ and integration, that

$$\int_\Omega \nabla u_n \nabla \tilde{v}_n = \int_\Omega g(\tilde{v}_n)\tilde{v}_n \to 0. \tag{7.16}$$

This in turn implies, by choosing $\phi = u_n$ in the second equation in (7.12), that also $\int_\Omega |u_n|^3 \to 0$, and by assumption (A2) follows that also $\int_\Omega G(\tilde{v}_n) \to 0$. This implies finally that $I(u_n, \tilde{v}_n) = \int_\Omega \nabla u_n \nabla \tilde{v}_n - \int_\Omega (\frac{1}{3}|u_n|^3 + G(\tilde{v}_n)) \to 0$; but this contradicts (7.11), and thus $(u, \tilde{v}) \neq (0, 0)$.

This completes the proof. □

8. Critical phenomena for the system

As pointed out for the scalar equation, critical growth is connected with several interesting phenomena. We now discuss these properties for the system

$$\begin{cases} -\Delta u = v^{q+1}, & \\ -\Delta v = u^{p+1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

8.1. *Critical growth: nonexistence of (positive) solutions*

First we present a result of E. Mitidieri [29] (see also van der Vorst [45]) which gives nonexistence of *positive* solutions for system (4.1) if the nonlinearities have critical growth.

THEOREM 8.1. *Let $\Omega \subset \mathbb{R}^N$ be a starshaped domain. Assume that*

$$\frac{1}{p+1} + \frac{1}{q+1} \leqslant 1 - \frac{2}{N}. \tag{8.2}$$

Then system (8.1) *has no positive solution.*

PROOF. The proof relies on a Pohozaev type identity which is a modification of an identity by Pucci and Serrin [37].

LEMMA 8.2. (See Corollary 2.1 and (2.5) in [29].) *Let (u, v) (with $u, v \in C^2(\overline{\Omega})$) be a solution of system* (8.1). *Then*

$$\begin{aligned} &\int_\Omega \big(\Delta u(x, \nabla v) + \Delta v(x, \nabla u)\big)\, dx \\ &\quad = \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}(x, \nabla v) + \frac{\partial v}{\partial n}(x, \nabla u) - (\nabla u, \nabla v)(x, n) \right) ds \\ &\qquad + (N-2) \int_\Omega (\nabla u, \nabla v)\, dx. \end{aligned} \tag{8.3}$$

Applying this identity to system (8.1) we obtain

$$N\int_\Omega\left(\frac{1}{p+1}|u|^{p+1}+\frac{1}{q+1}|v|^{q+1}\right)dx = (N-2)\int_\Omega(\nabla u,\nabla v)\,dx+\int_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n)\,ds. \tag{8.4}$$

On the other hand, multiplying the first equation of (8.1) by v and the second by u and integrating, we obtain, for any $a\in(0,1)$

$$\int_\Omega\nabla u\nabla v=a\int_\Omega|v|^{q+1}\,dx+(1-a)\int_\Omega|u|^{p+1}\,dx. \tag{8.5}$$

Now choose

$$a=\frac{N}{(N-2)(p+1)}$$

and hence by (8.2)

$$1-a\geqslant\frac{N}{(N-2)(q+1)}$$

which yields

$$(N-2)\int_\Omega(\nabla u,\nabla v)\,dx\geqslant N\int_\Omega\left(\frac{1}{p+1}|u|^{p+1}+\frac{1}{q+1}|v|^{q+1}\right)dx.$$

Hence, by (8.4), we get

$$0\geqslant\int_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n)\,ds.$$

Since u and v are positive solutions, we have by the maximum principle that

$$\frac{\partial u}{\partial n}<0\quad\text{and}\quad\frac{\partial v}{\partial n}<0\quad\text{on }\partial\Omega,$$

and since $(x,n)>0$ by the assumption that Ω is starshaped, we obtain a contradiction. □

8.2. *Critical growth: noncompactness and concentration*

We discuss further "phenomena of critical growth" for the model system (8.1). We first remark that by "solving" the first equation for v (assuming that $v>0$) we get $v=(-\Delta u)^{1/q}$, and then we obtain by the second equation

$$\begin{cases}-\Delta((-\Delta u)^{1/q})=u^p & \text{in }\Omega,\\ u=\Delta u=0 & \text{on }\partial\Omega,\end{cases} \tag{8.6}$$

which is equivalent to system (8.1).

The variational formulation for equation (8.6) leads to the minimization problem

$$\inf_{u\in S_{p+1}} \int_\Omega |\Delta u|^{\frac{q+1}{q}}\, dx, \tag{8.7}$$

where

$$S_{p+1} = \left\{u \in W^{2,\frac{q+1}{q}}(\Omega) \;\middle|\; u = \Delta u = 0 \text{ on } \partial\Omega,\ \|u\|_{L^{p+1}} = 1\right\}.$$

The Sobolev embedding theorem gives in this case

$$W^{2,\frac{q+1}{q}}(\Omega) \subset L^{p+1}(\Omega), \quad \text{for } p+1 \leqslant \frac{\frac{q+1}{q}N}{N - 2\frac{q+1}{q}}$$

which is equivalent to

$$\frac{1}{p+1} + \frac{1}{q+1} \geqslant 1 - \frac{2}{N},$$

i.e. the critical hyperbola!

We note that by Theorem 8.1 the infimum in (8.7) cannot be attained in starshaped domains, since otherwise there would exist a solution to system (8.1).

To understand the concentration behavior of minimizing sequences, it is important to study the situation in $\mathbb{R}^N$, as described in the case of the Laplacian in Section 2.4. One has the following result by P.L. Lions [31], which is proved by the methods of *concentration-compactness*:

THEOREM 8.3. *Let*

$$m = \inf\left\{\int_{\mathbb{R}^N} |\Delta u|^{\frac{q+1}{q}}\, dx \;\middle|\; u \in \mathcal{D}^{2,\frac{q+1}{q}},\ \int_{\mathbb{R}^N} |u|^{p+1} = 1\right\}, \tag{8.8}$$

where $\mathcal{D}^{2,\frac{q+1}{q}}$ *is the completion of* $C_0^\infty(\mathbb{R}^N)$ *in the norm* $\|\Delta\cdot\|_{\frac{q+1}{q}}$. *Then every minimizing sequence* (u_n) *of* (8.8) *is relatively compact in* $\mathcal{D}^{2,\frac{q+1}{q}}$ *up to translation and dilation, i.e. there exist* $(y_n) \in \mathbb{R}^N$, $(\sigma_n) \in (0,\infty)$ *such that the new minimizing sequence*

$$\tilde{u}_n = \sigma_n^{-N/(p+1)} u_n(\cdot - y_n/\sigma_n)$$

is relatively compact in $\mathcal{D}^{2,\frac{q+1}{q}}$. *In particular, there exists a minimum of* (8.8).

8.3. *Critical growth: instantons*

The fact that the infimum m in (8.8) is attained, and that there is a dilation invariance, says that we have again a family of *instantons*, given by the minimizers of (8.8). However, in contrast to the case of the Laplacian described in Section 2.4, these instantons are not explicitly known. In [23] Hulshof and van der Vorst show that the minimizer (ground state) of (8.8) is (up to translation and dilation) unique; furthermore, they show that the ground state is positive, radially symmetric and decreasing in r. Thus, all ground states of (8.8) are given by

$$u_{\epsilon,x_0} = \epsilon^{-\frac{N}{p+1}} u\left(\frac{x - x_0}{\epsilon}\right),$$

where u is the unique "normalized" ground state with $u(0) = 1$. Furthermore, Hulshof and van der Vorst [24] derive the precise asymptotic behavior of the normalized ground state.

8.4. *Brezis–Nirenberg type results for systems*

In Section 2.4 we have discussed the famous result of Brezis and Nirenberg [9], in which it is shown that the existence of solutions for elliptic equation with critical growth terms can be recovered by adding a suitable lower order perturbation. In the proof it is crucial to have the explicit form of the instantons in order to prove by explicit estimates that due to the perturbation the noncompactness level of the functional is avoided.

Hulshof, Mitidieri and van der Vorst [25] consider the corresponding problem for the following perturbed critical system

$$\begin{cases} -\Delta u = \mu v + v^{q+1}, & \\ -\Delta v = \lambda u + u^{p+1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.9}$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{8.10}$$

They give (rather complicated) conditions on μ and λ such that system (8.9) has nontrivial solutions. The proof is based on the *dual variational method*, cf. Ekeland and Temam [12]. To show that the noncompactness levels of the corresponding functional are avoided, they use the asymptotic behavior of the normalized ground state mentioned above.

9. Systems in two dimensions

We have seen in Section 3 that critical growth for the scalar equation in dimension $N = 2$ is governed by the Trudinger embedding and is of exponential type. For systems in two dimensions, one would like to obtain in analogy to the critical hyperbola in $N \geqslant 3$ a "critical curve" describing the maximal growth for the combined nonlinearities. This can be obtained using again *Lorentz spaces*.

9.1. *Exponential critical hyperbola*

In this section we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain, and consider again the system

$$\text{(S2)} \qquad \begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = 0 \quad \text{and} \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

For the scalar equation $-\Delta u = f(u)$ in Ω, $u = 0$ on $\partial\Omega$, critical growth is given by the Trudinger–Moser inequality, i.e. $F(u) \sim e^{|u|^2}$. For the system, we look for an analogue of the critical hyperbola for $N \geqslant 3$. By considering the functional

$$J(u, v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega F(u) \, dx - \int_\Omega G(v) \, dx$$

on the space $H_0^1 \times H_0^1$ one sees (as in the scalar case) that the nonlinearities $G(v) \sim e^{|v|^2}$ and $F(u) \sim e^{|u|^2}$ lie on this "critical curve". We assume now that $F(t)$ and $G(t)$ have "exponential polynomial growth", i.e.

$$F(t) \sim e^{|t|^p}, \quad \text{and} \quad G(t) \sim e^{|t|^q}, \quad \text{for some } 1 < p, q < +\infty.$$

We look for a relation between p and q such that the pair (F, G) becomes critical. We prove:

THEOREM 9.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then we have an "exponential critical curve" given by*

$$\big(F(s), G(s)\big) = \big(e^{|s|^p}, e^{|s|^q}\big), \quad \textit{with } \frac{1}{p} + \frac{1}{q} = 1.$$

REMARK 9.2. One might try, as in the case $N \geqslant 3$, to work with spaces like $W^{1,\alpha} \times W^{1,\beta}$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and, e.g., $0 < \alpha < 2$. However, note that then

$$W^{1,\alpha}(\Omega) \subset L^r(\Omega), \quad \text{with } r = \frac{\alpha N}{N - \alpha},$$

and furthermore, since $\beta = \frac{\alpha}{\alpha - 1} > 2$, we get

$$W_0^{1,\beta}(\Omega) \subset L^\infty(\Omega),$$

i.e. we find a maximal growth of polynomial type for F,

$$\big|F(s)\big| \leqslant |s|^{\frac{N\alpha}{N-\alpha}},$$

and *no growth restriction* on the nonlinearity $G(s)$. So, this choice of spaces brings us for any $(\alpha,\beta)\neq(2,2)$ immediately outside the range of exponential nonlinearities.

Thus, to treat the problem, we need to work with spaces which lie "very close" to the space $W^{1,2}$, that is, we look for an "interpolation space" which lies between $W^{1,2}$ and $W^{1,2+\delta}$, for every $\delta>0$. We have introduced such a class of spaces in Section 7.2, namely the *Sobolev–Lorentz spaces*.

PROOF. We consider the functional

$$J(u,v)=\int_\Omega \nabla u\nabla v\,dx-\int_\Omega F(u)-\int_\Omega G(v). \tag{9.1}$$

We want to consider the term $\int_\Omega \nabla u\nabla v\,dx$ on a product of Lorentz spaces, i.e. we want to estimate, using the Hölder inequality on Lorentz spaces:

$$\left|\int_\Omega \nabla u\nabla v\,dx\right| \leqslant \|\nabla u\|_{L(2,q)}\|\nabla v\|_{L(2,q')}, \qquad \frac{1}{q}+\frac{1}{q'}=1.$$

By Theorem C, Section 7.2, we have that

$$u\in W^1L(2,q) \quad\Rightarrow\quad e^{|u|^{\frac{q}{q-1}}}\in L^1, \quad \text{and}$$

$$v\in W^1L(2,q') \quad\Rightarrow\quad e^{|v|^{\frac{q'}{q'-1}}}\in L^1.$$

Thus the maximal growth allowed for $F(s)=\int_0^s f(t)\,dt$ and $G(s)=\int_0^s g(t)\,dt$ is given by

$$F(u)\sim e^{|u|^p}, \quad p=q'=\frac{q}{q-1}, \quad \text{and} \quad G(v)\sim e^{|v|^q}.$$

□

9.2. *Existence for the subcritical problem in dimension $N=2$*

In this section we show that for *subcritical growth* with respect to the exponential critical hyperbola defined in Theorem 9.1 we have compactness, and hence existence of solutions. We make the following assumptions on f and g (our aim is to give simple assumptions, at the expense of greater generality):

(A1) f and g are continuous functions, with $f(s)=o(s)$ and $g(s)=o(s)$ near the origin.

(A2) There exist constants $\mu>2$, $\nu>2$ and $s_0>0$ such that

$$0<\mu F(s)\leqslant sf(s), \quad \text{and} \quad 0<\nu G(s)\leqslant sg(s), \quad \forall |s|>0.$$

(A3) There exist constants $s_1 > 0$ and $M > 0$ such that

$$0 < G(s) \leqslant Mg(s) \quad \text{for all } |s| \geqslant s_1,$$
$$0 < F(s) \leqslant Mf(s) \quad \text{for all } |s| \geqslant s_1,$$

We remark that assumption (A3) implies that f and g have at least exponential growth.

(A4) f has at most critical growth, i.e. there exist constants a_1, a_2 and d such that

$$f(s) \leqslant a_1 + a_2 e^{d|s|^p}.$$

(A5) g is subcritical, i.e. for all $\delta > 0$ holds:

$$\lim_{|s|\to\infty} \frac{g(s)}{e^{\delta|s|^q}} = 0, \quad q = \frac{p}{p-1}.$$

EXAMPLE.

$$F(s) = e^{|s|^p} - 1 - |s|^p, \qquad G(s) = e^{|s|^\beta} - 1 - |s|^\beta, \quad \text{with } 1 < \beta < q.$$

We consider the space $E = W_0^1 L(2,q) \times W_0^1 L(2,p)$.

THEOREM 9.3. *Under assumptions* (A1)–(A5), *system* (S2) *has a nontrivial positive* (*weak*) *solution* $(u, v) \in E$.

PROOF. The proof follows the lines of Section 5.4 (see also [16] and [17]). □

We note that the functional J given in (9.1) is strongly indefinite in the origin, being positive and negative definite on infinite-dimensional subspaces. If working on a Hilbert space H, one can find suitable subspaces $H^+ \oplus H^- = H$ such that $I|_{H^+}$ is positive definite and $J|_{H^-}$ is negative definite near the origin. Since we are working on the Banach space E, the subspaces H^+, H^- have to be replaced by infinite-dimensional manifolds. As in (5.10) we define

(a) *The tilde map*.

Consider the bilinear map $\int_\Omega \nabla u \nabla v\, dx$ on the space $W_0^1 L(2,q) \times W_0^1 L(2,p)$, where $p = \frac{q}{q-1}$.

For $u \in W_0^1 L(2,q)$ denote with $\tilde{u} \in W_0^1 L(2,p)$ the unique element such that

$$\sup_{\{v \in W_0^1 L(2,p):\ \|\nabla v\|_{2,p} = \|\nabla u\|_{2,q}\}} \int_\Omega \nabla u \nabla v\, dx = \int_\Omega \nabla u \nabla \tilde{u}\, dx = \|\nabla u\|_{2,q} \|\nabla \tilde{u}\|_{2,p}$$

and hence

$$\int_\Omega \nabla u \nabla \tilde{u}\, dx = \|\nabla u\|_{2,q}^2 = \|\nabla \tilde{u}\|_{2,p}^2. \tag{9.2}$$

The existence and uniqueness of $\tilde{u}$ follows from the reflexivity and convexity of $W_0^1 L(2,q)$, see [17].

We can thus define the "tilde-map": $W_0^1 L(2,q) \to W_0^1 L(2,p)$, $u \mapsto \tilde{u}$. This map is clearly continuous; it is nonlinear, but positively homogeneous: $\widetilde{\rho u} = \rho \tilde{u}$, for all $\rho \geqslant 0$.

On the product space $E = W_0^1 L(2,q) \times W_0^1(2,p)$ we can now define two continuous submanifolds

$$E^+ = \{(u,\tilde{u}){:}\ u \in W_0^1 L(2,q)\}, \qquad E^- = \{(u,-\tilde{u}){:}\ u \in W_0^1 L(2,q)\}.$$

As remarked in Lemma 5.14, the nonlinear manifolds E^+ and E^- have a *linear* structure with respect to the following notion of *tilde-sum*:

$$(u,\tilde{v}) \mathbin{\tilde{+}} (y,\tilde{z}) := (u+y, \widetilde{v+z}),$$

and one has

$$E = E^+ \mathbin{\tilde{\oplus}} E^-, \quad \text{with norm } \|w\|_E^2 = \|(u,\tilde{v})\|_E^2 = \|\nabla u\|_{2,q}^2 + \|\nabla \tilde{v}\|_{2,p}^2.$$

(b) *Linking structure*.

Next, we verify that the functional $J(u,v)$ has a linking structure in $(0,0)$:

First, using (A1) and (A4) one estimates, for given $\epsilon > 0$

$$F(s) \leqslant \epsilon s^2 + c|s|^3 e^{d|s|^p}, \qquad G(s) \leqslant \epsilon s^2 + c|s|^3 e^{d|s|^q}, \quad \forall s \in \mathbb{R}.$$

CLAIM 1. There exist $\rho > 0$ and $\sigma > 0$ such that $J(u,\tilde{u}) \geqslant \sigma$ for $\|(u,\tilde{u})\|_E = \rho$; indeed, using (7.10) and

$$\begin{aligned} J(u,\tilde{u}) &= \int_\Omega \nabla u \nabla \tilde{u} - \int_\Omega F(u) - \int_\Omega G(\tilde{u}) \\ &\geqslant \frac{1}{2}\|\nabla u\|_{2,q}^2 - \varepsilon \int_\Omega |u|^2 - c\|u\|_6^3 \left(\int_\Omega e^{2d|u|^p}\right)^{1/2} \\ &\quad + \frac{1}{2}\|\nabla \tilde{u}\|_{2,p}^2 - \varepsilon \int_\Omega |\tilde{u}|^2 - c\|\tilde{u}\|_6^3 \left(\int_\Omega e^{2d|\tilde{u}|^q}\right)^{1/2}. \end{aligned}$$

Now, we use that by Theorem D, Section 7.2

$$\|u\|_6 \leqslant d_3 \|\nabla u\|_{2,q}, \quad \text{and} \quad \int_\Omega e^{2|u|^p} \leqslant c, \quad \text{if } \|\nabla u\|_{2,q} \leqslant \theta_1,$$

$$\|\tilde{u}\|_6 \leqslant d_4 \|\nabla \tilde{u}\|_{2,p}, \quad \text{and} \quad \int_\Omega e^{2|\tilde{u}|^q} \leqslant c, \quad \text{if } \|\nabla \tilde{u}\|_{2,p} \leqslant \theta_2.$$

With these estimates the claim follows easily.

Next, fix $e_1 \in W_0^1 L(2,q)$ and $\tilde{e}_1 \in W_0^1 L(2,p)$ with $\|\nabla e_1\|_{2,q} = \|\nabla \tilde{e}_1\|_{2,p} = 1$, and let

$$Q = \{r(e_1,\tilde{e}_1) \mathbin{\tilde{+}} w;\ w \in E^-,\ \|w\|_E \leqslant R_0,\ 0 \leqslant r \leqslant R_1\}.$$

CLAIM 2. There exist $R_0, R_1 > 0$ such that $J(z) \leqslant 0$, $\forall z \in \partial Q$, where ∂Q denotes the boundary of Q in $\mathbb{R}(e_1, \tilde{e}_1) \widetilde{+} E^-$:

(i) For $(u, \tilde{v}) \in \partial Q \cap E^-$ we have $(u, \tilde{v}) = (u, -\tilde{u})$ and hence

$$J(u, -\tilde{u}) = -\int_\Omega \nabla u \nabla \tilde{u} - \int F(u) - \int G(-\tilde{u}) \leqslant -\|\nabla u\|_{2,q}^2 \leqslant 0.$$

(ii) Let $(u, \tilde{v}) = r(e_1, \tilde{e}_1) \widetilde{+} (w, -\tilde{w}) = (re_1 + w, \widetilde{re_1 - w}) \in \partial Q$, with $\|(w, -\tilde{w})\|_E = R_0, 0 \leqslant r \leqslant R_1$.

First set $R_1 = 1$. Then

$$\begin{aligned} J(u, \tilde{v}) &\leqslant \int_\Omega \nabla(re_1 + w)\nabla(\widetilde{re_1 - w}) \\ &= \int_\Omega \nabla(w - re_1)\nabla(\widetilde{w - re_1}) - \int_\Omega \nabla(2re_1)\nabla(\widetilde{w - re_1}) \\ &\leqslant -\big\|\nabla(w - re_1)\big\|_{2,q}^2 + 2\|\nabla re_1\|_{2,q}\big\|\nabla(\widetilde{w - re_1})\big\|_{2,p} \\ &\leqslant -\|\nabla w\|_{2,q}^2 - \|\nabla re_1\|_{2,q}^2 + 2\|\nabla w\|_{2,q}\|\nabla re_1\|_{2,q} \\ &\quad + 2\|\nabla re_1\|_{2,q}\big(\|\nabla w\|_{2,q} + \|\nabla re_1\|_{2,q}\big) \\ &\leqslant -\|\nabla w\|_{2,q}^2 + 4r\|\nabla w\|_{2,q} + r^2 \leqslant 0, \end{aligned}$$

for $2\|\nabla w\|_{2,q}^2 = \|\nabla w\|_{2,q}^2 + \|\nabla \tilde{w}\|_{2,p}^2 = \|(w, -\tilde{w})\|_E^2 = \overline{R}_0^2$ sufficiently large.

Note that this estimate now holds for all $\rho \geqslant 1$, with $0 \leqslant r \leqslant \rho$ and $\|(w, -\tilde{w})\|_E^2 = \rho\overline{R}_0$.

(iii) Let $z = \rho(e_1, \tilde{e}_1) \widetilde{+} \rho(w, -\tilde{w}) \in \partial Q$, with $\|(w, -\tilde{w})\|_E \leqslant \overline{R}_0$. Then by (A2), for $\theta = \min\{\mu, \nu\} > 2$

$$\begin{aligned} J(u, \tilde{v}) &= \int_\Omega \nabla(\rho e_1 + \rho w)\nabla(\widetilde{\rho e_1 - \rho w}) - \int_\Omega F(\rho e_1 + \rho w) + G(\widetilde{\rho e_1 - \rho w}) \\ &\leqslant \rho^2\big\|\nabla(e_1 + w)\big\|_{2,q}\big\|\nabla(e_1 - w)\big\|_{2,q} - c\int_\Omega |\rho e_1 + \rho w|^\theta + c_1 \\ &\quad - c\int_\Omega |\widetilde{\rho e_1 - \rho w}|^\theta + c_1 \\ &\leqslant \rho^2\big(\|\nabla e_1\|_{2,q} + \|\nabla w\|_{2,q}\big)^2 \\ &\quad - c\rho^\theta\left\{\int_\Omega |e_1 + w|^\theta + \int_\Omega |\widetilde{e_1 - w}|^\theta\right\} + 2c_1. \end{aligned}$$

It follows that

$$J(u, \tilde{v}) \leqslant \rho^2(1 + \overline{R}_0)^2 - c\rho^\theta\delta_0 + 2c_1 \leqslant 0 \tag{9.3}$$

for $\rho \geqslant R_1$ sufficiently large, where

$$\delta_0 = \inf_{\|(w,-\tilde{w})\|_E \leqslant \overline{R}_0} \left\{ \int_\Omega |e_1 + w|^\theta + \int_\Omega |\widetilde{e_1 - w}|^\theta \right\} > 0;$$

indeed, if $\delta_0 = 0$ we would find a sequence w_n with $\|(w_n, -\tilde{w}_n)\| \leqslant \overline{R}^0$ and $\int_\Omega |e_1 + w_n|^\theta + |\widetilde{e_1 - w_n}|^\theta \to 0$. By the compact embeddings $W^1L(2,q) \subset L^\theta$ and $W^1\widetilde{L(2,p)} \subset \widetilde{L^\theta}$ we get strongly convergent subsequences $e_1 + w_n \to e_1 + w = 0$ and $\widetilde{e_1 - w_n} \to \widetilde{e_1 - w} = 0$, i.e. $w = e_1$ and $w = -e_1$: contradiction.

Finally, defining $R_0 = R_1\overline{R}_0$, the claim holds.

(c) *Palais–Smale sequences are bounded*.

Let $(u_n, \tilde{v}_n) \in E$ with $|J(u_n, \tilde{v}_n)| \leqslant d$, and

$$\left|J'(u_n, \tilde{v}_n)\big[(\phi, \tilde{\psi})\big]\right| \leqslant \varepsilon_n \left\|(\phi, \tilde{\psi})\right\|_E, \quad \varepsilon_n \to 0,\ \forall(\phi, \tilde{\psi}) \in E. \tag{9.4}$$

Then $\|(u_n, \tilde{v}_n)\|_E \leqslant c$.

Indeed, choosing $(\phi, \tilde{\psi}) = (u_n, \tilde{v}_n) = z_n$ in (9.4) we get, using (A2)

$$\begin{aligned}\int_\Omega f(u_n)u_n + \int_\Omega g(\tilde{v}_n)\tilde{v}_n &\leqslant 2\left|\int_\Omega \nabla u_n \nabla \tilde{v}_n\right| + \varepsilon_n \left\|(u_n, \tilde{v}_n)\right\|_E \\ &\leqslant 2d + 2\int_\Omega F(u_n) + 2\int_\Omega G(\tilde{v}_n) + \varepsilon_n \left\|(u_n, \tilde{v}_n)\right\|_E \\ &\leqslant 2d + \frac{2}{\mu}\int_\Omega f(u_n)u_n \\ &\quad + \frac{2}{\nu}\int_\Omega g(\tilde{v}_n)\tilde{v}_n + \varepsilon_n \left\|(u_n, \tilde{v}_n)\right\|_E\end{aligned}$$

from which we get

$$\int_\Omega f(u_n)u_n \leqslant c + \varepsilon_n \left\|(u_n, \tilde{v}_n)\right\|_E, \qquad \int_\Omega g(\tilde{v}_n)\tilde{v}_n \leqslant c + \varepsilon_n \left\|(u_n, \tilde{v}_n)\right\|_E \tag{9.5}$$

and then also

$$\int_\Omega F(u_n) \leqslant c, \qquad \int_\Omega G(\tilde{v}_n) \leqslant c. \tag{9.6}$$

Next, taking $(\phi, \tilde{\psi}) = (v_n, 0)$ and $(\phi, \tilde{\psi}) = (0, \tilde{u}_n)$ in (9.4) we have

$$\|\nabla v_n\|_{2,q}^2 \leqslant \int_\Omega f(u_n)v_n + \varepsilon_n \left\|(v_n, 0)\right\|_E, \tag{9.7}$$

and

$$\|\nabla \tilde{u}_n\|_{2,p}^2 \leqslant \int_\Omega g(\tilde{v}_n)\tilde{u}_n + \varepsilon_n \big\|(0,\tilde{u}_n)\big\|_E. \tag{9.8}$$

Setting $V_n = \frac{v_n}{\|\nabla v_n\|_{2,q}}$ and $\widetilde{U}_n = \frac{\tilde{u}_n}{\|\nabla \tilde{u}_n\|_{2,p}}$ we obtain

$$\|\nabla v_n\|_{2,q} \leqslant \int_\Omega f(u_n)V_n + \varepsilon_n \quad \text{and} \quad \|\nabla \tilde{u}_n\|_{2,p} \leqslant \int_\Omega g(\tilde{v}_n)\widetilde{U}_n + \varepsilon_n. \tag{9.9}$$

We now use the following inequality: for any $\alpha > 1$ (and setting $\alpha' = \frac{\alpha}{\alpha-1}$) holds:

$$st \leqslant \begin{cases} (e^{t^\alpha} - 1) + s(\log^+ s)^{1/\alpha}, & \text{for all } t \geqslant 0 \text{ and } s \geqslant e^{(\frac{1}{\alpha})^{\alpha'}}, \qquad \text{(I)} \\ (e^{t^\alpha} - 1) + \frac{\alpha-1}{\alpha^{\alpha'}} s^{\alpha'}, & \text{for all } t \geqslant 0 \text{ and } 0 \leqslant s \leqslant e^{(\frac{1}{\alpha})^{\alpha'}}. \qquad \text{(II)} \end{cases} \tag{9.10}$$

PROOF. For fixed $s > 0$, consider $\sup_{t\geqslant 0}\{st - (e^{t^\alpha} - 1)\}$, and let t_s denote the (unique) point where the supremum is attained; then $s = \alpha t_s^{\alpha-1} e^{t_s^\alpha}$. We consider the cases:

(i) $t_s \geqslant (\frac{1}{\alpha})^{\frac{1}{\alpha-1}}$: then $s = \alpha t_s^{\alpha-1} e^{t_s^\alpha} \geqslant e^{t_s^\alpha}$ and hence $(\log s)^{\frac{1}{\alpha}} \geqslant t_s$, and then

$$\sup_{t\geqslant 0}\big\{st - (e^{t^\alpha} - 1)\big\} = st_s - \big(e^{t_s^\alpha} - 1\big) \leqslant st_s \leqslant s(\log s)^{1/\alpha}, \text{ hence (I).}$$

(ii) $0 \leqslant t_s \leqslant (\frac{1}{\alpha})^{\frac{1}{\alpha-1}}$ and $s \geqslant e^{(\frac{1}{\alpha})^{\alpha'}}$: then $st_s \leqslant s(\frac{1}{\alpha})^{\frac{1}{\alpha-1}} \leqslant s(\log^+ s)^{\frac{1}{\alpha}}$, by the assumption on s, hence (I).

(iii) $0 \leqslant t_s \leqslant (\frac{1}{\alpha})^{\frac{1}{1-\alpha}}$ and $s \leqslant e^{(\frac{1}{\alpha})^{\alpha'}}$: then (II) holds; in fact (II) holds always, since as in (I) $st \leqslant t^\alpha + \frac{\alpha-1}{\alpha^{\alpha'}} s^{\alpha'} \leqslant (e^{t^\alpha} - 1) + \frac{\alpha-1}{\alpha^{\alpha'}} s^{\alpha'}$, for all $s, t \geqslant 0$.

We apply the above inequality to the estimates in (9.9).

Applying inequality (9.10) with $\alpha = p$ and $t = |V_n(x)|$, $s = |f(u_n(x))|$, to the first estimate in (9.9):

$$\begin{aligned} \|\nabla v_n\|_{2,q} &\leqslant \int_\Omega f(u_n)V_n + \varepsilon_n \\ &\leqslant \int_\Omega \big(e^{|V_n|^p} - 1\big) + \int_\Omega \big|f(u_n)\big|\big[\log^+\big|f(u_n)\big|\big]^{1/p} + \varepsilon_n \\ &\leqslant c + \int_\Omega f(u_n)u_n + \varepsilon_n \\ &\leqslant c + \varepsilon_n + \varepsilon_n \big\|(u_n, \tilde{v}_n)\big\|_E. \end{aligned}$$

Applying inequality (9.10) with $\alpha = q$ and $t = |\widetilde{U}_n|$, $s = |g(v_n)|$, to the second estimate in (9.9), and using (A5) and (9.5) yields

$$\begin{aligned}\|\nabla \tilde{u}_n\|_{2,p} &\leqslant \int_\Omega g(\tilde{v}_n)\widetilde{U}_N + \varepsilon_n \\ &\leqslant \int_\Omega \bigl(e^{\widetilde{U}_n^q} - 1\bigr) + \int_\Omega \bigl|g(\tilde{v}_n)\bigr|\bigl[\log^+\bigl|g(\tilde{v}_n)\bigr|\bigr]^{1/q} + \varepsilon_n \\ &\leqslant c + \int_\Omega g(\tilde{v}_n)\tilde{v}_n + \varepsilon_n \\ &\leqslant c + \varepsilon_n \bigl\|(u_n, \tilde{v}_n)\bigr\|_E + \varepsilon_n. \end{aligned} \tag{9.11}$$

Joining the two inequalities yields the boundedness of $\|(u_n, \tilde{v}_n)\|_E$.

(d) *Finite-dimensional approximation.*

Note that the functional J is strongly indefinite on the space E (i.e. positive and negative definite on infinite-dimensional manifolds), and hence the standard linking theorems cannot be applied. We therefore consider an approximate problem on finite-dimensional spaces (Galerkin approximation):

Denote by $(e_i)_{i\in\mathbb{N}}$ an orthonormal set of eigenfunctions corresponding to the eigenvalues (λ_i), $i \in \mathbb{N}$, of $(-\Delta, H_0^1(\Omega))$, and set

$$\begin{aligned} E_n^+ &= \operatorname{span}\bigl\{(e_i, \tilde{e}_i) \bigm| i = 1, \dots, n\bigr\}, \\ E_n^- &= \operatorname{span}\bigl\{(e_i, -\tilde{e}_i) \bigm| i = 1, \dots, n\bigr\}, \\ E_n &= E_n^- \mathbin{\widetilde{\oplus}} E_n^-. \end{aligned}$$

Set now $Q_n = Q \cap E_n$, Q as in (b) above, and define the family of mappings

$$\Gamma_n = \bigl\{\gamma \in C\bigl(Q_n, E_n^- \mathbin{\widetilde{\oplus}} \bigl[(e_1, \tilde{e}_1)\bigr]\bigr) \bigm| \gamma(z) = z \text{ on } \partial Q_n\bigr\}$$

and set

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} J\bigl(\gamma(z)\bigr).$$

It is now quite standard (see [38,17]) to conclude that:

PROPOSITION 9.4. *For each $n \in \mathbb{N}$ the functional $J_n = J|_{E_n}$ has a critical point $z_n = (u_n, \tilde{v}_n) \in E_n$ at level c_n, with*

$$J(z_n) = c_n \in [\sigma, R_1] \tag{9.12}$$

and

$$J'(z_n)\bigl[(\phi, \tilde{\psi})\bigr] = 0, \quad \textit{for all } (\phi, \tilde{\psi}) \in E_n, \tag{9.13}$$

and hence

$$\begin{cases} \int_\Omega \nabla u_n \nabla \tilde{\psi} = \int_\Omega g(\tilde{v}_n)\tilde{\psi}, \\ \int_\Omega \nabla \tilde{v}_n \nabla \phi = \int_\Omega f(u_n)\phi, \end{cases} \quad \forall (\phi, \tilde{\psi}) \in E_n. \tag{9.14}$$

(e) *Limit* $n \to \infty$.

By (d) we find a sequence $(u_n, \tilde{v}_n) \in E_n$ with

$$J(u_n, \tilde{v}_n) \to c \in [\sigma, R_1] \quad \text{and} \quad J_n'(u_n, \tilde{v}_n) = 0, \quad \text{in } E_n,$$

and by (c) we have $\|(u_n, \tilde{v}_n)\|_E \leqslant c$. Then $(u_n, \tilde{v}_n) \rightharpoonup (u, \tilde{v})$ in E. Furthermore, we may assume that

$$\tilde{v}_n \to \tilde{v} \quad \text{in } L^\alpha, \text{ for all } \alpha \geqslant 1. \tag{9.15}$$

Indeed, we have for any small $\delta > 0$

$$W_0^1 L(2, q) \subset W_0^1 L(2-\delta, 2-\delta) = W_0^{1,2-\delta} \subset L^{\frac{(2-\delta)2}{\delta}},$$

and hence a compact embedding into L^α, for all $1 \leqslant \alpha < \frac{(2-\delta)2}{\delta}$.

Using (9.5), (9.6) and assumption (A3) one concludes now as in [14, Lemma 2.1], that

$$\int_\Omega f(u_n) \to \int_\Omega f(u), \qquad \int_\Omega g(\tilde{v}_n) \to \int_\Omega g(\tilde{v}).$$

Thus, in (9.14) we can take the limit $n \to \infty$ to obtain

$$\begin{cases} \int_\Omega \nabla u \nabla \tilde{\psi} = \int_\Omega g(\tilde{v})\tilde{\psi}, \\ \int_\Omega \nabla \tilde{v} \nabla \phi = \int_\Omega f(u)\phi, \end{cases} \quad \forall (\phi, \tilde{\psi}) \in \bigcup E_n = E. \tag{9.16}$$

Hence $(u, \tilde{v}) \in E$ is a (weak) solution of (9.16).

Finally, we prove that $(u, \tilde{v}) \in E$ is nontrivial. Assume by contradiction that $u = 0$, which implies that also $v = 0$. Since g is subcritical, we obtain by (A5), for all $\delta > 0$

$$|g(t)| \leqslant c_\delta e^{\delta |t|^q}, \quad \forall t \in \mathbb{R}.$$

Now we choose $\tilde{\psi} = \tilde{v}_n$ in the first equation of (9.16), and estimate by Hölder

$$\left| \int_\Omega g(\tilde{v}_n)\tilde{v}_n \right| \leqslant c_\delta \left\| e^{\delta |\tilde{v}_n|^q} \right\|_{L^\beta} \|\tilde{v}_n\|_{L^{\alpha}} \leqslant d_\delta \|\tilde{v}_n\|_{L^\alpha}, \tag{9.17}$$

where we have used that $\|\nabla \tilde{v}_n\|_{2,p} \leqslant c$, and hence by Theorem D, Section 7.2 above, for $\beta > 1$ sufficiently small:

$$\left\| e^{\delta |\tilde{v}_n|^q} \right\|_{L^\beta} = \int_\Omega e^{\delta \beta |\tilde{v}_n|^q} \leqslant c.$$

Since $\|\tilde{v}_n\|_{L^\alpha} \to 0$ by (9.15), we conclude that $\int g(\tilde{v}_n)\tilde{v}_n \to 0$ by (9.17), and hence by the first equation in (9.14) that

$$\int_\Omega \nabla u_n \nabla \tilde{v}_n \to 0. \tag{9.18}$$

This in turn implies, by choosing $\phi = u_n$ in the second equation in (9.14), that also $\int_\Omega f(u_n)u_n \to 0$. By assumption (A2) we now conclude that

$$\int_\Omega F(u_n) \to 0, \quad \text{and} \quad \int_\Omega G(u_n) \to 0. \tag{9.19}$$

Finally, by (9.18) and (9.19) we now obtain that $J(u_n, \tilde{v}_n) = \int_\Omega \nabla u_n \nabla \tilde{v}_n - \int_\Omega F(u_n) + G(\tilde{v}_n) \to 0$; but this contradicts (9.12), and thus $(u, \tilde{v}) \neq (0, 0)$.

This completes the proof. □

9.3. *Critical systems in dimension $N = 2$*

For the "critical" system (S2) not much is known. Indeed, we can state the following

OPEN PROBLEMS.
- Loss of compactness and concentration phenomena for systems with critical growth, i.e. when the exponents lie on the "exponential critical hyperbola";
- nonexistence of (radial, positive?) solutions for certain model equations with critical growth;
- existence of instantons, or optimal concentrating sequences;
- group invariance and Pohozaev type identities.

Concerning existence results for systems in $N = 2$ with critical growth, only the following result is known, see [16], in which both nonlinearities have critical growth with respect to $H_0^1(\Omega)$.

Let d denote the inner radius of the set Ω, that is d is equal to the radius of the largest open ball contained in Ω. Recall that we say that a function h has critical growth at $+\infty$ w.r.t. to $H_0^1(\Omega)$ if there exists $\gamma_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{h(t)}{e^{\gamma t^2}} = 0, \quad \forall \gamma > \gamma_0, \quad \text{and} \quad \lim_{t\to+\infty} \frac{h(t)}{e^{\gamma t^2}} = +\infty, \quad \forall \gamma < \gamma_0; \tag{9.20}$$

in this case we say that γ_0 is the *exponential critical exponent* of h.

THEOREM 9.5. *Assume that f and g satisfy assumptions* (A1)–(A3) *in Section* 9.2, *and that f and g have critical growth with exponential critical exponents α_0, respectively, β_0. Furthermore suppose that*

(A4) $$\lim_{t\to+\infty}\frac{tf(t)}{e^{\alpha_0 t^2}} > \frac{4}{d^2\sqrt{\alpha_0\beta_0}} \quad \textit{and} \quad \lim_{t\to+\infty}\frac{tg(t)}{e^{\beta_0 t^2}} > \frac{4}{d^2\sqrt{\alpha_0\beta_0}}.$$

Then system (S2) *possesses a nontrivial weak solution* $(u, v) \in E$.

PROOF. The proof is a combination of the proof of Theorem 3.2 for the scalar equation and the proof of Theorem 9.3 for the subcritical system. We refer the interested reader to [16]. □

Note that this theorem gives an existence result for two-dimensional systems in which both nonlinearities have *the same* critical growth. This corresponds to the case in which the exponential exponents lie on the diagonal of the exponential critical hyperbola. Existence of solutions for critical cases which are not on the diagonal remains an open problem.

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CHAPTER 4

Nonlinear Eigenvalue Problem with Quantization

Takashi Suzuki

Division of Mathematical Science, Department of Systems Innovation, Graduate School of Science, Osaka University, Japan
E-mail: *suzuki@sigmath.es.osaka-u.ac.jp*

Futoshi Takahashi

Department of Mathematics, Faculty of Science, Osaka City University, Japan
E-mail: *futoshi@sci.osaka-cu.ac.jp*

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1. Energy quantization

Quantized blowup mechanism is widely observed in nonlinear problems derived from physical principles and phenomena. Energy quantization is a typical case described by the Dirichlet norm. In this section, we introduce several examples and then study the harmonic map case in detail.

1.1. *Quantized blowup mechanism*

Energy quantization arises if the problem is provided with the scaling invariance of energy, e.g., harmonic map, semilinear elliptic equation involved by the critical Sobolev exponent, and H-systems. In more precise, energy identity with bubbling occurs to the noncompact solution sequence in these cases.

1.1.1. *Harmonic map.* Given a compact Riemannian surface (Σ, g) and a compact Riemannian manifold $N \hookrightarrow \mathbf{R}^n$, we put

$$H^1(\Sigma, N) = \left\{ u \in H^1(\Sigma, \mathbf{R}^n) \;\middle|\; u(x) \in N, \text{ a.e. } x \right\}$$

$$E(u) = E_\Sigma(u) = \int_\Sigma \left| \nabla u(x) \right|^2 dv_\Sigma(x),$$

where $dv_\Sigma = dv_\Sigma(x)$ denotes the area element of (Σ, g). We say that $u : \Sigma \to N$ is a harmonic map if it is a solution to the Euler–Lagrange equation of $E = E(u)$ defined for $u \in H^1(\Sigma, N)$, i.e.,

$$-\Delta u = A(u)(\nabla u, \nabla u), \tag{1.1}$$

where $A(u)(\cdot,\cdot)$ denotes the second fundamental form of $N \hookrightarrow \mathbf{R}^n$. Let S^2 be the standard two-dimensional sphere. Then, there is a quantized blowup mechanism of the harmonic map sequence defined on a 2-dimensional domain described as follows [70,97,98,49,99, 121,87].

THEOREM 1.1. *Let $\{u_k\}_k$ be a harmonic map sequence satisfying*

$$E_0 = \sup_k E(u_k) < +\infty.$$

Then, passing to a subsequence we assume $u_k \rightharpoonup u$ in $H^1(\Sigma, N)$ weakly to some map $u \in H^1(\Sigma, N)$. This u is a harmonic map, and there exist

- *p-sequences of points $\{x_k^1\}, \ldots, \{x_k^p\}$ in Σ,*
- *p-sequences of positive numbers $\{\delta_k^1\}, \ldots, \{\delta_k^p\}$ converging to 0,*
- *p-nonconstant harmonic maps $\{\omega^1\}, \ldots, \{\omega^p\} : S^2 \to N$*

satisfying the following, where $E_0(\omega) = \int_{S^2} |\nabla \omega|^2 \, dv_{S^2}$:

$$\lim_{k\to\infty} E(u_k) = E(u) + \sum_{j=1}^{p} E_0(\omega^j),$$

$$\lim_{k\to\infty} \max_{i\neq j} \left\{ \frac{\delta_k^i}{\delta_k^j}, \frac{\delta_k^j}{\delta_k^i}, \frac{|x_k^i - x_k^j|}{\delta_k^i + \delta_k^j} \right\} = +\infty,$$

$$\lim_{k\to\infty} \left\| u_k - u - \sum_{j=1}^{p} \left\{ \omega^j \left(\frac{\cdot - x_k^j}{\delta_k^j} \right) - \omega^j(\infty) \right\} \right\|_{H^1(\Sigma,N)} = 0. \tag{1.2}$$

The first equality of (1.2) is an energy identity, which says that there is no unaccounted energy loss during the iterated rescaling process near the point of singularity, sometimes referred to as the bubbling process, and that the only reason for failure of strong convergence to the weak limit is the formation of several bubbles due to the nonconstant harmonic maps $\omega^j : S^2 \to N$ ($j = 1, \ldots, p$). There is a possibility that some of $\{x_k^j\}_k$ ($j = 1, \ldots, p$) converge to the same point, and this process is classified into two cases, the separated bubbles and the bubbles on bubbles. These properties were proved by Jost [70] (Lemma 4.3.1, p. 127), and later Qing [98] extended the result for the sequence of approximate harmonic maps satisfying

$$-\Delta u_k = A(u_k)(\nabla u_k, \nabla u_k) + f_k \tag{1.3}$$

with $\|f_k\|_2 \leqslant C$, when N is a standard sphere. The general case of N is also known [138, 49].

1.1.2. *Elliptic problem with Sobolev exponent.* A form of the Sobolev imbedding theorem is formulated by $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, where $\Omega \subset \mathbf{R}^m$ is an open set with $m \geqslant 3$ and $2^* = \frac{2m}{m-2}$. If Ω is bounded and $1 < q < 2^* - 1$, then we obtain the compact imbedding $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$, which results in the existence of infinitely many solutions to

$$-\Delta u = |u|^{p-1} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

while there is no solution if $q \geqslant 2^* - 1$ and Ω is star-shaped [95,100,90]. Several existence and nonexistence results of the solution to

$$-\Delta u = \lambda u + |u|^{2^*-2} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

are also known in accordance with the shape of the domain and the value $\lambda \in \mathbf{R}$ [22,4,90]. Actually, it is the Euler–Lagrange equation of

$$E_\lambda(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2 \right) dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx$$

defined for $u \in H_0^1(\Omega)$, whereby the control of Palais–Smale sequence is a key ingredient of the study. Here, we say that $\{u_k\}_k \subset H_0^1(\Omega)$ is a Palais–Smale sequence to E_λ if $E'_\lambda(u_k) \to 0$ in $H^{-1}(\Omega) = H_0^1(\Omega)'$, i.e.,

$$-\Delta u_k = \lambda u_k + |u_k|^{2^*-1} u_k + f_k \quad \text{in } \Omega, \qquad u_k = 0 \quad \text{on } \partial\Omega,$$
$$\|f_k\|_{H^{-1}(\Omega)} \to 0.$$

We obtain an energy identity and bubbling similarly to those of the harmonic map sequence.

THEOREM 1.2. (See [116].) *If $\{u_k\}_k \subset H_0^1(\Omega)$ is a Palais–Smale sequence to the above defined E_λ, such that $\sup_k E_\lambda(u_k) < +\infty$, then there are $p \in \mathbf{N}$, $\delta_k^j \downarrow 0$, and $x_k^j \in \Omega$ $(1 \leqslant j \leqslant p)$, such that, passing to a subsequence,*

$$\left\| u_k - u^0 - \left(\delta_k^j\right)^{-\frac{n-2}{2}} \sum_{j=1}^p \omega^j \left(\frac{\cdot - x_k^j}{\delta_k^j} \right) \right\|_{H^1(\Omega)} \to 0,$$
$$E_\lambda(u_k) \to E_\lambda\left(u^0\right) + \sum_{j=1}^p E_0(\omega^j)$$

as $k \to \infty$, where $\omega^j = \omega^j(x)$ $(1 \leqslant j \leqslant p)$ are solutions to

$$-\Delta \omega = |\omega|^{2^*-2} \omega \quad \text{in } \mathbf{R}^m$$

satisfying $\omega \in L^{2^*}(\mathbf{R}^m)$ and $\nabla \omega \in L^2(\mathbf{R}^m; \mathbf{R}^m)$,

$$E_0(\omega) = \frac{1}{2} \int_{\mathbf{R}^m} |\nabla \omega|^2 \, dx - \frac{1}{2^*} \int_{\mathbf{R}^m} |\omega|^{2^*} \, dx,$$

and $u^0 = u^0(x)$ is a solution to

$$-\Delta u = \lambda u + |u|^{2^*-2} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

1.1.3. *H-systems.* Parametric representation of the surface of constant mean curvature is described by $u : B = \{(x, y) \mid x^2 + y^2 < 1\} \to \mathbf{R}^3$ satisfying

$$\Delta u = 2H u_x \times u_y, \quad |u_x|^2 - |u_y|^2 = u_x \cdot u_y = 0 \quad \text{in } B$$
$$u|_{\partial B} : \text{monotonic parametrization of } \gamma \text{ fixing three points},$$

where $\gamma \subset \mathbf{R}^3$ is a given oriented rectifiable Jordan curve, H is a constant, and $|\ |$, $\times$, and $\cdot$ are three-dimensional vector length, outer product, and inner product, respectively. This is the Euler–Lagrange equation of $E_H = E_H(u)$ defined for $u \in H^1(B) \cap C(\overline{B})$ that is a monotonic parametrization of γ fixing three points, where

$$E_H(u) = \frac{1}{2}\int_B |\nabla u|^2\,dz + \frac{2H}{3}\int_\Omega u \cdot u_x \times u_y\,dz$$

for $dz = dx\,dy$. Existence of the solution, on the other hand, is associated with that of the Dirichlet problem

$$\Delta u = 2Hu_x \times u_y + f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

and there is an energy quantization of the noncompact Palais–Smale sequence [18,117–119].

In more precise, we define $E_H(u)$ for $u \in H_0^1(\Omega; \mathbf{R}^3)$, and call $\{u_k\}_k \subset H_0^1(\Omega; \mathbf{R}^3)$ a Palais–Smale sequence if

$$\Delta u_k = 2Hu_{kx} \times u_{ky} + f_k \quad \text{in } \Omega, \qquad u_k = 0 \quad \text{on } \partial\Omega$$

holds with $f_k \to 0$ in $H^{-1}(\Omega; \mathbf{R}^3) = H_0^1(\Omega; \mathbf{R}^3)'$. Then, we obtain a similar result to this $\{u_k\}_k$.

THEOREM 1.3. (See [19].) *If $\{u_k\}_k \subset H_0^1(\Omega; \mathbf{R}^3)$ is a Palais–Smale sequence to the above defined E_H satisfying*

$$\sup_k \int_\Omega |\nabla u_k|^2\,dz < +\infty,$$

then, there are $p \in \mathbf{N}$, $\delta_k^j \downarrow 0$, and $x_k^j \in \Omega$ $(1 \leqslant j \leqslant p)$ such that

$$\left\| u^k - u^0 - \sum_{j=1}^p \omega^j\left(\frac{\cdot - x_k^j}{\delta_k^j}\right)\right\|_{H^1(\Omega)} \to 0,$$

$$\int_\Omega |\nabla u_k|^2\,dz \to \int_\Omega \left|\nabla u^0\right|^2 dz + \sum_{j=1}^p \int_{\mathbf{R}^2} |\nabla \omega^j|^2\,dz,$$

$$V\left(u^j\right) \to V\left(u^0\right) + \sum_{j=1}^p V_0(\omega^j)$$

as $k \to \infty$, where $\omega^j = \omega^j(x)$ $(1 \leqslant j \leqslant p)$ are solutions to

$$\Delta\omega = 2H\omega_x \times \omega_y \quad \text{in } \mathbf{R}^2, \qquad \omega(\infty) = 0, \qquad \int_{\mathbf{R}^2} |\nabla\omega|^2\,dz < +\infty,$$

$u^0 = u^0(x)$ *is a solution to*

$$\Delta u = 2H u_x \times u_y \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and

$$V(u) = \frac{1}{3}\int_\Omega u \cdot u_x \times u_y \, dz, \qquad V_0(\omega) = \frac{1}{3}\int_{\mathbf{R}^2} \omega \cdot \omega_x \times \omega_y \, dz.$$

The Palais–Smale sequence is defined also to the harmonic map. It is formulated by $\{u_k\}$ satisfying (1.3) with $f_k \to 0$ in $H^{-1}(\Sigma, N)$. Then, its energy gap is eliminated under the additional assumption $\|f_k\|_2 = O(1)$ [93]. In contrast with Theorem 1.3, this is not removable condition, e.g., each $\alpha > 0$ admits a sequence of smooth maps $u_k : S^2 \to S^2$ ($k = 1, 2, \ldots$) satisfying $E'(u_k) \to 0$, the first two relations of (1.2) with $p = 1$, and $E(u_k) \to E(u) + E(\omega^1) + \alpha$ [97,49,138].

1.2. *Harmonic map*

The proof of Theorems 1.1–1.3 shares a common argument, and here we describe the harmonic map case, reformulating the problem with more generality.

Thus, (M, g) denotes an m-dimensional compact Riemannian manifold with smooth boundary ∂M, and N is a compact Riemannian manifold without boundary. By Nash's theorem this N is isometrically imbedded in $\mathbf{R}^n$ for large n. We define the Sobolev space composed of a class of the mappings from M to N provided with the finite energy as in the previous section, i.e.,

$$H^1(M, N) = \left\{u \in H^1(M, \mathbf{R}^n) \mid u(x) \in N, \ \text{a.e. } x\right\},$$
$$E(u) = E_M(u) = \int_M \left|\nabla u(x)\right|^2 dv_M(x),$$

where dv_M is a volume element of (M, g).

Harmonic map defined in the previous section is indicated by the weakly harmonic map in this general formulation. More precisely, a map $u \in H^1(M, N)$ is called a *weakly harmonic map* if

$$\frac{d}{d\varepsilon} E\big(\Pi(u + \varepsilon\phi)\big)\big|_{\varepsilon=0} = 0$$

for any $\phi \in C_0^\infty(M, \mathbf{R}^n)$, where $\Pi : U \to N$ is a smooth nearest point projection from some tubular neighborhood U of N to N. This is equivalent to saying that u is a weak solution of the Euler–Lagrange equation (1.1) on M, sometimes called the *harmonic map equation,*

$$-\Delta u = A(u)(\nabla u, \nabla u),$$

where Δ and $A(y)(\cdot,\cdot)$ denote the Laplace–Beltrami operator on (M, g) and the second fundamental form of the imbedding $N \hookrightarrow \mathbf{R}^n$ at $y \in N$, respectively. In a typical case that N is the round sphere S^n, it takes the form

$$-\Delta u = |\nabla u|^2 u. \tag{1.4}$$

A map $u \in H^1(M, N)$ is called a *minimizing harmonic map* if

$$E(u) \leqslant E(v)$$

holds for any $v \in H^1(M, N)$ satisfying $u = v$ on ∂M. We call $u \in H^1(M, N)$, on the other hand, a *stationary harmonic map* if it is weakly harmonic, and

$$\frac{d}{dt} E(u \circ \Phi_t)\Big|_{t=0} = 0 \tag{1.5}$$

for any smooth family $\{\Phi_t\}_{|t|\ll 1}$ of diffeomorphisms of M such that $\Phi_0 = \mathrm{Id}$. Any minimizing harmonic map or weakly C^2 harmonic map is stationary.

1.2.1. *Monotonicity formula.* If $M = \Omega \subset \mathbf{R}^m$ is a bounded domain, we can take $\Phi_t(x) = x + t\xi(x)$ for $\xi \in C_0^\infty(\Omega, \mathbf{R}^m)$. Then, (1.5) implies

$$\int_\Omega \sum_{i,j=1}^m \bigl(\delta_{ij} |\nabla u|^2 - 2D_i u D_j u\bigr) D_i \xi^j \, dx = 0. \tag{1.6}$$

Putting $\xi(x) = x - z$ in (1.6), we obtain the *monotonicity formula*,

$$\begin{aligned} &R^{2-m} \int_{B_R(z)} |\nabla u|^2 \, dx - r^{2-m} \int_{B_r(z)} |\nabla u|^2 \, dx \\ &\quad = 2 \int_{B_R(z)\setminus B_r(z)} |x - z|^{2-m} |\nabla u|^2 \, dx \quad (0 < r < R) \end{aligned} \tag{1.7}$$

if $B_R(z) = B(z, R) \Subset \Omega$, and therefore, the *rescaled energy*

$$r^{2-m} \int_{B_r(z)} |\nabla u|^2 \, dx$$

is a monotone increasing function of r. This formula takes an important role in the study of local regularity.

More precisely, given $u \in H^1(M, N)$, we put

$$\mathrm{reg}(u) = \{z \in M \mid u \text{ is } C^\infty \text{ in a neighbourhood of } z\},$$

$$\mathrm{sing}(u) = M \setminus \mathrm{reg}(u).$$

Since reg(u) is open, sing(u) is relatively closed in M. It can happen that sing$(u) \neq \emptyset$ even if u is a minimizing harmonic map. For example, $u : B^m \mapsto S^{m-1}$ defined by $u(x) = \frac{x}{|x|}$ is an energy minimizing map with sing$(u) = \{0\}$ if $m \geqslant 3$ [78]. Here and henceforth, B^m denotes the m-dimensional unit open ball in $\mathbf{R}^m$.

Several results are known concerning sing(u). In the following, $\mathcal{H}^s$ and $\dim_H$ stand for the s-dimensional Hausdorff measure and the Hausdorff dimension, respectively. Recall that m denotes the dimension of M.

(1) If $u \in H^1(M, N)$ is a minimizing harmonic map and $m \geqslant 3$, then it holds that $\dim_H$ sing$(u) \leqslant m - 3$. If $m = 3$, then sing(u) is discrete [113].
(2) There is a weakly harmonic map $u \in H^1(B^3, S^2)$ such that sing$(u) = \overline{B^3}$ [101].
(3) If $u \in H^1(M, N)$ is a stationary harmonic map, then it holds that $\mathcal{H}^{m-2}(\text{sing}(u)) = 0$ [50,13].
(4) Weakly harmonic map for $m = 2$ is smooth [66,67].

See [79,87,93,99,102] and the references therein for the related work.

1.2.2. *Hardy-BMO structure.* Here, we describe the proof of the final item of the above list in the case of $M = B^2$ and $N = S^n$, that is, any weakly harmonic map $u \in H^1(B^2, S^n)$ is smooth in B^2. In fact, first, we obtain

$$\sum_{j=1}^{n+1} u^j \nabla u^j = \vec{0}$$

by $\sum_{j=1}^{n+1} |u^j|^2 = 1$ for $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, and therefore,

$$\nabla u^i = \sum_{j=1}^{n+1} \big((u^j)^2 \nabla u^i - u^j u^i \nabla u^j\big) = \sum_{j=1}^{n+1} u^j \vec{b}^{ji}$$

for $\vec{b}^{ji} = u^j \nabla u^i - u^i \nabla u^j \in \mathbf{R}^2$.

It follows that

$$\nabla \cdot \vec{b}^{ji} = \nabla \cdot \big(u^j \nabla u^i - u^i \nabla u^j\big) = u^j \Delta u^i - u^i \Delta u^j = 0$$

from (1.4), and hence

$$\Delta u^i = \nabla \cdot \nabla u^i = \sum_{j=1}^{n+1} \nabla u^j \cdot \vec{b}^{ji} \tag{1.8}$$

and

$$\vec{b}^{ji} = \left(\frac{\partial \phi^{ji}}{\partial y}, -\frac{\partial \phi^{ji}}{\partial x}\right)$$

for some $\phi^{ji} \in H^1(B^2, \mathbf{R})$. This implies

$$\Delta u^i = \sum_{j=1}^{n+1} \left\{ \frac{\partial u^j}{\partial x} \frac{\partial \phi^{ji}}{\partial y} - \frac{\partial u^j}{\partial y} \frac{\partial \phi^{ji}}{\partial x} \right\}.$$

The right-hand side of the above equality is provided with the Jacobian structure. It is observed in H-systems first, and leads to the continuity of u^i [143]. Actually, each term of them belongs to the local Hardy space $\mathcal{H}^1_{\mathrm{loc}}(\mathbf{R}^2)$, and then Hardy-BMO paring is applicable [40]. Then, this continuous u is smooth from the standard elliptic regularity.

1.2.3. *Energy concentration.* Now, we describe the proof of Theorem 1.1. Thus, (Σ, g) denotes a compact Riemannian surface without boundary. First, we show a weak version called rough estimate.

THEOREM 1.4. (See [103].) *Under the assumption of Theorem* 1.1, $u : \Sigma \to N$ *is a harmonic map*. *Furthermore, there is* $\varepsilon_0 > 0$ *such that, passing to a subsequence,*

$$u_k \to u \quad \textit{locally uniformly in } \Sigma \setminus \{x_1, \ldots, x_\ell\},$$
$$|\nabla u_k|^2 \, dv_g \rightharpoonup |\nabla u|^2 \, dv_g + \sum_{j=1}^{\ell} m_j \delta_{x_j} \quad \textit{in the sense of measure}, \tag{1.9}$$

where $\{x_1, \ldots, x_\ell\} \subset \Sigma$ *and* $m_j \geqslant \varepsilon_0$.

Since Σ is two-dimensional, any weak solution to (1.1) is regular, and therefore, the weak limit u of $\{u_k\}_k$ is a harmonic map. More strongly, there is a singularity vanishing theorem described as follows.

THEOREM 1.5. (See [103].) *If* $u : D \setminus \{x_0\} \subset \Sigma \to N$ *is harmonic and*

$$\int_D |\nabla u|^2 \, dv_g < +\infty,$$

then there is a harmonic map $\tilde{u} : D \to N$ *such that* $\tilde{u}|_{D \setminus \{x_0\}} = u$, *where* $D \subset \Sigma$ *is an open set and* $x_0 \in D$.

Henceforth, $D(x, r)$ denotes the geodesic disc on Σ with radius $r > 0$ and center $x \in \Sigma$. Then, we obtain an energy decay estimate [103]. Thus, there is $\varepsilon_0 > 0$ and $\theta \in (0, 1)$ determined by (Σ, g) and N such that if $u : \Sigma \to N$ is harmonic and

$$\int_{D_1} |\nabla u|^2 \, dv_g \leqslant \varepsilon_0,$$

then it holds that

$$\int_{D(x,r)} |\nabla u|^2 \, dv_g \leqslant \theta \int_{D(x,2r)} |\nabla u|^2 \, dv_g,$$

where $D_r = D(x_0, r)$ with $x_0 \in \Sigma$, $x \in D_{1/2}$, and $0 < r < 1/4$. This estimate, combined with Morrey's Dirichlet growth theorem [57], guarantees ε-regularity, so that

$$E(2r) = \int_{D_{2r}} |\nabla u|^2 \, dv_g \leqslant \varepsilon_0$$

implies

$$\sup_{D_r} |\nabla u|^2 \leqslant C r^{-2} E(2r) \tag{1.10}$$

with a constant $C > 0$ independent of u and $r \in (0, 1)$. If $\{u_k\}_k$ is a harmonic map sequence satisfying

$$\int_{D_1} |\nabla u_k|^2 \, dv_g \leqslant \varepsilon_0,$$

therefore, by (1.1) we obtain a subsequence, denoted by the same symbol, and a harmonic map $u : D_1 \to N$ such that

$$u_k \to u \quad \text{strongly in } H^1(D_r, N) \text{ and } C^0(D_r, N),$$

where $r \in (0, 1)$. This ε-compactness guarantees the rough estimate as follows.

PROOF OF THEOREM 1.4. Given $\delta > 0$, we put

$$\mathcal{A}_{\delta,k} = \left\{ x \in \Sigma \;\middle|\; \int_{D(x,\delta)} |\nabla u_k|^2 \, dv_g \geqslant \varepsilon_0 \right\}.$$

Then, by Vitali's covering theorem [115], there is $\mathcal{A}'_{\delta,k} \subset \mathcal{A}_{\delta,k}$ such that

$$\left\{ D(x,\delta) \mid x \in \mathcal{A}'_{\delta,k} \right\}$$

is a disjoint family and $\mathcal{A}_{\delta,k} \subset \bigcup_{x \in \mathcal{A}'_{\delta,k}} D_{3\delta}(x) \equiv \Omega(\delta, k)$. This implies

$$\varepsilon_0 \cdot \sharp \mathcal{A}'_{\delta,k} \leqslant \int_{\bigcup_{x \in \mathcal{A}'_{\delta,k}} D(x,\delta)} |\nabla u_k|^2 \, dv_g \leqslant E_0,$$

and hence $\sharp \mathcal{A}'_{\delta,k} \leqslant E_0/\varepsilon_0$. Passing to a subsequence, therefore, we obtain $\mathcal{A}'_{\delta,k} = \{x_k^1, \ldots, x_k^\ell\}$ with $\ell \leqslant E_0/\varepsilon_0$ and $x_k^j \to x_j \in \Sigma$.

We have, on the other hand,

$$x \notin \Omega(\delta,k) \quad \Rightarrow \quad \int_{D_\delta(x)} |\nabla u_k|^2\, dv_g < \varepsilon_0,$$

and therefore, applying the diagonal argument to $\delta_k \downarrow 0$, we take a subsequence, still denoted by the same symbol, such that $u_k \to u$ in $C^1(\Sigma \setminus \{x_1,\dots,x_\ell\})$ for $\bigcap_k \Omega(\delta_k,k) = \{x_1,\dots,x_\ell\}$. This u is harmonic in $\Sigma \setminus \{x_1,\dots,x_\ell\}$ and satisfies

$$\int_\Sigma |\nabla u|^2\, dv_g < +\infty$$

by Fatou's lemma, and therefore, $\{x_1,\dots,x_\ell\}$ are removable singular points of u by Theorem 1.5.

We obtain, furthermore,

$$|\nabla u_k|^2\, dv_g \rightharpoonup |\nabla u|^2\, dv_g + \sum_{j=1}^{\ell} m_j \delta_{x_j}$$

in the sense of measure. In case of $m_j < \varepsilon_0$, therefore, there is $r_0 > 0$ sufficiently small such that

$$\int_{D(x_j,2r_0)} |\nabla u_k|^2\, dv_g = \int_{D(x_j,2r_0)} |\nabla u|^2\, dv_g + m_j + o(1) < \varepsilon_0$$

for $k \gg 1$, and this implies that x_j is not a blowup point of $\{u_k\}_k$ by (1.10). Thus, we obtain (1.9) with $m_j \geqslant \varepsilon_0$. □

The above lemma or the ε-compactness guarantees the energy gap lemma described as follows.

LEMMA 1.2.1. *It holds that*

$$\inf\left\{ \int_{S^2} |\nabla\omega|^2 dv_g \,\middle|\, \omega : S^2 \to N \textit{ nonconstant harmonic map} \right\} > 0.$$

1.2.4. *Energy quantization.* We proceed to the proof of Theorem 1.1. Since $\partial\Sigma = \emptyset$, there is no boundary effect on the blowing-up behavior of the solution sequence. Furthermore, the blowup profile has been localized around each concentration point. The scaling limit is described by $\tilde{\Sigma} = S^2$ with standard metric dv_{S^2}, $\infty \notin \mathcal{S} = \{x_1,\dots,x_\ell\}$, and $N = S^n$, where ∞ denotes the north pole.

First, the following lemma is obtained by the above mentioned Jacobian structure, where $0 < \varepsilon_0 \ll 1$ is an absolute constant and D_R is the geodesic disc with the radius $R > 0$.

LEMMA 1.2.2. *If $u : D_R \to S^n$ is a harmonic map satisfying*

$$\int_{D_R} |\nabla u|^2 \, dv_g \leqslant \varepsilon_0$$

and $v \in H^1(\Omega, \mathbf{R}^{n+1})$ is a solution to

$$\Delta v = 0 \quad in\ \Omega, \qquad v = u \quad in\ \partial\Omega,$$

then it holds that

$$\int_\Omega |\nabla u|^2 \, dv_g \leqslant C \int_\Omega |\nabla v|^2 \, dv_g$$

with a constant $C > 0$ independent of $\Omega \Subset D_R$.

PROOF. Since the target space of this harmonic map is a sphere, we obtain (1.8), i.e.,

$$\Delta u^i = \sum_{j=1}^{n+1} \nabla u^j \cdot \vec{b}^{ji}$$

for $i = 1, \dots, n+1$, where $\vec{b}^{ji} = u^j \nabla u^i - u^i \nabla u^j$. Then, subtracting the equation of v, we have

$$\Delta (u - v)^i = \sum_{j=1}^{n+1} \nabla (u - v)^j \cdot \vec{b}^{ji} + \sum_{j=1}^{n+1} \nabla v^j \cdot \vec{b}^{ji}, \tag{1.11}$$

and define $w^i \in W_0^{1,1}(\Omega)$ by

$$\Delta w^i = \sum_{j=1}^{n+1} \nabla (u - v)^j \cdot \vec{b}^{ji} \quad \text{in } \Omega, \qquad w^i = 0 \quad \text{on } \partial\Omega. \tag{1.12}$$

Since $(\vec{b}^{ji})_{j=1,\dots,n+1}$ and $(\nabla (u - v)^j)_{j=1,\dots,n+1}$ are divergence and curl-free L^2-vector fields, respectively, the right-hand side of (1.12) is regarded as an element in the local Hardy space $\mathcal{H}^1_{\mathrm{loc}}(\mathbf{R}^2)$ [40], which guarantees

$$\left\| \nabla w^i \right\|_2 \leqslant C \sum_{j=1}^{n+1} \left\| \nabla (u - v)^j \right\|_2 \cdot \left\| \vec{b}^{ji} \right\|_2 \tag{1.13}$$

with a constant $C > 0$ independent of Ω [16,30].

Multiplying (1.11) by $(u-v)^i$ and integrating over Ω, we have

$$\begin{aligned}
&\int_\Omega \left|\nabla(u-v)\right|^2 dv_g \\
&\quad = \sum_{i=1}^{n+1}\int_\Omega \nabla w^i\cdot\nabla(u-v)^i\,dv_g - \sum_{i,j=1}^{n+1}\int_\Omega \left(\nabla v^j\cdot\vec{b}^{ji}\right)(u-v)^i\,dv_g \\
&\quad \leqslant C\varepsilon_0^{1/2}\int_\Omega \left|\nabla(u-v)\right|^2 dv_g + C\int_\Omega |\nabla u|\cdot|\nabla v|\,dv_g
\end{aligned}$$

by

$$\left\|\vec{b}^{ji}\right\|_2 \leqslant C\|\nabla u\|_2 \leqslant C\varepsilon_0^{1/2} \quad \text{and} \quad \|u-v\|_\infty \leqslant C,$$

where the maximum principle to the harmonic function is applied. This and the inequality

$$\int_\Omega |\nabla u|^2\,dv_g \leqslant 2\int_\Omega \left|\nabla(u-v)\right|^2 dv_g + 2\int_\Omega |\nabla v|^2\,dv_g$$

lead to the conclusion when ε_0 is small enough. □

By the proof of Theorem 1.4, it holds that

$$\mathcal{S} = \bigcap_{r>0}\left\{z\in\Sigma \;\middle|\; \liminf_{k\to\infty}\int_{D_r(z)} |\nabla u_k|^2\,dv_g > \varepsilon_0\right\}.$$

We apply a hierarchical argument, dividing the proof in several steps. From later on, we do not mention the process of subtracting subsequences unless otherwise stated.

STEP 1. We extract the principal bubble at each blowup point, putting

$$\delta = \frac{1}{2}\min\left\{|x_i - x_j| \;\middle|\; i\neq j\right\}.$$

Thus, we define the *concentration function* of $\{u_k\}_k$ near $x_1\in\mathcal{S}$ by

$$Q_k^1(t) = \sup_{z\in D_\delta(x_1)}\int_{D_t(z)} |\nabla u_k|^2\,dv_g.$$

It is continuous, monotone increasing in $t\geqslant 0$, and satisfies $Q_k^1(0)=0$ and $Q_k^1(\delta)>\varepsilon_0$ for $k\gg 1$, and therefore, there exist $x_k^1\in D_\delta(x_1)$ and $\delta_k^1\in(0,\delta)$ such that

$$Q_k^1(\delta_k^1) = \int_{D_{\delta_k^1}(x_k^1)} |\nabla u_k|^2\,dv_g = \frac{\varepsilon_0}{2}.$$

Then, it is easy to see that $x_k^1 \to x_1$ and $\delta_k^1 \to 0$ as $k \to \infty$.

The rescaled $\tilde{u}_k^1(x) = u_k(\delta_k^1 x + x_k^1)$ is a harmonic map satisfying

$$\int_{D_1(z)} \left|\nabla \tilde{u}_k^1\right|^2 d\tilde{v}_g \leqslant \frac{\varepsilon_0}{2}$$

for all $z \in \tilde{D}_\delta^1(x_k) = (\delta_k^1)^{-1}\{D_\delta(x_1) - x_k^1\}$ with the equality for $z = 0$, where $d\tilde{v}_g$ denotes the rescaled metric of dv_g. Then, we apply the ε-compactness to this $\{\tilde{u}_k^1\}_k$, and obtain

$$\tilde{u}_k^1 \to \omega^1 \quad \text{in } H^1_{\mathrm{loc}} \cap C^0_{\mathrm{loc}}\big(S^2 \setminus \{\infty\}, S^n\big)$$

with ω^1 extended as a harmonic map from S^2 to N satisfying

$$\int_{D_1} \left|\nabla \omega^1\right|^2 dv_{S^2} = \frac{\varepsilon_0}{2} \quad \text{and} \quad \int_{S^2} \left|\nabla \omega^1\right|^2 dv_{S^2} \leqslant \sup_k E(u_k) < +\infty.$$

This ω^1 is called a *bubble* at the blowup point x_1.

Repeating this procedure, we obtain

- sequences of points $\{x_k^j\} \subset D_\delta(x_j)$ satisfying $\lim_{k\to\infty} x_k^j = x_j$,
- sequences of positive numbers $\{\delta_k^j\}$ converging to 0,
- nonconstant harmonic map $\omega^j : S^2 \to S^n$

such that $\tilde{u}_k^j \to \omega^j$ in $H^1_{\mathrm{loc}} \cap C^0_{\mathrm{loc}}(S^2 \setminus \{\infty\}, S^n)$ for $j = 1, \dots, \ell$, where

$$\tilde{u}_k^j(x) = u_k\big(\delta_k^j x + x_k^j\big).$$

STEP 2. We estimate the energy difference between u_k and the principal bubbles, putting

$$v_k(x) = u_k(x) - \sum_{j=1}^{\ell}\left[\omega^j\left(\frac{x - x_k^j}{\delta_k^j}\right) - \omega^j(\infty)\right]. \tag{1.14}$$

It is easy to see that $v_k \rightharpoonup u$ weakly in $H^1(\Sigma, S^n)$. Now, we show

$$\int_\Omega |\nabla v_k|^2 dv_g = \int_\Omega |\nabla u_k|^2 dv_g - \sum_{j;\ x_j \in \Omega} \int_{S^2} \left|\nabla \omega^j\right|^2 dv_{S^2} + o(1) \tag{1.15}$$

for each open set $\Omega \subset \Sigma$.

PROOF OF (1.15). Given $j = 1, \dots, \ell$, we set

$$\int_{D_\delta(x_j)} |\nabla v_k|^2 dv_g = I + II$$

for

$$I = \int_{D_{R_j \delta_k^j}(x_k^j)} |\nabla v_k|^2 \, dv_g$$

$$II = \int_{D_\delta(x_j) \setminus D_{R_j \delta_k^j}(x_k^j)} |\nabla v_k|^2 \, dv_g$$

and $R_j > 0$. Since $\tilde{u}_k^j \to \omega^j$ strongly in $H^1_{\mathrm{loc}}(S^2 \setminus \{\infty\}, S^n)$, first, we obtain

$$\begin{aligned} I &= \int_{D_{R_j}} |\nabla \tilde{u}_k|^2 \, d\tilde{v}_g - \int_{D_{R_j}} \left|\nabla \omega^j\right|^2 dv_{S^2} + o(1) \\ &= \int_{D_{R_j \delta_k^j}(x_k^j)} |\nabla u_k|^2 \, dv_g - \int_{S^2} \left|\nabla \omega^j\right|^2 dv_{S^2} + o(1). \end{aligned}$$

Next, it follows that

$$II = \int_{D_\delta(x_j) \setminus D_{R_j \delta_k^j}(x_k^j)} |\nabla u_k|^2 \, dv_g + o(1)$$

from $R_j \gg 1$, and therefore,

$$\int_{D_\delta(x_j)} |\nabla v_k|^2 \, dv_g = \int_{D_\delta(x_j)} |\nabla u_k|^2 \, dv_g - \int_{S^2} |\nabla \omega^j|^2 \, dv_{S^2} + o(1).$$

Finally, we have the H^1-strong convergence $u_k \to u$ on $\Omega \setminus \bigcup_{j=1}^{\ell} D_\delta(x_j)$, and hence (1.15) is proven. □

STEP 3. To search for further bubbles, first, we assume the case

$$\liminf_{k \to \infty} \int_\Sigma |\nabla v_k|^2 \, dv_g = \int_\Sigma |\nabla u|^2 \, dv_g.$$

Then, v_k converges strongly to u in $H^1(\Sigma, S^n)$, and the energy identity holds by (1.15). Moreover, since $x_k^j \to x_j$ and $|x_i - x_j| > \delta$ for $i \neq j$, the second equation of (1.2) also holds, and thus, the proof of theorem is complete with $p = \ell$.

We have, otherwise, $\liminf_{k \to \infty} \int_\Sigma |\nabla v_k|^2 \, dv_g > \int_\Sigma |\nabla u|^2 \, dv_g$, and define the concentration set of $\{v_k\}$ by

$$\mathcal{S}' = \bigcap_{r > 0} \left\{ z \in \Sigma \;\middle|\; \liminf_{k \to \infty} \int_{D_r(z)} |\nabla v_k|^2 \, dv_g > \varepsilon_0 \right\}.$$

Since

$$\int_{D_r(z)} |\nabla v_k|^2 \, dv_g < \int_{D_r(z)} |\nabla u_k|^2 \, dv_g$$

by (1.15), we obtain $\mathcal{S}' \subset \mathcal{S}$ and assume $\mathcal{S}' = \{x_1, \ldots, x_{\ell'}\}$ with $1 \leqslant \ell' \leqslant \ell$.

Taking $x_1 \in \mathcal{S}'$, now we detect the second bubble. First, we obtain

$$\liminf_{r \downarrow 0} \lim_{k \to \infty} \int_{D_r(x_1)} |\nabla v_k|^2 \, dv_g \geqslant \varepsilon_0 > 0.$$

Then, using the concentration function of v_k defined by

$$Q_k^{\ell+1}(t) = \sup_{z \in D_\delta(x_1)} \int_{D_t(z)} |\nabla v_k|^2 \, dv_g,$$

we choose

- a sequence of points $\{x_k^{\ell+1}\} \subset D_\delta(x_1)$ satisfying $\lim_{k\to\infty} x_k^{\ell+1} = x_1$,
- a sequence of positive numbers $\{\delta_k^{\ell+1}\}$ converging to 0

such that

$$Q_k^{\ell+1}\big(\delta_k^{\ell+1}\big) = \int_{D_{\delta_k^{\ell+1}}(x_k^{\ell+1})} |\nabla v_k|^2 \, dv_g = \frac{\varepsilon_0}{2}.$$

Here, we obtain $\delta_k^{\ell+1} > \delta_k^1$ by (1.15).

We use the rescaled map $\tilde{v}_k^{\ell+1}(x) = v_k(\delta_k^{\ell+1} x + x_k^{\ell+1})$ defined for $x \in \tilde{D}_\delta^{\ell+1}(x_k) = (\delta_k^{\ell+1})^{-1}(D_\delta(x_1) - x_k^{\ell+1})$ similarly to Step 1. Actually, it holds that

$$\int_{D_1(z)} \big|\nabla \tilde{v}_k^{\ell+1}\big|^2 \, d\tilde{v}_g \leqslant \frac{\varepsilon_0}{2}$$

for all $z \in \tilde{D}_\delta^{\ell+1}(x_k)$ with the equality when $z = 0$, and

$$\tilde{v}_k^{\ell+1} \rightharpoonup \omega^{\ell+1} \quad \text{weakly in } H^1_{\mathrm{loc}}\big(S^2 \setminus \{\infty\}, S^n\big)$$

for some $\omega^{\ell+1}$. Although these $\{v_k^{\ell+1}\}_k$ and $\{\tilde{v}_k^{\ell+1}\}_k$ are not harmonic, we can derive a variant of ε-compactness to the latter.

For this purpose, first, we note the following.

LEMMA 1.2.3. *It holds that*

$$\max\left\{ \frac{\delta_k^j}{\delta_k^{\ell+1}}, \frac{\delta_k^{\ell+1}}{\delta_k^j}, \frac{|x_k^j - x_k^{\ell+1}|}{\delta_k^j + \delta_k^{\ell+1}} \right\} \to \infty$$

for $j = 1, \ldots, \ell$.

PROOF. The assertion is obvious for $j \neq 1$, because

$$\lim_{k\to\infty} \left|x_k^j - x_k^{\ell+1}\right| = |x_j - x_1| > \delta.$$

If this is not the case for $j = 1$, there exists $R > 1$ such that

$$R^{-1} \leqslant \frac{\delta_k^{\ell+1}}{\delta_k^1} \leqslant R \quad \text{and} \quad \frac{|x_k^1 - x_k^{\ell+1}|}{\delta_k^1 + \delta_k^{\ell+1}} \leqslant R.$$

This implies

$$\begin{aligned} \frac{\varepsilon_0}{2} = Q_k^{\ell+1}\bigl(\delta_k^{\ell+1}\bigr) &= \int_{D_{\delta_k^{\ell+1}}(x_k^{\ell+1})} |\nabla v_k|^2 \, dv_g \leqslant \int_{D_{L\delta_k^1}(x_k^1)} |\nabla v_k|^2 \, dv_g \\ &= \int_{D_L} \left|\nabla\bigl(\tilde{u}_k^1 - \omega^1\bigr)\right|^2 d\tilde{v}_g \to 0 \end{aligned}$$

for some $L = L(R) > 0$ ($L(R) = 2R + R^2$ would suffice) by the H^1-strong local convergence of $\tilde{u}_k^1 \to \omega^1$. This is a contradiction. □

Now, we show the following lemma.

LEMMA 1.2.4. *The above defined $\omega^{\ell+1}$ is a nonconstant harmonic map*: $S^2 \to S^n$, *and* $\tilde{v}_k^{\ell+1} \to \omega^{\ell+1}$ *strongly in* $H^1_{\mathrm{loc}}(S^2 \setminus \{\infty\}, S^n)$.

PROOF. We distinguish two cases of "separated bubbles" (Case 1) and "bubbles on bubbles" (Case 2) indicated by Figures 1 and 2 of [97], respectively, regarding $\delta_k^{\ell+1} > \delta_k^1$.

CASE 1. There exists $R > 1$ such that

$$R^{-1} \leqslant \frac{\delta_k^{\ell+1}}{\delta_k^1} \leqslant R \quad \text{and} \quad \frac{|x_k^1 - x_k^{\ell+1}|}{\delta_k^1 + \delta_k^{\ell+1}} \to +\infty$$

or

$$\frac{\delta_k^{\ell+1}}{\delta_k^1} \to +\infty \quad \text{and} \quad \frac{|x_k^1 - x_k^{\ell+1}|}{\delta_k^1 + \delta_k^{\ell+1}} \to +\infty.$$

In this case, two bubbles ω^1 and $\omega^{\ell+1}$ are geometrically separated in spite of $\lim_{k\to\infty} x_k^1 = \lim_{k\to\infty} x_k^{\ell+1} = x_1$, and therefore, there is $L \gg 1$ such that

$$D_{\delta_k^1 L}(x_k^1) \cap D_{\delta_k^{\ell+1} L}\bigl(x_k^{\ell+1}\bigr) = \emptyset$$

for $k \gg 1$, regardless with $\delta_k^{\ell+1} \sim \delta_k^1$ or $\delta_k^{\ell+1} \gg \delta_k^1$. This implies

$$D_L \cap \left(\frac{x_k^{\ell+1} - x_k^1}{\delta_k^1} + \frac{\delta_k^{\ell+1}}{\delta_k^1} D_L \right) = \emptyset$$

and therefore,

$$\begin{aligned} \int_{D_L} \left| \nabla \omega^1 \left(\frac{x_k^{\ell+1} - x_k^1 - \delta_k^{\ell+1} \cdot}{\delta_k^1} \right) \right|^2 d\tilde{v}_g &= \int_{\frac{x_k^{\ell+1} - x_k^1}{\delta_k^1} + \frac{\delta_k^{\ell+1}}{\delta_k^1} D_L} |\nabla \omega^1|^2 \, dv_g \\ &\leqslant \int_{S^2 \setminus D_L} \left| \nabla \omega^1 \right|^2 dv_{S^2} = o(1) \end{aligned}$$

for $L \gg 1$. Thus, it holds that

$$\omega^1 \left(\frac{x_k^{\ell+1} - x_k^1 - \delta_k^{\ell+1} \cdot}{\delta_k^1} \right) \to \omega^1(\infty) \quad \text{strongly in } H^1_{\mathrm{loc}}\left(S^2 \setminus \{\infty\}, S^n\right),$$

passing to a subsequence.

In the case of $j \neq 1$,

$$D_{\delta_k^j L}\left(x_k^j\right) \cap D_{\delta_k^{\ell+1} L}\left(x_k^{\ell+1}\right) = \emptyset,$$

is obvious, and hence it holds that

$$\omega^j \left(\frac{x_k^{\ell+1} - x_k^j - \delta_k^{\ell+1} \cdot}{\delta_k^j} \right) \to \omega^j(\infty) \quad \text{strongly in } H^1_{\mathrm{loc}}\left(S^2 \setminus \{\infty\}, S^n\right)$$

similarly. Since we can apply the ε-compactness to

$$\begin{aligned} \tilde{u}_k^{\ell+1}(x) &= u_k\left(\delta_k^{\ell+1} x + x_k^{\ell+1}\right) \\ &= v_k\left(\delta_k^{\ell+1} x + x_k^{\ell+1}\right) + \sum_{j=1}^{\ell} \left[\omega^j \left(\frac{\delta_k^{\ell+1} x + x_k^{\ell+1} - x_k^j}{\delta_k^j} \right) - \omega^j(\infty) \right] \end{aligned}$$

the proof of this lemma is complete, where (1.14) is used to derive the above equality.

CASE 2. There exists $M > 0$ such that

$$\frac{\delta_k^{\ell+1}}{\delta_k^1} \to +\infty \quad \text{and} \quad \frac{|x_k^1 - x_k^{\ell+1}|}{\delta_k^{\ell+1}} \leqslant M. \tag{1.16}$$

In this case, we obtain

$$\int_{S^2\setminus D_\alpha}\left|\nabla\omega^1\left(\frac{x_k^{\ell+1}-x_k^1-\delta_k^{\ell+1}\cdot}{\delta_k^1}\right)\right|^2 dv_{S^2}$$
$$=\int_{S^2\setminus(\frac{x_k^{\ell+1}-x_k^1}{\delta_k^1}+\frac{\delta_k^{\ell+1}}{\delta_k^1}D_\alpha)}\left|\nabla\omega^1\right|^2 dv_{S^2}=o(1)$$

as $k\to\infty$ by $\frac{\delta_k^{\ell+1}}{\delta_k^1}\to+\infty$, where $\alpha>0$ is arbitrary. For $j\neq 1$, on the other hand, there is the strong $H^1_{\rm loc}$ convergence of $\nabla\omega^j\big(\frac{x_k^{\ell+1}-x_k^j-\delta_k^{\ell+1}\cdot}{\delta_k^j}\big)\to 0$. Since ε-compactness is applicable to the above defined $\{\tilde{u}_k^{\ell+1}\}_k$, it follows that $\tilde{u}_k^{\ell+1}\to\omega^{\ell+1}$ strongly in $H^1_{\rm loc}(S\setminus(\{\infty\}\cup D_\alpha))$.

From the first bubbling process it follows that

$$\int_{D_{\beta\delta_k^1}(x_k^1)}|\nabla v_k|^2\,dv_g=o(1),$$

$$\int_{D_{\beta\delta_k^1}(x_k^1)}\left|\nabla\omega^{\ell+1}\left(\frac{\cdot-x_k^{\ell+1}}{\delta_k^{\ell+1}}\right)\right|^2 dv_g$$
$$\leqslant\int_{D_{\frac{\beta\delta_k^1}{\delta_k^{\ell+1}}}(x_k^1-x_k^{\ell+1})}\left|\nabla\omega^{\ell+1}\right|^2 dv_{S^2}=o(1)$$

for $\beta\gg 1$, and therefore,

$$\int_{D_R}\left|\nabla\left(\tilde{v}_k-\omega^{\ell+1}\right)\right|^2 d\tilde{v}_g=\int_{D_\alpha}+\int_{D_R\setminus D_\alpha}$$
$$=\int_{A_k(\alpha,\beta)}\left|\nabla\left(v_k-\omega^{\ell+1}\left(\frac{\cdot-x_n^{\ell+1}}{\delta_k^{\ell+1}}\right)\right)\right|^2 dv_g+o(1)$$

holds for any $R\gg 1$, where

$$A_k(\alpha,\beta)=D_{\alpha\delta_k^{\ell+1}}\left(x_k^1\right)\setminus D_{\beta\delta_k^1}\left(x_k^1\right). \tag{1.17}$$

This $A=A_k(\alpha,\beta)$ is connecting two bubbles ω^1 and $\omega^{\ell+1}$, and is called the *neck region*. Since $\frac{\delta_k^{\ell+1}}{\delta_k^1}\to+\infty$ is supposed in this case, this region appears always.

However, we can assume

$$\int_{A_k(\alpha,\beta)}\left|\nabla\left(v_k-\omega^{\ell+1}\left(\frac{\cdot-x_k^{\ell+1}}{\delta_k^{\ell+1}}\right)\right)\right|^2 dv_g\leqslant\int_{A_k(\alpha,\beta)}|\nabla u_k|^2\,dv_g<\varepsilon_0$$

by making $\beta \gg 1 \gg \alpha > 0$, and then Lemma 1.2.2 guarantees

$$\int_{A_k(\alpha,\beta)} |\nabla u_k|^2 \, dv_g \leqslant C \int_{A_k(\alpha,\beta)} |\nabla w_k|^2 \, dv_g \tag{1.18}$$

if

$$\Delta w_k = 0 \quad \text{in } A_k(\alpha,\beta), \qquad w_k = u_k \quad \text{on } \partial A_k(\alpha,\beta).$$

To estimate the right-hand side of (1.18), we define $f_k = f_k(x)$ and $g_k = g_k(x)$ by

$$\begin{aligned}
&\Delta f_k = 0 \quad \text{in } A_k(\alpha,\beta),\\
&f_k = u_k - \omega^{\ell+1}(P) \quad \text{on } \partial D_{\alpha\delta_k^{\ell+1}}\big(x_k^1\big),\\
&f_k = u_k - \omega^1(\infty) \quad \text{on } \partial D_{\beta\delta_k^1}\big(x_k^1\big),
\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}
&\Delta g_k = 0 \quad \text{in } A_k(\alpha,\beta),\\
&g_k = \omega^{\ell+1}(P) \quad \text{on } \partial D_{\alpha\delta_k^{\ell+1}}\big(x_k^1\big),\\
&g_k = \omega^1(\infty) \quad \text{on } \partial D_{\beta\delta_k^1}\big(x_k^1\big),
\end{aligned} \tag{1.20}$$

where $P = \lim_{k\to\infty} \frac{x_k^1 - x_k^{\ell+1}}{\delta_k^{\ell+1}}$. Since $w_k = f_k + g_k$, it holds that

$$\int_{A_k(\alpha,\beta)} |\nabla w_k|^2 \, dv_g \leqslant 2\int_{A_k(\alpha,\beta)} |\nabla f_k|^2 \, dv_g + 2\int_{A_k(\alpha,\beta)} |\nabla g_k|^2 \, dv_g,$$

while

$$\int_{A_k(\alpha,\beta)} |\nabla f_k|^2 \, dv_g = o(1), \tag{1.21}$$

$$\int_{A_k(\alpha,\beta)} |\nabla g_k|^2 \, dv_g \leqslant C \frac{|\omega^1(\infty) - \omega^{\ell+1}(P)|^2}{\log\big(\frac{\alpha\delta_k^{\ell+1}}{\beta\delta_k^1}\big)} \tag{1.22}$$

are proven. In this case, we obtain

$$\int_{A_k(\alpha,\beta)} |\nabla u_k|^2 \, dv_g \leqslant C \int_{A_k(\alpha,\beta)} |\nabla w_k|^2 \, dv_g = o(1),$$

and therefore,

$$\int_{D_R} \big|\nabla\big(v_k\big(x_k^{\ell+1} + \delta_k^{\ell+1}\cdot\big) - \omega^{\ell+1}\big)\big|^2 \, dv_g = o(1),$$

which completes the proof of this lemma.

To show the above estimates, we abbreviate

$$\partial A_\alpha^k = \partial D_{\alpha\delta_k^{l+1}}\big(x_k^1\big) \quad \text{and} \quad \partial A_\beta^k = \partial D_{\beta\delta_k^1}\big(x_k^1\big)$$

for simplicity. First, from the strong convergence $\tilde{u}_k^{\ell+1} \to \omega^{\ell+1}$ in $H^1_{\mathrm{loc}}(S^2 \setminus (\{\infty\} \cup D_\alpha))$, it follows that

$$\left\| u_k - \omega^{\ell+1}\left(\frac{\cdot - x_k^{\ell+1}}{\delta_k^{\ell+1}}\right)\right\|_{W^{1/2,2}(\partial A_\alpha^k)} = o(1).$$

Similarly, since $\tilde{u}_k^1 \to \omega^1$ strongly in $H^1(D_\beta)$, we obtain

$$\left\| u_k - \omega^{1}\left(\frac{\cdot - x_k^{1}}{\delta_k^{1}}\right)\right\|_{W^{1/2,2}(\partial A_\beta^k)} = o(1).$$

We have, on the other hand,

$$\omega^{\ell+1}\left(\frac{x_k^1 - x_k^{\ell+1}}{\delta_k^{\ell+1}}\right) \to \omega^{\ell+1}(P) \quad \text{and} \quad \omega^1\left(\frac{\cdot - x_k^1}{\delta_k^1}\right) \to \omega^1(\infty)$$

by (1.16), and therefore, (1.21) by the elliptic estimate to (1.19).

Since the boundary value of g_k is a constant in (1.20), on the other hand, it follows that $g_k = A_k \log|x| + B_k$, where

$$A_k = \frac{\omega^{\ell+1}(P) - \omega^1(\infty)}{\log\big(\frac{\alpha\delta_k^{\ell+1}}{\beta\delta_k^1}\big)},$$

$$B_k = \frac{\omega^1(\infty)\log\alpha\delta_k^{\ell+1} - \omega^{\ell+1}(P)\log\beta\delta_k^1}{\log\big(\frac{\alpha\delta_k^{l+1}}{\beta\delta_k^1}\big)}.$$

This implies

$$\int_{A_k(\alpha,\beta)} |\nabla g_k|^2\, dv_g \leqslant C\int_{R_1}^{R_2} \big|g'(r)\big|^2 r\, dr = C A_k^2 \log\left(\frac{R_2}{R_1}\right)$$

for $R_2 = \alpha\delta_k^{l+1}$ and $R_1 = \beta\delta_k^1$, and hence (1.22). □

STEP 4. We have extracted the principal and the second bubbles $\{\omega^j\}_{j=1,\dots,\ell}$ and $\{\omega^j\}_{j=\ell+1,\dots,\ell'}$, respectively. If there is no energy gap by these bubbles, the proof is done. Otherwise, we apply the hierarchical argument and detect third bubbles by the bubbling process of Steps 2 and 3. By Lemma 1.2.1, on the other hand, each ω^i has the energy bounded from below, and therefore, this process terminates in a finite time. Eventually, we obtain the energy identity and the proof is complete.

2. Mass quantization

Mass quantization is concerned with the L^1 norm. It is associated with a different exponent from that of energy quantization, but is derived from the scaling invariance of the problem similarly, where the location of blowup points are prescribed by the linear part. Section 2.1 describes this profile for the complex Ginzburg–Landau equation case. Then, we take a problem involving two-dimensional Laplacian and the exponential nonlinearity in the second section. We study mass quantization of this problem with its higher-dimensional analogy in the final section.

2.1. *Ginzburg–Landau equation*

The complex Ginzburg–Landau equation is described by

$$-\Delta v = \frac{1}{\varepsilon^2}\left(1-|v|^2\right)v \quad \text{in } \Omega, \qquad v = g \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbf{R}^2 \cong \mathbf{C}$ is a simply-connected bounded domain with smooth boundary $\partial\Omega \cong S^1$, $v : \Omega \to \mathbf{C}$, and $g = g(x) : \partial\Omega \to S^1 = \mathbf{R}/(2\pi\mathbf{Z})$ is a given smooth function. It is the Euler–Lagrange equation of

$$E_\varepsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega \left(1-|v|^2\right)^2 dx$$

defined for $v \in H^1_g(\Omega) = \{v \in H^1(\Omega;\mathbf{C}) \mid v = g \text{ on } \partial\Omega\}$. Actually, E_ε is compact on $H^1_g(\Omega)$ and each critical point of E_ε corresponds to a solution to (2.1).

This functional was introduced by V. Ginzburg and L. Landau in 1950 in the study of super-conductivity. Now, it is considered to describe the energy state of a super-conducting sample on a macroscopic scale. Many variants have been used in various fields so far. Mass quantization, on the other hand, is observed in $0 < \varepsilon \ll 1$, called strong repulsive case in physics.

Any solution $v = v(x)$ to (2.1) is smooth on $\overline{\Omega}$, and is provided with the following estimates, where $C = C(\Omega, g)$ is a constant determined by Ω and g:

$$|v| \leqslant 1, \qquad |\nabla v| \leqslant C\varepsilon^{-1} \quad \text{in } \Omega,$$

$$\Omega\text{: star-shaped} \quad \Rightarrow \quad \frac{1}{4\varepsilon^2}\int_\Omega \left(1-|v|^2\right)^2 dx \leqslant C. \tag{2.2}$$

In fact, we obtain

$$\frac{1}{2}\Delta\left(|v|^2-1\right) \geqslant \frac{1}{\varepsilon^2}|v|^2\left(|v|^2-1\right) \quad \text{in } \Omega, \qquad |v| = 1 \quad \text{on } \partial\Omega,$$

and $|v| \leqslant 1$ follows from the maximum principle. From this inequality and the elliptic estimate, we have $\|v\|_{W^{2,p}} \leqslant C\varepsilon^{-2}$, and therefore, $\|v\|_{W^{1,p}} \leqslant C\varepsilon^{-1}$ by the interpolation theorem, where $1 < p < \infty$. Letting $p > 2$, we obtain $|\nabla v| \leqslant C\varepsilon^{-1}$ in Ω. The final inequality of (2.2) is obtained by the Pohozaev identity

$$\begin{aligned} &\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 ds + \frac{1}{2\varepsilon^2}\int_{\Omega}\left(1-|v|^2\right)^2 dx \\ &\quad = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial g}{\partial\tau}\right|^2 ds - \int_{\partial\Omega}(x\cdot\tau)\frac{\partial v}{\partial\nu}\frac{\partial g}{\partial\tau}\,ds, \end{aligned}$$

where ν and τ denote the unit normal and tangential vectors to $\partial\Omega$, respectively, such that (ν,τ) forms a right-hand system.

The energy minimizing solution to (2.1) attains

$$\kappa_\varepsilon = \inf\left\{E_\varepsilon(v) \mid v \in H^1_g(\Omega)\right\}.$$

Its existence is obvious although it may not be unique. If Ω is a unit disc and $g(\theta) = \exp(\imath\, d\theta)$ for $d \in \mathbf{Z}$, one obtains a solution to (2.1), assuming $v_{d,\varepsilon}(r,\theta) = f_{d,\varepsilon}(r)\exp(\imath\, d\theta)$, where $f = f_{d,\varepsilon}$ is a solution to

$$\begin{aligned} &r^2 f'' + rf' - d^2 f + \frac{r^2}{\varepsilon^2} f(1-f^2) = 0 \quad (0<r<1), \\ &f(0) = 0, \qquad f(1) = 1. \end{aligned}$$

Estimating $E_\varepsilon(v_{d,\varepsilon})$, we can show that this $v_{d,\varepsilon}$ with $|d| \geqslant 2$ does not attain κ_ε in the case of $0 < \varepsilon \ll 1$. Multiple existence result for the general case is obtained by the method of variation [2,3].

Asymptotic behavior of the solution to (2.1) as $\varepsilon \downarrow 0$ is, thus, associated with d, the degree (winding number) of the mapping $g : \partial\Omega \to S^1$. This asymptotics is actually described by the renormalized energy defined for ℓ-points $a_1, \dots a_\ell \in \Omega$ and ℓ-integers $d_1, \dots, d_\ell \in \mathbf{Z}$ satisfying $d = \sum_{i=1}^{\ell} d_i$. In more precise, we take the scalar-valued $\Phi = \Phi(x)$ satisfying

$$\Delta\Phi = \sum_{i=1}^{\ell} 2\pi d_i \delta_{a_i} \quad \text{in } \Omega, \qquad \frac{\partial\Phi}{\partial\nu} = g \wedge g_\tau \quad \text{on } \partial\Omega, \qquad \int_{\partial\Omega}\Phi\, ds = 0,$$

and introduce the harmonic function $R = R(x)$ by

$$R(x) = \Phi(x) - \sum_{i=1}^{\ell} d_i \log|x - a_i|, \tag{2.3}$$

where $\wedge$ denotes the wedge product in $\mathbf{R}^2$. Then, the renormalized energy associated to the configuration $(a_1, \ldots, a_\ell)$ is defined by

$$W_g(a_1, \ldots, a_\ell) = -\pi \sum_{i<j} d_i d_j \log|a_i - a_j| \\ - \pi \sum_{i=1}^{\ell} d_i R(a_i) + \frac{1}{2} \int_{\partial\Omega} \Phi(g \wedge g_\tau)\, ds. \tag{2.4}$$

Thus, it is a function defined on $\overbrace{\Omega \times \cdots \times \Omega}^{\ell \text{ times}} \setminus \Delta$, where Δ denotes the diagonal part:

$$\Delta = \big\{(a_1, \ldots, a_\ell) \in \Omega^\ell \;\big|\; a_i = a_j \text{ for some } i \neq j\big\}.$$

We note that this W_g depends on the choice of $(d_1, \ldots, d_\ell)$.

Profile of concentration may be called mass quantization in this case.

THEOREM 2.1. (See [15].) *Let $\Omega \subset \mathbf{R}^2$ be a star-shaped bounded domain with smooth boundary $\partial\Omega$, and $v_\varepsilon = v_\varepsilon(x)$ be a solution to* (2.1) *for* $0 < \varepsilon \ll 1$. *Then, there is a constant $C > 0$ determined by g and Ω such that*

$$E_\varepsilon(v_\varepsilon) \leqslant C\big(|\log \varepsilon| + 1\big).$$

Any $\varepsilon_k \downarrow 0$ admits $\{\varepsilon'_k\} \subset \{\varepsilon_k\}$ such that for some ℓ distinct points $a_1, \ldots, a_\ell$ in Ω, ℓ integers $d_1, \ldots, d_\ell$ different from 0, *it holds that*

$$v_{\varepsilon'_k} \to v_* \quad \text{in } C^\infty_{\mathrm{loc}}(\Omega \setminus \mathcal{S}; \mathbf{C}) \cap C^{1,\alpha}(\overline{\Omega} \setminus \mathcal{S}; \mathbf{C}),$$

where $\mathcal{S} = \{a_1, \ldots, a_\ell\}$, $0 < \alpha < 1$, and $v_* : \overline{\Omega} \setminus \mathcal{S} \to S^1$ is a harmonic map defined by

$$v_*(z) = \exp\big(\imath\phi(z)\big) \prod_{i=1}^{\ell} \left(\frac{z - a_i}{|z - a_i|}\right)^{d_i} \tag{2.5}$$

using the harmonic function $\phi = \phi(z)$ on $\overline{\Omega}$ compatible to $v_* = g$ on $\partial\Omega$. We obtain, furthermore,

$$\frac{1}{4(\varepsilon'_k)^2}\big(1 - |v_{\varepsilon'_k}|^2\big)^2\, dx \rightharpoonup \frac{\pi}{2} \sum_{i=1}^{\ell} d_i^2\, \delta_{a_i}(dx)$$

in the sense of measures on $\overline{\Omega}$, and the configuration $(a_i)_{i=1,\ldots,\ell}$ is a critical point of the renormalized energy W_g:

$$DW_g(a_1, \ldots, a_\ell) = 0 \in \mathbf{R}^{2\ell},$$

which is equivalent to

$$\nabla_x \big(\Phi(x) - d_i \log|x - a_i|\big)_{x=a_i} = 0 \quad (i = 1, \dots, \ell).$$

Concerning the energy minimizing solution $\underline{v}_\varepsilon \in H^1_g(\Omega)$ which attains κ_ε, the above result is sharpened as follows. First, in the case of $d = 0$, there is smooth $\psi : \partial\Omega \to \mathbf{R}$ such that $g = \exp(\imath\psi)$. We define $v_* = \exp(\imath\varphi_*) \in H^1_g(\Omega)$ by

$$\Delta\varphi_* = 0 \quad \text{in } \Omega, \qquad \varphi_* = \psi \quad \text{on } \partial\Omega \tag{2.6}$$

similarly. Then, it follows that

$$\kappa_\varepsilon \equiv E_\varepsilon(\underline{v}_\varepsilon) \leqslant E_\varepsilon(v_*) = \frac{1}{2}\int_\Omega |\nabla v_*|^2\,dx = \frac{1}{2}\int_\Omega |\nabla\varphi_*|^2\,dx,$$

and then $\|\underline{v}_\varepsilon - v_*\|_\infty = O(\varepsilon^2)$ and $\underline{v}_\varepsilon \to v_*$ in H^1 are proven [14,120]. In the other case of $d > 0$, we obtain $d_i = 1$ in (2.5) and the map $\underline{v}_\varepsilon$ has exactly $d = \deg(g)$ zeros for $0 < \varepsilon \ll 1$. Furthermore, the configuration $(a_1, \dots, a_d)$ minimizes the renormalized energy on $\Omega^d \setminus \Delta$ and it holds that

$$\kappa_\varepsilon = \pi d|\log\varepsilon| + W_g(a_1, \dots, a_d) + d\gamma_0 + o(1)$$

as $\varepsilon \downarrow 0$, where $\gamma_0 > 0$ is an absolute constant.

In the nonminimizing case, the multiplicity d_i may not be $+1$ and the opposite sign of vortices can coexist.

2.2. *Exponential nonlinearity*

Emden–Fowler equation with exponential nonlinearity,

$$-\Delta v = \sigma e^v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{2.7}$$

arises in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion [55,27,11], where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $\sigma > 0$ is a parameter. In the case of $n = 2$, it is also associated with the theories of turbulence and self-dual gauge [82,80,148,134]. Actually, this equation with $n = 2$ is provided with complex and geometric structures, which results in the mass quantization of the blowup family of solutions [126,128].

2.2.1. *Complex structure.* Putting $u = v + \log\sigma$, we obtain

$$-\Delta u = e^u \quad \text{in } \Omega. \tag{2.8}$$

If we identify $x=(x_1,x_2)\in\Omega$ to $z=x_1+\imath x_2\in\mathbf{C}$, then (2.8) means

$$u_{z\overline{z}}=-\frac{1}{4}e^u$$

for $\overline{z}=x_1-\imath x_2$. This implies

$$s_{\overline{z}}=u_{zz\overline{z}}-u_z u_{z\overline{z}}=-\frac{1}{4}e^u u_z+\frac{1}{4}u_z e^u=0$$

for

$$s=u_{zz}-\frac{1}{2}u_z^2, \tag{2.9}$$

and therefore, $s=s(z)$ is a holomorphic function of $z\in\Omega\subset\mathbf{C}$.

Regarding (2.9) as a Riccati equation of u, we obtain

$$\varphi_{zz}+\frac{1}{2}s\varphi=0 \tag{2.10}$$

for $\varphi=e^{-u/2}$. Here, we take $x^*=(x_1^*,x_2^*)\in\Omega$, and define a fundamental system of solutions to the linear equation (2.10), denoted by $\{\varphi_1(z),\varphi_2(z)\}$, such that

$$\varphi_1|_{z=z^*}=\left.\frac{\partial\varphi_2}{\partial z}\right|_{z=z^*}=1 \quad\text{and}\quad \left.\frac{\partial\varphi_1}{\partial z}\right|_{z=z^*}=\varphi_2|_{z=z^*}=0, \tag{2.11}$$

where $z^*=x_1^*+\imath x_2^*$. This $\{\varphi_1(z),\varphi_2(z)\}$ is composed of analytic functions of $z\in\Omega$, and it holds that

$$\varphi=e^{-u/2}=\overline{f_1}(\overline{z})\varphi_1(z)+\overline{f_2}(\overline{z})\varphi_2(z) \tag{2.12}$$

for some functions $\overline{f_1}$ and $\overline{f_2}$ of $\overline{z}$.

These $\overline{f_1}(\overline{z})$, $\overline{f_2}(\overline{z})$ are prescribed by the Wronskian. Since

$$W(\varphi_1,\varphi_2)\equiv\varphi_1\varphi_{2z}-\varphi_{1z}\varphi_2=1,$$

it holds that

$$\overline{f_1}(\overline{z})=W(\varphi,\varphi_2)=\varphi\varphi_{2z}-\varphi_z\varphi_2,$$
$$\overline{f_2}(\overline{z})=W(\varphi_1,\varphi)=\varphi_1\varphi_z-\varphi_{1z}\varphi,$$

and the left-hand side is independent of z. Taking $z=z^*$ in the right-hand side, therefore, we obtain

$$\overline{f_1}(\overline{z})=\varphi(z^*,\overline{z}) \quad\text{and}\quad \overline{f_2}(\overline{z})=\varphi_z(z^*,\overline{z}). \tag{2.13}$$

Since φ is real-valued, it holds that

$$\varphi_{\overline{z}\overline{z}} + \frac{1}{2}\overline{s}\varphi = 0 \tag{2.14}$$

for $\overline{s} = \overline{s}(\overline{z})$ defined by $\overline{s}(\overline{z}) = \overline{s(z)}$. This relation is valid to $\varphi = \overline{f_1}(\overline{z}), \overline{f_2}(\overline{z})$ defined by (2.13), while $\{\overline{\varphi}_1, \overline{\varphi}_2\}$ forms a fundamental system of solutions satisfying

$$\overline{\varphi}_1|_{\overline{z}=\overline{z^*}} = \frac{\partial\overline{\varphi}_2}{\partial\overline{z}}\Big|_{\overline{z}=\overline{z}^*} = 1 \quad \text{and} \quad \frac{\partial\overline{\varphi_1}}{\partial\overline{z}}\Big|_{\overline{z}=\overline{z}^*} = \overline{\varphi_2}|_{\overline{z}=\overline{z}^*} = 0.$$

Thus, $\overline{f_1}(\overline{z})$ and $\overline{f_2}(\overline{z})$ are linear combinations of $\overline{\varphi}_1(\overline{z})$ and $\overline{\varphi}_2(\overline{z})$.

If the above prescribed $x^* = (x_1^*, x_2^*) \in \Omega$ is a critical point of u, then it holds that

$$\begin{aligned}
\overline{f_1}(\overline{z}^*) &= \varphi(z^*, \overline{z}^*) = e^{-u/2}\big|_{x=x^*}, \\
\frac{\partial\overline{f_1}}{\partial\overline{z}}(\overline{z}^*) &= \varphi_{\overline{z}}(z^*, \overline{z}^*) = \frac{\partial}{\partial\overline{z}}e^{-u/2}\big|_{x=x^*} = 0, \\
\overline{f_2}(\overline{z}^*) &= \varphi_{\overline{z}}(z^*, \overline{z}^*) = \frac{\partial}{\partial\overline{z}}e^{-u/2}\big|_{x=x^*} = 0, \\
\frac{\partial\overline{f_2}}{\partial\overline{z}}(\overline{z}^*) &= \varphi_{z\overline{z}}(z^*, \overline{z}^*) = \frac{1}{4}\Delta e^{-u/2}\big|_{x=x^*} = -\frac{1}{8}e^{-u/2}\Delta u\big|_{x=x^*}, \\
&= \frac{1}{8}e^{u/2}\big|_{x=x^*}
\end{aligned}$$

and therefore, we obtain $\overline{f_1}(\overline{z}) = c\overline{\varphi_1}(\overline{z})$ and $\overline{f_2}(\overline{z}) = \frac{1}{8}c^{-1}\overline{\varphi_2}(\overline{z})$ for $c = e^{-u/2}|_{x=x^*}$. This means $f_1 = c\varphi_1$ and $f_2 = \frac{c^{-1}}{8}\varphi_2$, and therefore, it holds that

$$e^{-u/2} = c|\varphi_1|^2 + \frac{c^{-1}}{8}|\varphi_2|^2 \tag{2.15}$$

by (2.12). Writing $\psi_1 = c^{1/2}8^{1/4}\varphi_1$ and $\psi_2 = c^{-1/2}8^{-1/4}\varphi_2$, we have

$$\begin{aligned}
&W(\psi_1, \psi_2) = W(\varphi_1, \varphi_2) = 1, \\
&\left(\frac{1}{8}\right)^{1/2} e^{u/2} = \left\{c\left(\frac{1}{8}\right)^{-1/2}|\varphi_1|^2 + c^{-1}\left(\frac{1}{8}\right)^{1/2}|\varphi_2|^2\right\}^{-1} \\
&\qquad\qquad = \frac{1}{|\psi_1|^2 + |\psi_2|^2},
\end{aligned}$$

and therefore,

$$\frac{|F'|}{1+|F|^2} = \frac{W(\psi_1, \psi_2)}{|\psi_1|^2 + |\psi_2|^2} = \left(\frac{1}{8}\right)^{1/2} e^{u/2} \tag{2.16}$$

for $F = \psi_2/\psi_1$. This means that (2.7) is reduced to finding an analytic function $F = F(z)$ of $z \in \Omega \subset \mathbf{C}$ such that

$$\rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2} \tag{2.17}$$

by $u = v + \log \sigma$ and $v|_{\partial\Omega} = 0$, where

$$\rho(F) = \frac{|F'|}{1 + |F|^2}.$$

The above defined $F = F(z)$ is a quotient of two linearly independent solutions to (2.10), and therefore, it holds that

$$\{F; z\} = -\frac{1}{2}s,$$

where

$$\{F; z\} = \frac{3}{4}\left(\frac{F''}{F'}\right)^2 - \frac{1}{2}\frac{F'''}{F'}$$

is the Schwarzian derivative.

2.2.2. *Geometric structure.* It is known that $\rho(F)$ describes the spherical derivative of $F = F(z)$. More precisely, if $d\Sigma^2$ denote the standard metric of the Riemannian sphere $\overline{\mathbf{C}}$ with the south pole $(0, 0, 0)$ and the north pole $(0, 0, 1)$, and if $\tau : \overline{\mathbf{C}} \to \mathbf{C} \cup \{\infty\}$ denotes the stereographic projection, then the conformal transformation $\overline{F} = \tau^{-1} \circ F$ induces the relation

$$\frac{d\Sigma}{ds} = \rho(F), \tag{2.18}$$

where $ds^2 = dx_1^2 + dx_2^2$ denotes the Euclidean metric on $\mathbf{C} \cong \mathbf{R}^2$. In particular, $\rho(F)$ is invariant under $O(3)$ transformation of $\overline{\mathbf{C}}$.

If $\omega \Subset \Omega$ is a sub-domain, then the immersed length of $\overline{F}(\partial\omega)$ and the immersed area of $\overline{F}(\omega)$ on $\overline{\mathbf{C}}$ are defined by

$$\ell_1(\partial\omega) = \int_{\partial\omega} \rho(F)\, ds \quad \text{and} \quad m_1(\omega) = \int_{\omega} \rho(F)^2\, dx,$$

respectively, and therefore, it follows that

$$\ell_1(\partial\omega)^2 \geqslant 4m_1(\omega)\big(\pi - m_1(\omega)\big) \tag{2.19}$$

from the isoperimetric inequality. Putting

$$\ell(\partial\omega) = \int_{\partial\omega} p^{1/2}\,ds = 8^{1/2}\int_{\partial\omega} \rho(F)\,ds,$$
$$m(\omega) = \int_{\omega} p\,dx = 8\int_{\omega} \rho(F)^2\,dx$$

with $p = e^u$, we obtain

$$\ell(\partial\omega)^2 \geqslant \frac{1}{2}m(\omega)\bigl(8\pi - m(\omega)\bigr) \tag{2.20}$$

by (2.16) and (2.19).

Relation (2.20) is a form of Bol's inequality on the surface $\mathcal{M}$ with the Gaussian curvature less than or equal to $1/2$. More precisely, (2.20) describes this inequality for nonparametric $\mathcal{M}$, where $p = p(x) > 0$ is a C^2 function defined on the domain $\Omega \subset \mathbf{R}^2$ of which boundary is composed of a finite number of Jordan curves,

$$-\Delta \log p \leqslant p \quad \text{in } \Omega, \tag{2.21}$$

and $\omega \Subset \Omega$ is a sub-domain with the boundary $\partial\omega$ locally homeomorphic to a line [7]. This geometric isoperimetric inequality induces an analytic isoperimetric inequality concerning the first eigenvalue of the Laplace–Beltrami operator, by spherically decreasing rearrangement with respect to $d\Sigma = p(x)^{1/2}\,ds$.

THEOREM 2.2. (See [8].) *If $\Omega \subset \mathbf{R}^2$ is a bounded open set with the boundary $\partial\Omega$ composed of a finite number of Jordan curves and $p = p(x) > 0$ is a continuous function on $\overline{\Omega}$ which is C^2 in Ω and satisfies* (2.21)*, then it holds that*

$$\lambda \equiv \int_{\Omega} p < 8\pi \quad \Rightarrow \quad \nu_1(p,\Omega) \geqslant \nu_1(p^*,\Omega^*), \tag{2.22}$$

where

$$\nu_1(p,\Omega) = \inf\left\{ \int_{\Omega} |\nabla v|^2\,dx \;\middle|\; v \in H_0^1(\Omega),\ \int_{\Omega} v^2 p\,dx = 1 \right\}, \tag{2.23}$$

$p^* = \sigma^* e^{v^*}$, $\Omega^* = \{x \in \mathbf{R}^2 \mid |x| < 1\}$, *and*

$$-\Delta v^* = \sigma^* e^{v^*} \quad \text{in } \Omega^*, \qquad v^* = 0 \quad \text{on } \partial\Omega^*,$$
$$\int_{\Omega^*} \sigma^* e^{v^*} = \lambda. \tag{2.24}$$

If the equality holds in (2.22)*, then Ω is a disc and $p = p(x)$ is a radially symmetric function satisfying $-\Delta \log p = p$ in Ω.*

The value $\nu_1(p,\Omega)$ defined by (2.23) is the first eigenvalue of

$$-\Delta\varphi = \nu p\varphi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{2.25}$$

while the above introduced (σ^*, v^*) exists uniquely for each $\lambda \in (0, 8\pi)$. This $\nu = \nu_1(p,\Omega)$ is attained by the eigenfunction $\varphi \in H_0^1(\Omega)$ sufficiently smooth in Ω, $\varphi > 0$ in Ω, and $\varphi = 0$ on $\partial\Omega$. We put $\Omega_t = \{x \in \Omega \mid \varphi(x) > t\}$ for $t > 0$ and take the open concentric disc D_t^* of Ω^* satisfying

$$\int_{D_t^*} p^*\,dx = \int_{D_t} p\,dx.$$

Then, we define Bandle's spherically decreasing rearrangement φ^* of φ by

$$\varphi^*(x) = \sup\{t \mid x \in D_t^*\},$$

where $x \in \Omega^*$. This φ^* is radially symmetric, $\varphi^* > 0$ in Ω^*, and $\varphi^* = 0$ on $\partial\Omega^*$. Since $\varphi \mapsto \varphi^*$ is equi-measurable, the relation

$$\int_\Omega \varphi^2 p\,dx = \int_{\Omega^*} \varphi^{*2} p^*\,dx \tag{2.26}$$

holds, while the decrease of Dirichlet integral,

$$\int_\Omega |\nabla\varphi|^2\,dx \geqslant \int_{\Omega^*} |\nabla\varphi^*|^2\,dx \tag{2.27}$$

is achieved by Bol's inequality as is described in the following chapter. Thus we obtain (2.22).

2.2.3. *Radial solutions.* From the general theory [56], any classical solution to (2.7) is radially symmetric if Ω is the unit ball. To classify such a solution for $n = 2$, first, we study

$$v'' + \frac{1}{r}v' + \sigma e^v = 0 \quad (0 < r < \infty), \qquad v'(0) = 0. \tag{2.28}$$

If $v_0 = v_0(r)$ is a solution to this problem, then so is

$$v(r) = v_0\bigl(e^{\alpha/2} r\bigr) + \alpha$$

for $\alpha \in \mathbf{R}$. Thus, we shall assign a special solution $v_0(r)$ to (2.28) and chose α by the boundary condition, i.e.,

$$v_0\bigl(e^{\alpha/2}\bigr) + \alpha = 0. \tag{2.29}$$

For this purpose, we deduce

$$\frac{d^2}{ds^2}(v+2s)+\sigma e^{v+2s}=0$$

from (2.28) using $s=\log r$, and obtain the one-dimensional case,

$$u''+\sigma e^u=0 \quad (-\infty<s<\infty), \tag{2.30}$$

where $u=v+2s$. This equation implies

$$\left\{u''-\frac{1}{2}(u')^2\right\}'=0,$$

and we take the case

$$u''-\frac{1}{2}(u')^2=-2. \tag{2.31}$$

Actually, (2.31) is reduced to the logistic equation

$$\ell'=(1-\ell)\ell$$

by $\ell=\frac{2-u'(s/2)}{4}$, and we can assign a solution

$$\ell(s)=\frac{1}{2}\left(1+\tanh\frac{s}{2}\right).$$

This $\ell=\ell(s)$ induces $v_0=v_0(r)$ defined by

$$v_0+2s=-2\log\cosh s+\log\frac{2}{\sigma},$$

i.e.,

$$v_0(r)=\log\left\{\frac{8/\sigma}{(r^2+1)^2}\right\}$$

as a special solution to (2.28). Then, (2.29) is reduced to the algebraic equation

$$\frac{8}{\sigma}=\frac{(e^\alpha+1)^2}{e^\alpha},$$

and thus, we have classified the solutions to (2.7) for

$$\Omega=\Omega^*\equiv\left\{x\in\mathbf{R}^2 \;\middle|\; |x|<1\right\}.$$

Actually, they are described explicitly, i.e.,

$$v = v_\pm^{*\sigma}(x) = \log\left\{\frac{8\beta_\pm/\sigma}{(1+\beta_\pm|x|^2)^2}\right\}, \qquad \beta_\pm = \frac{4}{\sigma}\left\{1 - \frac{\sigma}{4} \pm \left(1 - \frac{\sigma}{2}\right)^{1/2}\right\}.$$

It holds that

$$v_+^{*\sigma} = v_-^{*\sigma} = 2\log\frac{2}{1+|x|^2}$$

for $\sigma = 2$, and the number of solutions to (2.7) $\Omega = \Omega^*$ is two, one, and zero according to $0 < \sigma < 2$, $\sigma = 2$, and $\sigma > 2$, respectively. Total set of solutions $\mathcal{C}^* = \{(\sigma, v)\}$, on the other hand, forms a one-dimensional manifold in $\mathbf{R}_+ \times C(\overline{\Omega}^*)$. We obtain

$$\lim_{\sigma\downarrow 0} v_-^{*\sigma}(x) = 0 \quad \text{uniformly in } x \in \overline{\Omega}^*,$$

$$\lim_{\sigma\downarrow 0} v_+^{*\sigma}(x) = 4\log\frac{1}{|x|} \quad \text{locally uniformly in } x \in \overline{\Omega}^* \setminus \{0\},$$

and therefore, the endpoints of $\mathcal{C}^*$ are $(0,0)$ and $(0, v_*)$ with $v_* = 4\log\frac{1}{|x|}$. Thus, $v_* = v_*(x)$ is a singular limit of the solution.

It is convenient to write these radially symmetric solutions as

$$\left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \left(\frac{e^u}{8}\right)^{1/2} = \frac{\mu^{1/2}}{|x|^2 + \mu} \tag{2.32}$$

with

$$\mu^{1/2} = \mu_\pm^{1/2} = \left(\frac{2}{\sigma}\right)^{1/2}\left\{1 \mp \sqrt{1 - \frac{\sigma}{2}}\right\}.$$

In fact, in this case their Liouville integrals (2.16) are described by $F(z) = Cz$ with

$$C = \mu^{-2} = C_\pm = \left\{\frac{1}{\sigma}\{4 - \sigma \pm 2\sqrt{4 - 2\sigma}\}\right\}^{1/2},$$

and the length of $\overline{F}(\partial\Omega^*)$ and the area of $\overline{F}(\Omega^*)$ are equal to

$$\ell_1(\partial\Omega^*) = \int_{\partial\Omega^*}\left(\frac{e^u}{8}\right)^{1/2} ds = 2\pi\left(\frac{\sigma}{8}\right)^{1/2}$$

and

$$m_1(\Omega^*) = \int_{\Omega^*}\frac{e^u}{8}\,dx = \frac{1}{8}\int_{\Omega^*} p\,dx = \frac{\lambda}{8},$$

respectively. Therefore, λ grows from 0 to 8π monotonously along the branch $\mathcal{C}^*$ connecting $(0,0)$ and $(0, v_*)$. Furthermore, the bending point $\sigma = 2$ of $\mathcal{C}^*$ corresponds to $\lambda = 4\pi$, while σ increases first from 0 to 2, and then decreases from 2 to 0.

If we take λ as a control parameter and eliminate σ in (2.7), then it follows that

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{2.33}$$

This λ casts a physical parameter derived from the inverse temperature and the coupling constant in turbulence and self-dual gauge, respectively [24,25,82,128,133,148].

2.2.4. *Laplace–Beltrami operator.* From the general theory of bifurcation [6,44,126], the above described profile of $\mathcal{C}^*$ guarantees the linearized stability of $v_-^{*\sigma}$ as a solution to (2.7) for $\Omega = \Omega^*$. This means that the first eigenvalue of the self-adjoint operator

$$L_-^{*\sigma} = -\Delta - \sigma e^{v_-^{*\sigma}}$$

in $L^2(\Omega^*)$ with the domain $(H^2 \cap H_0^1)(\Omega^*)$ is positive if $0 < \sigma < 2$, and zero if $\sigma = 2$. These properties are equivalent to $\nu_1(p^*, \Omega^*) > 1$ and $\nu_1(p^*, \Omega^*) = 1$ for $0 < \sigma < 2$ and $\sigma = 2$, respectively, where $\nu_1 = \nu_1(p^*, \Omega^*)$ denotes the first eigenvalue of

$$-\Delta\varphi = \nu p^* \varphi \quad \text{in } \Omega^*, \qquad \varphi = 0 \quad \text{on } \partial\Omega^* \tag{2.34}$$

for $p^* = \sigma e^{v_-^{*\sigma}}$.

This $\nu_1(p^*, \Omega^*)$ is equal to $\nu_1(p, \Omega)$ of (2.23) for $(p, \Omega) = (p^*, \Omega^*)$, and by the above consideration it holds that

$$0 < \lambda < 4\pi \quad \Rightarrow \quad \nu_1(p^*, \Omega^*) > 1, \tag{2.35}$$

where

$$p^*(x) = \frac{8\mu}{(|x|^2 + \mu)^2} \tag{2.36}$$

with $\mu > 0$ determined by

$$\lambda = \int_{\Omega^*} p^*(x)\, dx. \tag{2.37}$$

Thus, we obtain

$$0 < \lambda < 4\pi \quad \Rightarrow \quad \nu_1(p, \Omega) > 1 \tag{2.38}$$

by Theorem 2.2 if $\Omega \subset \mathbf{R}^2$ is a domain with smooth boundary $\partial\Omega$, $p = p(x) > 0$ is a C^2 function on $\overline{\Omega}$ satisfying (2.21), and $\int_\Omega p\, dx = \lambda$.

We can confirm, on the other hand, (2.35) directly, using the associated Legendre equation. More precisely, putting

$$\varphi(x)=\Phi(\xi)e^{\imath m\theta}, \qquad x=re^{\imath\theta}, \qquad \xi=\frac{\mu-r^2}{\mu+r^2}, \qquad \Lambda=1/\nu,$$

we obtain the associated Legendre equation [8]

$$\begin{aligned}&\big[(1-\xi^2)\Phi_\xi\big]_\xi+\big[2/\Lambda-m^2/(1-\xi^2)\big]\Phi=0 \quad (\xi_\mu<\xi<1),\\ &\Phi(1)=1, \qquad \Phi(\xi_\mu)=0\end{aligned} \tag{2.39}$$

by (2.34), where $\xi_\mu=(\mu-1)/(\mu+1)$. Thus, if $\Phi=\Phi(\xi)$ denotes a solution to the first equation of (2.39) for $\Lambda=1$, $m=0$, and $\Phi(1)=1$, then $\nu_1(p^*,\Omega^*)>1$ is equivalent to

$$\Phi(\xi)>0 \quad (\xi_\mu<\xi<1).$$

Since such Φ is given by $P_0(\xi)=\xi$, this means $\xi_\mu>0$, and therefore, we can reproduce (2.35) by

$$\lambda<4\pi \quad \Leftrightarrow \quad \mu>1 \quad \Leftrightarrow \quad \xi_\mu>0.$$

The associated Legendre equation appears when one adopts the polar coordinate to get the eigenvalues of three dimensional Laplacian written in the Cartesian coordinate. To understand the reason why this equation arises in the study of (2.34), we recall that $p^*=p^*(x)$ of (2.36) is associated with the Liouville integral $F(z)=\mu^{-2}z$ by $(p^*/8)^{1/2}=\rho(F)$. Using the stereographic projection: $\tau:\overline{\mathbf{C}}\to\mathbf{C}\cup\{\infty\}$, therefore, $\overline{\varphi}=\varphi\circ\tau$ satisfies

$$-\Delta_{S^2}\overline{\varphi}=\frac{\nu}{8}\overline{\varphi} \quad \text{in } \hat{\omega}, \qquad \varphi=0 \quad \text{on } \partial\hat{\omega}, \tag{2.40}$$

where $\varphi=\varphi(x)$ is a solution to (2.34). Here, Δ_{S^2} is the Laplace–Beltrami operator and $\hat{\omega}\subset\overline{\mathbf{C}}$ is a disc with the center $(0,0,0)$. In other words, $\nu_1(p^*,\Omega^*)$ defined by (2.36)–(2.37) is nothing but the first eigenvalue of the Laplace–Beltrami operator $-\Delta_{S^2}$ defined on $\omega\subset S^2$ with $\cdot|_{\partial\omega}=0$, where S^2 and $\omega\subset S^2$ denote the round sphere with total area 8π and an immersed disc with total area $\lambda\in(0,8\pi)$, respectively. Then, we obtain the associated Legendre equation using separation of variables to (2.40).

Spherically decreasing rearrangement described in the proof of Theorem 2.2 is reformulated as a Schwarz symmetrization on the round sphere in this context. Thus, given a positive C^2 function $p=p(x)$ defined on a domain $\Omega\subset\mathbf{R}^2$ with continuous extension to $\overline{\Omega}$ satisfying (2.21) and $\int_\Omega p(x)\,dx=\lambda\in(0,8\pi)$, we take an immersed disc $\omega\subset S^2$ with total area λ. Let $\varphi=\varphi(x)$ be a nonnegative C^2 function defined on $\overline{\Omega}$ satisfying $\varphi|_{\partial\Omega}=0$. Then, we put

$$\varphi^*(x)=\sup\{t \mid x\in\omega_t\} \tag{2.41}$$

for $x \in \omega$, where ω_t denotes the concentric disc of ω satisfying

$$\int_{\omega_t} dv = \int_{\{\varphi > t\}} p\, dx \tag{2.42}$$

and dv is the area element of S^2. Then, (2.26)–(2.27) reads;

$$\int_\Omega \varphi^2 p\, dx = \int_\omega \varphi^{*2}\, dv, \qquad \int_\Omega |\nabla\varphi|^2\, dx \geqslant \int_\omega |\nabla_{S^2}\varphi^*|^2\, dv. \tag{2.43}$$

2.2.5. *Spherically harmonic functions.* We recall that if $\Omega \subset \mathbf{R}^2$ is a domain, then (2.8) is equivalent to (2.16), i.e., $\rho(F) = (e^u/8)^{1/2}$, where $F = F(z)$ is an analytic function. This is regarded as an analogy of the harmonic case, that is, $\Delta u = 0$ in Ω if and only if $u = \operatorname{Re} F$, where $F = F(z)$ is an analytic function. In fact, we can derive the mean value theorem for this type of functions described below, and this property guarantees a Harnack type inequality [123]. In this sense, the function $u = u(x)$ satisfying $-\Delta u \leqslant e^u$ and $\Delta u \leqslant e^u$ may be called spherically sub-harmonic and super-harmonic, respectively.

THEOREM 2.3. (See [126,128].) *If $\Omega \subset \mathbf{R}^2$ is an open set and $u = u(x)$ is a C^2 function defined in Ω, then $-\Delta u \leqslant e^u$ in Ω if and only if*

$$u(x_0) \leqslant \frac{1}{|\partial B(x_0,R)|}\int_{\partial B(x_0,R)} u\, ds - 2\log\left\{1 - \frac{1}{8\pi}\int_{B(x_0,R)} e^u\, dx\right\}_+$$

for any $B(x_0,R) \Subset \Omega$. Similarly, $\Delta u \leqslant e^u$ in Ω if and only if

$$u(x_0) \geqslant \frac{1}{|\partial B(x_0,R)|}\int_{\partial B(x_0,R)} u\, ds - 2\log\left\{1 + \frac{1}{8\pi}\int_{B(x_0,R)} e^u\, dx\right\}$$

for any $B(x_0,R) \Subset \Omega$.

The first inequality of the above theorem implies the following fact, called Bandle's mean value theorem. The other application to $-\Delta u = K(x)e^u$ is given by [31,39].

THEOREM 2.4. (See [5].) *If $p = p(x) > 0$ is continuous on $\overline{B}$, C^2 in B, and satisfies*

$$-\Delta \log p \leqslant p \quad \text{in } B, \qquad \int_B p \leqslant 4\pi,$$

then it holds that

$$\frac{p(0)}{1 + r^2 p(0)/8} \leqslant \frac{1}{|\partial B_r|}\int_{\partial B_r} p^{1/2}\, ds, \tag{2.44}$$

where $B = B(0,R) \subset \mathbf{R}^2$, $B_r = B(0,r)$, and $r \in (0,R)$.

PROOF. Putting

$$u = \log p, \qquad m(r) = \int_{B_r} e^u \, dx < 8\pi,$$

we apply Theorem 2.3. It holds that

$$u(0) \leqslant \frac{1}{|\partial B_r|} \int_{\partial B_r} u \, ds - 2 \log\left(1 - \frac{m(r)}{8\pi}\right)$$

for $r \in (0, R)$. Writing $p = e^u$, we obtain

$$\begin{aligned} p(0) &\leqslant \left(1 - \frac{m(r)}{8\pi}\right)^{-2} \exp\left\{\frac{1}{|\partial B_r|} \int_{\partial B_r} u \, ds\right\} \\ &\leqslant \left(1 - \frac{m(r)}{8\pi}\right)^{-2} \frac{1}{|\partial B_r|} \int_{\partial B_r} p \, ds = \frac{1}{2\pi r}\left(1 - \frac{m(r)}{8\pi}\right)^{-2} m'(r) \end{aligned}$$

by Jensen's inequality, and hence it follows that

$$p(0) R^2 = 2 \int_0^R p(0) r \, dr \leqslant \frac{1}{\pi} \int_0^R \frac{m'(r)}{(1 - m(r)/(8\pi))^2} dr = 8m(8\pi - m)^{-1},$$

where

$$m = m(R) = \int_B p \, dx.$$

Bol's inequality, on the other hand, guarantees

$$\ell^2 \leqslant \frac{1}{2} m(8\pi - m)$$

for $\ell = \int_{\partial B} p^{1/2} \, ds$, and therefore, $m \leqslant m_-$ in the case of $m \leqslant 4\pi$, where $M = m_-$ is the smaller solution to

$$M^2 - 8\pi M + 2\ell^2 = 0,$$

i.e.,

$$m_- = 4\pi\left(1 - \sqrt{1 - j^2}\right) \quad \text{for } j = \ell/(2\sqrt{2}\pi).$$

Then, we obtain

$$p(0) R^2 \leqslant 8m_-(8\pi - m_-)^{-1}$$

and hence (2.44) for $r = R$. □

2.2.6. *Duality.* Problem (2.33) is the Euler–Lagrange equation of the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda \log \lambda - \lambda \tag{2.45}$$

defined for $v \in H_0^1(\Omega)$. The Trudinger–Moser inequality [85] guarantees that this functional is C^1, and is bounded from below if $\lambda = 8\pi$. Actually, there are several variants of this inequality [127].

This $\mathcal{J}_\lambda = \mathcal{J}_\lambda(v)$ is regarded as the dual form of a physically important functional, Helmholtz' free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\left\langle(-\Delta_D)^{-1}u, u\right\rangle$$

defined for $u \geqslant 0$ and $\int_\Omega u = \lambda$ [128]. The equilibrium with respect to $\mathcal{F}(u)$ is described by

$$(-\Delta_D)^{-1}u = \log u + \text{constant} \quad \text{in } \Omega, \qquad \|u\|_1 = \lambda. \tag{2.46}$$

We define, on the other hand, the Lagrangian by

$$L(u, v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle.$$

First, if $v \in H_0^1(\Omega)$ is a solution to (2.33) then $u = \frac{\lambda e^v}{\int_\Omega e^v}$ is a solution to (2.46), and conversely, if $u \geqslant 0$ is a solution to (2.46) then $v = (-\Delta_D)^{-1}u$ is a solution to (2.33). Next, there are unfolding Legendre transformation and the minimality in accordance with the Lagrangian formulated by

$$L|_{v=(-\Delta_D)^{-1}u} = \mathcal{F} \quad \text{and} \quad L|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} = \mathcal{J}_\lambda \tag{2.47}$$

and

$$L(u, v) \geqslant \max\left\{\mathcal{F}(u), \mathcal{J}_\lambda(v)\right\}, \tag{2.48}$$

respectively, where $u \geqslant 0$, $\|u\|_1 = \lambda$, and $v \in H_0^1(\Omega)$. We have, more precisely,

$$\inf\left\{L(u, v) \mid v \in H_0^1(\Omega)\right\} = \mathcal{F}(u),$$
$$\inf\left\{L(u, v) \mid u \geqslant 0,\ \|u\|_1 = \lambda\right\} = \mathcal{J}_\lambda(v),$$

and in particular,

$$\inf_{u,v} L(u, v) = \inf_v \mathcal{J}_\lambda(v) = \inf_u \mathcal{F}(u).$$

These profiles are the consequence of the abstract structure of Toland duality [135,136] observed in wide areas [128].

Decrease of this free energy together with mass conservation, on the other hand, is realized by the model (B) equation [61,68]. This is the Smoluchowski–Poisson equation concerning material transport under the self-attractive force, or a simplified system of chemotaxis in the context of mathematical biology, i.e.,

$$\begin{aligned} &u_t = \nabla \cdot (\nabla u - u \nabla v), \qquad -\Delta v = u \quad \text{in } \Omega \times (0,T), \\ &\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = v = 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{2.49}$$

In fact, this system is described by

$$u_t = \nabla \cdot \big(u \nabla \delta \mathcal{F}(u) \big), \qquad u \frac{\partial}{\partial \nu} \delta \mathcal{F}(u) \big|_{\partial\Omega} = 0$$

and hence it follows that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u = - \int_{\partial\Omega} u \frac{\partial}{\partial \nu} \delta \mathcal{F}(u) = 0, \\ &\frac{d}{dt} \mathcal{F}(u) = - \int_\Omega u \big| \nabla \delta \mathcal{F}(u) \big|^2 \leqslant 0. \end{aligned}$$

This means that the stationary state of (2.49) is defined by

$$u \geqslant 0, \qquad \|u\|_1 = \lambda, \qquad \delta \mathcal{F}(u) = \text{constant},$$

that is, (2.46), equivalent to (2.33) by the above mentioned transformation.

2.2.7. *Collapse formation.* There is a quantized blowup mechanism in the nonlinear eigenvalue problem (2.33) derived from (2.7), which is the origin of the formation of collapse with quantized mass to (2.49). A typical example of such a profile is the following theorem. See [127,128] for related topics.

THEOREM 2.5. (See [127].) *If the solution to*

$$\begin{aligned} &u_t = \nabla \cdot (\nabla u - u \nabla v), \qquad -\Delta v = u - \frac{1}{|\Omega|} \int_\Omega u \quad \textit{in } \Omega \times (0,T), \\ &\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \textit{on } \partial\Omega \times (0,T) \end{aligned}$$

blows-up at $t = T < +\infty$, then it holds that

$$u(x,t)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0)\,\delta_{x_0}(dx) + f(x)\,dx$$

in $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ *as* $t \uparrow T$, *where* $\Omega \subset \mathbf{R}^2$ *is a bounded domain with smooth boundary* $\partial\Omega$,

$$\mathcal{S} = \bigl\{x_0 \in \overline{\Omega} \bigm| \text{there exist } x_k \to x_0 \text{ and } t_k \uparrow T \text{ such that } u(x_k, t_k) \to +\infty\bigr\}$$

is the blowup set, $0 \leqslant f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$, *and*

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

Thus, we obtain

$$2\sharp(\mathcal{S} \cap \Omega) + \sharp(\mathcal{S} \cap \partial\Omega) \leqslant \|u_0\|_1/(4\pi). \tag{2.50}$$

We can show also that the equality in (2.50) is excluded [128].

2.3. *Method of scaling*

The quantized blowup mechanism to (2.7) is described as follows.

THEOREM 2.6. (See [88].) *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded domain with smooth boundary* $\partial\Omega$, *and* $\{(\sigma_k, v_k)\}_k$ *be a solution sequence to* (2.7) *such that* $\sigma_k \downarrow 0$. *Then, passing to a subsequence*, $\lambda_k = \int_\Omega \sigma_k e^{v_k}\, dx \to 8\pi\ell$ *with some* $\ell = 0, 1, \ldots, +\infty$. *According to this value, the solution behaves as follows*:

(1) $\ell = 0$: *uniform convergence to* 0, *i.e.*, $\|v_k\|_\infty \to 0$.

(2) $0 < \ell < +\infty$: ℓ*-point blowup, i.e., there exist* $x_j^* \in \Omega$ $(j = 1, \ldots, \ell)$ *and* $v_0 = v_0(x)$ *such that* $v_k \to v_0$ *locally uniformly in* $\overline{\Omega} \setminus \mathcal{S}$ *for* $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$. *We obtain*

$$v_0(x) = 8\pi \sum_{j=1}^{\ell} G\bigl(x, x_j^*\bigr),$$

$$\frac{1}{2}\nabla R\bigl(x_j^*\bigr) + \sum_{i \neq j} \nabla_x G\bigl(x_i^*, x_j^*\bigr) = 0 \quad (1 \leqslant j \leqslant \ell), \tag{2.51}$$

where $G = G(x, x')$ *is the Green's function*:

$$-\Delta G(\cdot, x') = \delta_{x'}(dx) \quad \text{in } \Omega, \qquad G(\cdot, x') = 0 \quad \text{on } \partial\Omega$$

defined for $x' \in \Omega$, *and*

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x'=x}$$

is the Robin function.

(3) $\ell = +\infty$: *entire blowup, i.e.*, $v_k \to +\infty$ *locally uniformly in* Ω.

The second case is crucial in the above theorem. Using (2.33), we can reformulate it as follows.

THEOREM 2.7. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and $\{(\lambda_k, v_k)\}$ be a solution sequence to* (2.33) *satisfying $\lambda_k \to \lambda_0 \in (0, +\infty)$. Then, $\lambda_0 = 8\pi\ell$ with $\ell \in \mathbf{N}$, and passing to a subsequence, we obtain $v_k \to v_0$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$ with $v_0 = v_0(x)$ satisfying* (*2.51*) *for $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$.*

To prove this case, first, we note that $\|v_k\|_{W^{1,q}} = O(1)$ holds by the L^1-estimate [114, 23], where $1 \leqslant q < 2 = \frac{n}{n-1}$ for $n = 2$. This implies a uniform boundary estimate indicated by

$$\|v_k\|_{L^\infty(\omega)} = O(1), \tag{2.52}$$

using the reflection argument combined with the Kelvin transformation [56,45], where $\omega = \hat{\omega} \cap \overline{\Omega}$ and $\hat{\omega}$ is an open set satisfying $\partial\Omega \subset \hat{\omega}$.

The original proof uses the complex structure [88]. In fact, we obtain

$$e^{-v/2} = c|\varphi_1|^2 + \frac{\sigma c^{-1}}{8}|\varphi_2|^2$$

in (2.7), using $s(z)$ of (2.9) with u replaced by v, where $\{\varphi_1(z), \varphi_2(z)\}$ is the fundamental system of solutions to (2.10) defined by (2.11) with z^* corresponding to a critical point of v, denoted by $x^* \in \Omega$. Thus, there is a family of holomorphic functions $\{s_k(z)\}$ defined by (2.9) for $u = v_k$, and this family is uniformly bounded on $\overline{\Omega}$ by (2.52) and the elliptic estimate. Passing to a subsequence, therefore, we obtain $s_k \to s_0$ locally uniformly in Ω.

Introducing the fundamental system of solutions $\{\varphi_{1k}(z), \varphi_{2k}(z)\}$ to (2.10) for $s = s_k(z)$, we take $x^* = x_k^*$ as a maximum point of v_k. Passing to a subsequence, the convergence $s_k \to s$ mentioned above guarantees those of $\varphi_{1k} \to \varphi_{10}$ and $\varphi_{2k} \to \varphi_{20}$ locally uniformly as analytic functions in Ω, because $\{x_k^*\}$ is in $\Omega \setminus \hat{\omega}$. Then, it holds that $c_k = \exp(-\|v_k\|_\infty/2) \to 0$ in the analogous relation to (2.15),

$$e^{-v_k/2} = c_k|\varphi_{1k}|^2 + \frac{\sigma_k c_k^{-1}}{8}|\varphi_{2k}|^2. \tag{2.53}$$

Since $\{v_k\}$ is bounded in $W^{1,q}(\Omega)$ for $1 \leqslant q < 2$, any blowup point of $\{v_k\}$ must be zero of the analytic function φ_{10}, and therefore, each blowup point is isolated. We obtain finiteness of the blowup points in this way, while classification of the singular limit, (2.51), is derived by residue analysis, more precisely, singularity vanishing of $s_0(z) = v_{0zz} - \frac{1}{2}v_{0z}^2$.

This concludes the proof of the theorem, but here we obtain $\sigma_k c_k^{-1} \approx 1$, i.e., $\|v_k\|_\infty \approx -2\log\sigma_k$ as $k \to \infty$. From the proof of Theorem 2.9 described below, on the other hand, each x_j^* takes a sequence $x_k^j \to x_j^*$, where x_k^j is a local maximum point of v_k. Thus, we can reformulate $x^* = x_k^j$ in (2.53), and consequently, the blowup rates $v_k(x_k^j) \to +\infty$ $(j = 1, \ldots, \ell)$ are proportional each other.

The other proof of the above theorem uses Theorems 2.8, 2.9 described in the following paragraph and the Pohozaev identity instead of the complex structure [81]. This argument is valid even to the nonhomogeneous coefficient case.

The second equality of (2.51) means that $(x_1^*, \ldots, x_\ell^*) \in \Omega \times \cdots \times \Omega$ is a critical point of

$$H = H(x_1, \ldots, x_\ell) = \frac{1}{2} \sum_i R(x_i) + \sum_{i<j} G(x_i, x_j).$$

If it is nondegenerate, then there is a local branch of solutions taking $(\sigma, v) = (0, v_0)$ as an endpoint for $v_0 = v_0(x)$ defined by the first relation of (2.51) [9]. First, the complex structure was used for this purpose, assuming that Ω is simply-connected and $\ell = 1$ [145, 84,125]. See [46,51] for later results.

Theorem 2.7 guarantees that the total degree of the solution set is constant in each component of $[0, +\infty) \setminus 8\pi\mathbf{N}$ [74]. It is actually determined by the genus of Ω and explicit formula is given by a detailed blowup analysis [32–34]. For example, if $\ell = 1$, then it holds that

$$\|v_k\|_\infty = -2\log\sigma_k + 2\log 8 - 8\pi R(x_0) + o(1).$$

We obtain also

$$v(x) = \sum_{i=1}^{2} \frac{a_i x_i}{8 + |x|^2} + b \cdot \frac{8 - |x|^2}{8 + |x|^2}$$

if $v = v(x)$ is a uniformly bounded solution to the linearized entire problem

$$-\Delta v = \frac{v}{\{1 + \frac{|x|^2}{8}\}^2} \quad \text{in } \mathbf{R}^2,$$

where $a_i, b \in \mathbf{R}$. See [58,106].

2.3.1. *Blowup analysis.* Most of the fundamental equations of physics are provided with self-similarity. Concerning (2.8) derived from the vorticity equation and the abelian Higgs theory, it is invariant under the transformation

$$u^\mu(x) = u(\mu x) + 2\log\mu,$$

where $\mu > 0$ is a constant. This causes the lack of compactness of the family of (approximate) solutions, and this mechanism is clarified by the blowup analysis of which ingredients are summarized as follows:

(1) scaling invariance of the problem,
(2) classification of the entire solution,
(3) control at infinity of the rescaled solution,
(4) hierarchical argument.

The following theorem, free from the boundary condition, is useful in such a study, because the effect of boundary conditions is usually lost in the scaling argument. It also deals with the nonhomogeneous coefficient case with the lack of the complex structure. Theorem 2.6 is associated with this theorem by $v_k - \log \sigma_k$. Actually, we obtain $v > 0$ in (2.7).

THEOREM 2.8. (See [21].) *If $\Omega \subset \mathbf{R}^2$ is a bounded domain and $v_k = v_k(x)$ $(k = 1, 2, \ldots)$ is a solution sequence to*

$$-\Delta v_k = V_k(x) e^{v_k} \quad \text{in } \Omega$$

with

$$0 \leqslant V_k(x) \leqslant C_1 \quad \text{in } \Omega, \qquad \int_\Omega e^{v_k} \leqslant C_2, \tag{2.54}$$

where C_1, C_2 are constants, then, passing to a subsequence, there arises the following alternatives:

(1) *$\{v_k\}$ is locally uniformly bounded in Ω.*
(2) *$v_k \to -\infty$ locally uniformly in Ω.*
(3) *There is a finite set $\mathcal{S} = \{x_j^*\} \subset \Omega$ and $m_j \geqslant 4\pi$ such that $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$ and*

$$V_k(x) e^{v_k}\, dx \rightharpoonup \sum_j m_j\, \delta_{x_j^*}(dx) \quad \text{in } \mathcal{M}(\Omega). \tag{2.55}$$

Furthermore, $\mathcal{S}$ is the blowup set of $\{v_k\}$ in Ω.

THEOREM 2.9. (See [75].) *In the third case of the above theorem, we obtain $m_j = 8\pi n_j$ for some $n_j \in \mathbf{N}$, provided that $V_k \to V$ uniformly on $\overline{\Omega}$.*

Boundedness of the Palais–Smale sequence relative to the Trudinger–Moser inequality does not follow always. Then, the above theorem is applied to compensate this difficulty in constructing nontrivial solutions to the mean field equation [122,48].

There are several differences between the energy quantization described in the previous chapter. First, the above blowup mechanism occurs only to the quantized values of mass, realized as the eigenvalue λ. Thus, we obtain the residual vanishing, the disappearance of the regular part of the limit measure in (2.55) under the control (2.54). Second, global structure described by the compactness of the domain manifold or the boundary condition excludes multiple bubbles, prescribing the location of blowup points.

The proof of these theorems are as follows [134,128]. First, we perform the prescaled analysis to prove Theorem 2.8. In more precise, if $\Omega \subset \mathbf{R}^2$ is a bounded domain, $f \in L^1(\Omega)$, and

$$-\Delta v = f(x) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then it holds that

$$\int_{\Omega} \exp\left(\frac{4\pi - \delta}{\|f\|_1}\left|v(x)\right|\right) dx \leqslant \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2,$$

where $0 < \delta < 4\pi$. This implies ε-regularity, stated as the following lemma, and then Theorem 2.8 is obtained by a standard argument.

LEMMA 2.3.1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, $K \subset \Omega$ a compact set, $c_1, c_2 > 0$, and $\varepsilon_0 \in (0, 4\pi)$. Then, there is $C > 0$ such that*

$$-\Delta v = V(x)e^v, \qquad 0 \leqslant V(x) \leqslant c_1 \quad \text{in } \Omega,$$

$$\|v^+\|_1 \leqslant c_2, \qquad \int_{\Omega} V(x)e^v \leqslant \varepsilon_0$$

implies $\|v^+\|_{L^\infty(K)} \leqslant C$.

Once Theorem 2.8 is proven, then Theorem 2.9 is reduced to the following case, where $B = B(0, R) \subset \mathbf{R}^2$ and $B_r = B(0, r)$.

THEOREM 2.10. *If*

$$-\Delta v_k = V_k(x)e^{v_k}, \qquad V_k(x) \geqslant 0 \quad \text{in } B,$$

$$V_k \to V \quad \text{in } C(\overline{B}),$$

$$\max_{\overline{B}} v_k \to +\infty,$$

$$\max_{\overline{B}\setminus B_r} v_k \to -\infty \quad (0 < r < R),$$

$$\lim_{k\to\infty} \int_B V_k(x)e^{v_k} = \alpha,$$

$$\int_B e^{v_k} \leqslant C_0,$$

then it holds that $\alpha \in 8\pi\mathcal{N}$.

There is actually the case of $\alpha = 8\pi\ell$ with $\ell \geqslant 2$ in the above theorem [37]. However, the conclusion $\alpha = 8\pi$ arises, provided that

$$\max_{\partial B} v_k - \min_{\partial B} v_k \leqslant C \quad \text{and} \quad \|\nabla V_k\|_\infty \leqslant C. \tag{2.56}$$

We obtain, furthermore,

$$\left| v_k(x) - \log \frac{e^{v_k(0)}}{(1 + \frac{V_k(0)}{8} e^{v_k(0)}|x|^2)^2} \right| \leqslant C \tag{2.57}$$

for $k = 1, 2, \dots$ and $x \in B$ in this case [74]. If (2.56) holds for $B = \Omega$, then $n_j = 1$ for any j in Theorem 2.9 and furthermore, the blowup points $x_1^*, \dots, x_\ell^*$ are prescribed by

$$\frac{1}{2}\nabla R\big(x_j^*\big) + \sum_{i \neq j} \nabla_x G\big(x_i^*, x_j^*\big) + \frac{1}{8\pi}\nabla \log V\big(x_j^*\big) = 0 \quad (1 \leqslant j \leqslant \ell)$$

similarly to the second equation of (2.51) [81]. See [89,91,106,127] for related results.

Theorem 2.10 is proven by the blowup analysis. Thus, we take $x_k \in B$ satisfying $v_k(x_k) = \|v_k\|_\infty$ with $x_k \to 0$, and put

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) + 2\log\delta_k,$$
$$\delta_k = e^{-v_k(x_k)/2} \to 0.$$

Then, it holds that

$$-\Delta\tilde{v}_k = V_k(\delta_k x + x_k)e^{\tilde{v}_k}, \qquad \tilde{v}_k \leqslant 0 = \tilde{v}_k(0) = 0 \quad \text{in } B(0, R/2\delta_k),$$
$$\int_{B(0,R/2\delta_k)} e^{\tilde{v}_k} \leqslant C_0,$$

and Theorem 2.8 is applicable to this $\{\tilde{v}_k\}$. Thus, $\{\tilde{v}_k\}$ is locally uniformly bounded in $\mathbf{R}^2$, and passing to a subsequence, we obtain $\tilde{v}_k \to \tilde{v}$ locally uniformly in $\mathbf{R}^2$ with

$$-\Delta\tilde{v} = V(0)e^{\tilde{v}}, \quad \tilde{v} \leqslant 0 = \tilde{v}(0) \quad \text{in } \mathbf{R}^2,$$
$$\int_{\mathbf{R}^2} e^{\tilde{v}} \leqslant C_0$$

by the elliptic regularity. From this we can infer $V(0) > 0$ and hence assume

$$a \leqslant V_k(x) \leqslant b \quad (x \in B)$$

for $k = 1, 2, \dots$ without loss of generality, where $a, b > 0$ are constants. We obtain, furthermore, $\tilde{v} = \tilde{v}(|x|)$ by the method of moving plane [36], which results in

$$\tilde{v}(x) = \log\frac{1}{(1 + \frac{V(0)}{8}|x|^2)^2}, \qquad \int_{\mathbf{R}^2} V(0)e^{\tilde{v}} = 8\pi. \tag{2.58}$$

2.3.2. Sup + Inf *inequality.* We have detected the principal collapse formed at the origin in the proof of Theorem 2.10. Now, we have to show the vanishing of residual parts by collecting the other collapses. This is done by the sup + inf inequality proven by Alexandroff's inequality originally. Alexandroff's inequality is also an isoperimetric inequality on surface described by its Gaussian curvature, regarded as a refinement of Bol's inequality [8]. Thus, we can show the following lemma.

LEMMA 2.3.2. (See [112].) *Let* $B = B(0,1) \subset \mathbf{R}^2$ *and* $a, b > 0$ *be constants. Then, there are* $C_0 > 0$ *and* $\alpha_0 > 4\pi$ *such that*

$$-\Delta v = V(x)e^v, \quad a \leqslant V(x) \leqslant b \quad \text{in } B,$$
$$\int_B V(x)e^v \leqslant \alpha_0$$

implies $v(0) \leqslant C_0$.

This lemma is regarded as a refinement of Lemma 2.3.1 under the cost of $V(x) \geqslant a$. If $V = V(x)$ is restricted to a compact family in $C(\overline{\Omega})$, which is sufficient for later arguments, then we can apply the blowup analysis for the proof. In this case, the above α_0 can be arbitrary in $\alpha_0 < 8\pi$ and furthermore, the case $a = 0$ is permitted.

Using the above lemma and the scaling invariance of the equation, next, we show the following lemma.

LEMMA 2.3.3. (See [112].) *If* $\Omega \subset \mathbf{R}^2$ *is a bounded domain,* $K \subset \Omega$ *is a compact set, and* $a, b > 0$ *are constants, then there are* $c_1 = c_1(a,b) \geqslant 1$ *and* $c_2 = c_2(a, b, \text{dist}\,(K, \partial\Omega)) > 0$ *such that*

$$-\Delta v = V(x)e^v, \quad a \leqslant V(x) \leqslant b \quad \text{in } \Omega$$

implies

$$\sup_K v + c_1 \inf_\Omega v \leqslant c_2. \tag{2.59}$$

In the other version of (2.59) proven by the blowup analysis [20], the condition $c_1 = 1$ is achieved under the cost of $\|\nabla V\|_\infty \leqslant C$. In any case, this sup + inf inequality induces the key estimate, again by the scaling.

LEMMA 2.3.4. (See [75].) *Given* $a, b > 0$ *and* $C_1 > 0$, *we obtain* $\gamma > 0$, $C_2 > 0$ *independent of* $0 < R_0 \leqslant R/4$ *such that*

$$-\Delta v = V(x)e^v, \quad a \leqslant V(x) \leqslant b \quad \text{in } B_R,$$
$$v(x) + 2\log|x| \leqslant C_1 \quad \text{in } B_R \setminus \overline{B_{R_0}}$$

implies

$$e^{v(x)} \leqslant C_2 e^{-\gamma v(0)} \cdot |x|^{-2(\gamma+1)}$$

for $2R_0 \leqslant |x| \leqslant R/2$.

2.3.3. *Residual vanishing.* To complete the proof of Theorem 2.10, first, we recall the blowup argument to detect the principal collapse. Using the diagonal argument, this process is refined. More precisely, we obtain $r_k^0 \to 0$ satisfying $r_k^0/\delta_k^0 \to +\infty$ and

$$\int_{B(x_k^0, 2r_k^0)} V_k(x)e^{v_k} \to 8\pi,$$

where $\|v_k\|_\infty = v_k(x_k^0) \to +\infty$, $x_k^0 \to 0$, and $\delta_k^0 = e^{-v_k(x_k^0)/2} \to 0$.

If

$$\sup\{v_k(x) + 2\log|x - x_k^0| \mid x \in B \setminus B(x_k^0, r_k^0)\} < +\infty, \tag{2.60}$$

then Lemma 2.3.4 is applicable. We have

$$e^{v_k(x)} \leqslant Ce^{-\gamma v_k(x_k^0)}|x - x_k^0|^{-2(\gamma+1)}$$

for $x \in B_{R/2} \setminus B(x_k^0, r_k^0)$, and therefore,

$$\begin{aligned}\int_{B_{R/2}\setminus B(x_k^0, r_k^0)} V_k(x)e^{v_k} &\leqslant b \cdot C \cdot (\delta_k^0)^{2\gamma} \cdot 2\pi \cdot \int_{r_k^0}^{+\infty} r^{-2(\gamma+1)} r\, dr \\ &= \frac{\pi bC}{\gamma}(\delta_k^0/r_k^0)^{2\gamma} \to 0.\end{aligned}$$

This implies

$$\int_B V_k(x)e^{v_k} \to 8\pi$$

because $v_k \to -\infty$ locally uniformly in $\overline{B} \setminus \{0\}$, and hence $\alpha = 8\pi$.

If (2.60) is not the case, then there is $x_k^1 \in \overline{B}$ such that

$$\sup_{x \in B\setminus B(x_k^0, r_k^0)} \{v_k(x) + 2\log|x - x_k^0|\} = v_k(x_k^1) + 2\log|x_k^1 - x_k^0| \to +\infty.$$

This implies $v_k(x_k^1) \to +\infty$ and $x_k^1 \to 0$. Furthermore, $\sigma_k^1 = d_k/\delta_k^1 \to +\infty$ for

$$d_k = |x_k^1 - x_k^0| \quad \text{and} \quad \delta_k^1 = e^{-v_k(x_k^1)/2}.$$

Given $|x| \leqslant \sigma_k^1/2$, we have

$$|\delta_k^1 x + x_k^1 - x_k^0| \geqslant |x_k^1 - x_k^0| - \delta_k^1|x| \geqslant \frac{1}{2}|x_k^1 - x_k^0|$$

and therefore,

$$\begin{aligned}
\tilde{v}_k^1(x) &\equiv v_k\big(\delta_k^1 x + x_k^1\big) + 2\log\delta_k^1 \\
&\leqslant v_k\big(x_k^1\big) + 2\log\big|x_k^1 - x_k^0\big| - 2\log\big|\delta_k^1 x + x_k^1 - x_k^0\big| + 2\log\delta_k^1 \\
&\leqslant v_k\big(x_k^1\big) + 2\log\delta_k^1 + 2\log\big|x_k^1 - x_k^0\big| - 2\log\frac{1}{2}\big|x_k^1 - x_k^0\big| \\
&= 2\log 2.
\end{aligned}$$

This implies

$$\begin{aligned}
&-\Delta\tilde{v}_k^1 = V_k\big(\delta_k^1 x + x_k^1\big)e^{\tilde{v}_k^1}, \quad \tilde{v}_k^1 \leqslant 2\log 2 \quad \text{in } B_{\sigma_k^1/2} \\
&\tilde{v}_k^1(0) = 0,
\end{aligned}$$

and passing to a subsequence, we obtain $\tilde{v}_k^1 \to \tilde{v}^1$ in $C_{\mathrm{loc}}^{1,\alpha}(\mathbf{R}^2)$ with $\tilde{v}^1 = \tilde{v}^1(x)$ satisfying

$$\begin{aligned}
&\tilde{v}^1(x) = \log\frac{a^2}{(1+\mu^2 a^2|x-\overline{x}|^2)^2}, \quad \tilde{v}^1(x) \leqslant 2\log 2 \quad (\text{for } x \in \mathbf{R}^2), \\
&\tilde{v}^1(0) = 0
\end{aligned}$$

for some $\mu, a > 0$ and $\overline{x} \in \mathbf{R}^2$.

This convergence allows us to reformulate x_k^1 and δ_k^1 by

$$v_k\big(x_k^1\big) = \|v_k\|_{L^\infty(B(x_k^1, 2r_k^1))} \to +\infty,$$

$\delta_k^1 = e^{-v_k(x_k^1)/2} \to 0$, and $r_k^1/\delta_k^1 \to +\infty$, where $r_k^1 = d_k/4$. Similarly to the above case, it follows that

$$\int_{B(x_k^1, 2r_k^1)} V_k(x)e^{v_k} \to 8\pi$$

with $B(x_k^1, 2r_k^1) \cap B(x_k^0, 2r_k^0) = \emptyset$.

We shall show $\alpha = 16\pi$, if

$$\sup\left\{v_k(x) + 2\log\min_{j=0,1}\big|x - x_k^j\big| \;\Big|\; x \in B\Big\backslash\bigcup_{j=0}^{1} B\big(x_k^j, r_k^j\big)\right\} < +\infty \tag{2.61}$$

is satisfied. It suffices to prove

$$\int_{B(x_k^0, 2d_k)} V_k(x)e^{v_k} \to 16\pi \tag{2.62}$$

because

$$\int_{B\setminus B(x_k^0,2d_k)} V_k(x)e^{v_k} \to 0$$

follows from (2.61) similarly. For this purpose, we take $\tilde{v}_k(x) = v_k(d_k x + x_k^0) + 2\log d_k$ and obtain

$$-\Delta\tilde{v}_k = V_k\bigl(d_k x + x_k^0\bigr)e^{\tilde{v}_k} \quad \text{in } d_k^{-1}\bigl(B - \{x_k^0\}\bigr).$$

We put, furthermore,

$$\tilde{x}_k^j = \frac{x_k^j - x_k^0}{d_k}, \qquad \tilde{\delta}_k^j = e^{-\tilde{v}_k(\tilde{x}_k^j)/2} = \frac{\delta_k^j}{d_k}, \qquad \tilde{r}_k^j = \frac{r_k^j}{d_k}$$

for $j = 0, 1$, and then it holds that

$$\begin{aligned}
&\frac{\tilde{r}_k^j}{\tilde{\delta}_k^j} = \frac{r_k^j}{\delta_k^j} \to +\infty, \\
&B\bigl(\tilde{x}_k^0, 2\tilde{r}_k^0\bigr) \cap B\bigl(\tilde{x}_k^1, 2\tilde{r}_k^1\bigr) = \emptyset, \\
&\sup\left\{\tilde{v}_k(x) + 2\log \min_{j=0,1}\bigl|x - \tilde{x}_k^j\bigr| \;\middle|\; x \in B_{R/d_k} \setminus \bigcup_{j=0}^{1} B\bigl(\tilde{x}_k^j, \tilde{r}_k^j\bigr)\right\} < +\infty, \\
&\int_{B(\tilde{x}_k^j,2\tilde{r}_k^j)} \tilde{V}_k(x)e^{\tilde{v}_k} \to 8\pi \quad (j = 0, 1)
\end{aligned} \tag{2.63}$$

for $\tilde{V}_k(x) = V_k(d_k x + x_k^0)$.

We obtain $\tilde{x}_k^0 = 0$ and $|\tilde{x}_k^1 - \tilde{x}_k^0| = 1$, and therefore, $\tilde{x}_k^1 \to \tilde{x}^1$ with $|\tilde{x}^1| = 1$, passing to a subsequence. The third relation of (2.63) and Theorem 2.8 now imply $\tilde{v}_k \to -\infty$ locally uniformly in $\mathbf{R}^2 \setminus \{0, \tilde{x}^1\}$. Therefore, if $\tilde{r}_k^1 \to \tilde{r}^1 > 0$, passing to a subsequence, then

$$\int_{B(\tilde{x}_k^1,1/2)} \tilde{V}_k(x)e^{\tilde{v}_k} \to 8\pi \tag{2.64}$$

and hence (2.62). If $\tilde{r}_k^1 \to 0$, we apply the scaling around $\tilde{x}_k^1$. Then, it holds that (2.64) by the third relation of (2.63).

If (2.61) is not the case, we continue the process and obtain $x_k^2 \to 0$ and $r_k^2 \to 0$ satisfying $v_k(x_k^2) = \|v_k\|_{L^\infty(B(x_k^2,2r_k^2))} \to +\infty$, $r_k^2/\delta_k^2 \to 0$, $B(x_k^i, 2r_k^i) \cap B(x_k^j, 2r_k^j) = \emptyset$ for $0 \leqslant i < j \leqslant 2$, and

$$\int_{B(x_k^2,2r_k^2)} V_k(x)e^{v_k} \to 8\pi,$$

where $\delta_k^2 = e^{-v_k(x_k^2)/2}$. To show that $\alpha = 24\pi$ in the case of

$$\sup\left\{ v_k(x) + 2\log \min_{0\leqslant j\leqslant 2} \left|x - x_k^j\right| \;\middle|\; x \in B \setminus \bigcup_{j=0}^{2} B\left(x_k^j, r_k^j\right) \right\} < +\infty,$$

we classify the rate $d_{i,j} = |x_k^i - x_k^j|$ of concentration to the origin for $0 \leqslant i < j \leqslant 2$. First, we show the residual vanishing inside the ball containing $B(x_k^j, 2r_k^j)$ with a proportional rate. These balls are contained in a larger ball, where the residual vanishing occurs similarly. We end-up this procedure in finitely many times, and obtain the conclusion.

Profile of $v_k = v_k(x)$ in the outer region $x \in B \setminus B(x_0, \delta_k)$ is almost similar to that of the Kelvin transformation of $v_k = v_k(x)$ on $B(x_0, \delta_k)$, under the assumption of (2.56). This is actually proven by the method of moving plane, and then (2.57) is obtained [74]. An alternative proof valid to degenerate $v(x) \geqslant 0$ is given by [10].

2.3.4. *Free boundary problem.* Putting $w = v + \log\lambda - \log\int_\Omega e^v$ in (2.33), we obtain

$$-\Delta w = e^w \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma = \partial\Omega, \qquad \int_\Omega e^w = \lambda. \tag{2.65}$$

Conversely, if $w = w(x)$ solves (2.65), then $v = w - w_\Gamma$ is a solution to (2.33). By Theorem 2.7, we can show the quantized blowup mechanism to (2.65).

THEOREM 2.11. *If $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\{(\lambda_k, w^k)\}$ is a solution sequence to* (2.65) *satisfying $\lambda_k \to \lambda_0$, then passing to a subsequence the following alternatives hold*:

(1) $\|w^k\|_\infty = O(1)$.

(2) $\sup_\Omega w^k \to -\infty$.

(3) $\lambda_0 = 8\pi\ell$ *for some $\ell \in \mathbf{N}$, and there exist $x_j^* \in \Omega$ ($j = 1, \dots, \ell$) satisfying the second relation of* (2.51) *and $x_k^j \to x_j^*$, such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \{x_1^*, \dots, x_\ell^*\}$, and*

$$e^{w^k}\,dx \rightharpoonup \sum_j 8\pi\,\delta_{x_j^*}(dx) \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

Thus, $\mathcal{S} = \{x_1^*, \dots, x_\ell^*\}$ is the blowup set of $\{w^k\}$.

PROOF. First, Theorems 2.8, 2.9 and their proof guarantee the following alternatives, passing to a subsequence.

(a) $\{w^k\}$ is locally uniformly bounded in Ω.

(b) $w^k \to -\infty$ locally uniformly in Ω.

(c) There is a finite set $\mathcal{S} \subset \Omega$ such that $w_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$, any $x_0 \in \mathcal{S}$ admits $x_k \to x_0$ with x_k a local maximum point of $w^k = w^k(x)$ satisfying $w^k(x_k) \to +\infty$, and

$$e^{w^k}\,dx \to \sum_{x_0 \in \mathcal{S}} m(x_0)\,\delta_{x_0}(dx) \quad \text{in } \mathcal{M}(\Omega)$$

for some $m(x_0) \in 8\pi\mathcal{N}$.

We obtain, on the other hand, $v_k = w^k - w_\Gamma^k \geqslant 0$ in Ω by the maximum principle, and also

$$\|v_k\|_{L^\infty(\omega)} \leqslant C \tag{2.66}$$

from the proof of Theorem 2.6, where ω is an $\overline{\Omega}$-neighbourhood of $\partial\Omega$. Since $w^k \geqslant w_\Gamma^k$, we have either $w_\Gamma^k \to -\infty$ or $w_\Gamma^k = O(1)$, passing to a subsequence.

Actually, $w_\Gamma^k = O(1)$ occurs in the case of (a), and then $\{v_k\}$ is uniformly bounded on $\overline{\Omega}$ by (2.66). Thus, the first case of the theorem arises. In the other case of $w_\Gamma^k \to -\infty$, if $\{v_k\}$ is uniformly bounded, then the second case of the theorem follows. If not, we obtain $\|v_k\|_\infty \to +\infty$, passing to a subsequence, and therefore, there arises the second case of Theorem 2.6, i.e., Theorem 2.7. This guarantees the third case of the theorem by

$$-\Delta v_k\,dx = -\Delta w^k\,dx = e^{w^k}\,dx \rightharpoonup \sum_j 8\pi\,\delta_{x_j^*}(dx) \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

The proof is complete. □

2.3.5. *Higher-dimensional case.* The problem (2.65) is regarded as a free boundary problem associated with plasma confinement, where $\{w > 0\}$ indicates the plasma region [54, 128]. Higher-dimensional mass quantization is observed in an analogous problem

$$-\Delta w = w_+^q \quad \text{in } \Omega, \qquad w = \text{constant on } \Gamma, \qquad \int_\Omega w_+^q = \lambda, \tag{2.67}$$

where $\Omega \subset \mathbf{R}^m$ ($m \geqslant 3$) is a bounded domain with smooth boundary $\partial\Omega = \Gamma$, and $q = \frac{m}{m-2}$. Furthermore, we can formulate it as the equilibrium self-gravitating fluid equation described by the field component [128].

Similarly to Theorem 2.11, we can prove the quantized blowup mechanism, where the quantized value $m_* > 0$ is defined by

$$m_* = \int_B U^q$$

for $U = U(x)$ satisfying

$$-\Delta U = U^q, \quad U > 0 \quad \text{in } B, \qquad U = 0 \quad \text{on } \partial B$$

with $B = B(0, R)$. This U is radially symmetric and exists uniquely for each $R > 0$, while m_* is independent of $R > 0$. In the following theorem, $G = G(x, x')$ denotes the Green's function of $-\Delta$ on Ω with the Dirichlet boundary condition and

$$R(x) = \left[G(x, x') - \Gamma(x - x')\right]_{x'=x},$$

where

$$\Gamma(x) = \frac{1}{\omega_m (m-2)|x|^{m-2}}$$

is the fundamental solution to $-\Delta$ and ω_m is the $(m-1)$-dimensional volume of the boundary of the unit ball in $\mathbf{R}^m$.

THEOREM 2.12. *If $\Omega \subset \mathbf{R}^m$ $(m \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\{(\lambda_k, w^k)\}$ is a solution sequence to* (2.67) *with $q = \frac{m}{m-2}$ satisfying $\lambda_k \to \lambda_0$, then passing to a subsequence the following alternatives hold*:

(1) $\|w^k\|_\infty = O(1)$.
(2) $\sup_\Omega w^k \to -\infty$.
(3) $\lambda_0 = m_* \ell$ *for some* $\ell \in \mathbf{N}$, *and there exist* $x_j^* \in \Omega$ *and* $x_k^j \to x_j^*$ $(j = 1, \ldots, \ell)$, *where* $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\} \subset \Omega$ *coincides with the blowup set of* $\{w^k\}$ *on* $\overline{\Omega}$ *satisfying the second relation of* (2.51), $x = x_k^j$ *is a local maximum point of* $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ *locally uniformly in* $\overline{\Omega} \setminus \mathcal{S}$, *and*

$$w^k(x)_+^q \, dx \rightharpoonup \sum_j m_* \delta_{x_j^*}(dx) \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

There is a partial answer concerning the actual existence of the solution sequence described in the above theorem [140,147,141]. Similarly to the two-dimensional case, we obtain $v_k = w^k - w_\Gamma^k \geqslant 0$ in Ω by the maximum principle. This $\{v_k\}$, furthermore, satisfies the boundary estimate (2.66). In fact, $v \in [0, \infty) \mapsto f(v) = (v + w_\Gamma)_+^q$ is locally Lipschitz continuous in (2.67), and also if the first two equations hold for $\Omega = B^c$ with $B = B(0, 1)$, then we obtain

$$-\Delta v = |y|^{-2} v_+^{\frac{m}{m-2}} \quad \text{in } B, \qquad v = \text{constant on } \partial B$$

and also

$$\int_B |y|^{-2} v_+^q \, dy \leqslant \int_B |y|^{-m} v_+^q \, dy = \int_{B^c} w_+^q \, dx$$

by the Kelvin transformation $v(y) = |x|^{m-2} w(x)$ with $y = x/|x|^2$. These structures are sufficient to guarantee (2.66) similarly to the two-dimensional case.

Local version comparable to Theorems 2.8, 2.9 also holds. Actually, there are ε-regularity, self-similarity, classification of the entire solution, and $\sup + \inf$ inequality, and these structures guarantee the following theorem. A slight difference to Theorem 2.11 is that the entire solution

$$-\Delta w = w_+^q, \quad w \leqslant w(0) = 1 \quad \text{in } \mathbf{R}^m, \qquad \int_{\mathbf{R}^m} w_+^q < +\infty \tag{2.68}$$

has a compact support, which, makes the argument simpler.

THEOREM 2.13. (See [139].) *If $\Omega \subset \mathbf{R}^m$ $(m \geqslant 3)$ is a bounded domain and $w = w^k$ $(k = 1, 2, \ldots)$ satisfies*

$$-\Delta w = w_+^q \quad \textit{in } \Omega, \qquad \int_\Omega w_+^q \leqslant C$$

for $q = \frac{m}{m-2}$ and $C > 0$, then, passing to a subsequence, we obtain the following alternatives.

(1) *$\{w^k\}$ is locally uniformly bounded in Ω.*
(2) *$w^k \to -\infty$ locally uniformly in Ω.*
(3) *There exist $\ell \in \mathbf{N}$, x_j^* $(j = 1, \ldots, \ell)$, and $x_k^j \to x_j^*$ such that $x = x_k^j$ is a local maximum point of $w^k = w^k(x)$, $w^k(x_k^j) \to +\infty$, $w^k \to -\infty$ locally uniformly in $\Omega \setminus \{x_1^*, \ldots, x_\ell^*\}$, and*

$$w^k(x)_+^q \, dx \rightharpoonup \sum_j m_* n_j \, \delta_{x_j^*}(dx) \quad \textit{in } \mathcal{M}(\Omega),$$

where $n_j \in \mathbf{N}$.

PROOF OF THEOREM 2.12. We have only to show the third case, assuming $w_\Gamma^k \to -\infty$ and $\|v_k\|_\infty \to +\infty$ for $v_k = w^k - w_\Gamma^k \geqslant 0$. Henceforth, we drop k for simplicity. Then, it holds that

$$w(x)_+^q \, dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0) \, \delta_{x_0}(dx)$$

in $\mathcal{M}(\overline{\Omega})$, where $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\} \subset \Omega$ is a set prescribed in the third case of Theorem 2.13, and $m(x_0) \in m_* \mathbf{N}$ for each $x_0 \in \mathcal{S}$. Thus, we have only to show $m(x_0) = m_*$ and the second relation of (2.51).

For this purpose, we apply the method of duality and scaling [128]. Thus, we take $u = w_+^q \geqslant 0$ and obtain

$$\int_\Omega u = \lambda, \qquad w - w_\Gamma = \int_\Omega G(\cdot, x') u(x') \, dx'.$$

This implies

$$\nabla w(x) = \int_\Omega \nabla_x G(x, x') u(x') \, dx'$$

and therefore,

$$\int_{\Omega}(\psi\cdot\nabla w)u=\iint_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)u(x')\,dx\,dx'$$

for $\psi\in C_0^\infty(\Omega)^m$, where the left-hand side is equal to

$$\int_{\Omega}(\psi\cdot\nabla w)u=\frac{1}{q+1}\int_{\Omega}\psi\cdot\nabla w_+^{q+1}=-\frac{1}{q+1}\int_{\Omega}w_+^{q+1}\nabla\cdot\psi. \tag{2.69}$$

Henceforth, $\varphi=\varphi_{x_0,R}$ denotes a smooth function supported by $B(x_0,R)$ and is equal to 1 on $\overline{B(x_0,R/2)}$. We put $\psi(x)=(x-a)\varphi(x)$ for $a\in\mathbf{R}^m$ and $\varphi=\varphi_{x_0,R}$, where $x_0\in\mathcal{S}$, $B(x_0,2R)\subset\Omega$, and $B(x_0,2R)\cap\mathcal{S}=\{x_0\}$. Then, it holds that

$$\nabla\cdot\psi=m\varphi+(x-a)\cdot\nabla\varphi,$$

and therefore,

$$\int_{\Omega}(\psi\cdot\nabla w)u=-\frac{m}{q+1}\int_{\Omega}w_+^{q+1}\varphi+o(1)$$

by (2.69). Thus, we obtain

$$\frac{m}{q+1}\int_{\Omega}w_+^{q+1}\varphi+\iint_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)u(x')\,dx\,dx'=o(1). \tag{2.70}$$

Using $\hat{\varphi}=\varphi_{x_0,2R}$, the second term of the left-hand side of (2.70) is equal to

$$\begin{aligned}
&\iint_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)u(x')\,dx\,dx'\\
&\quad=\iint_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')\,dx\,dx'\\
&\quad=\int_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')\hat{\varphi}(x')\,dx\,dx'\\
&\qquad+\iint_{\Omega\times\Omega}\psi(x)\cdot\nabla_x G(x,x')u(x)\hat{\varphi}(x)u(x')\bigl(1-\hat{\varphi}(x')\bigr)\,dx\,dx'.
\end{aligned}$$

The second term of the right-hand side of the above equality is equal to

$$m(x_0)(x_0-a)\cdot\sum_{x_0'\in\mathcal{S}\setminus\{x_0\}}m\bigl(x_0'\bigr)\nabla_x G\bigl(x_0,x_0'\bigr)+o(1),$$

while the method of symmetrization [127] is applied to the first term. Using

$$K(x,x') = G(x,x') - \Gamma(x-x'),$$

this term is thus equal to

$$\frac{1}{2}\iint_{\Omega\times\Omega} \rho_\psi^0(x,x')u^0(x)u^0(x')\,dx\,dx' + \iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x K(x,x')u^0(x)u^0(x')\,dx\,dx',$$

for $u^0 = u\hat{\varphi}$ and $\rho_\psi^0(x,x') = (\psi(x) - \psi(x'))\cdot\nabla\Gamma(x-x')$.

Since $K = K(x,x') \in C^{2,\theta}(\overline{\Omega}\times\Omega\cup\Omega\times\overline{\Omega})$, it holds that

$$\iint_{\Omega\times\Omega} \psi(x)\cdot\nabla_x K(x,x')u^0(x)u^0(x')\,dx\,dx' = m(x_0)^2(x_0 - a)\cdot\nabla_x K(x_0,x_0) + o(1).$$

We have, on the other hand,

$$\rho_\psi^0(x,x') = -(m-2)\Gamma(x-x') \quad \text{in } B(x_0,R/2)\times B(x_0,R/2)$$

and therefore,

$$\begin{aligned}\rho_\psi^0(x,x')u^0(x)u^0(x') = -(m-2)\Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x') &+ \rho_\psi^0(x,x')\big(1-\tilde{\varphi}(x)\big)\tilde{\varphi}(x')u^0(x)u^0(x') \\ &+ \rho_\psi^0(x,x')\tilde{\varphi}(x)\big(1-\tilde{\varphi}(x')\big)u^0(x)u^0(x'),\end{aligned}$$

where $\tilde{\varphi} = \varphi_{x_0,R/2}$ and $\tilde{u}^0 = u\tilde{\varphi}$. Here,

$$\left|\rho_\psi^0(x,x')\right| \leqslant C\Gamma(x-x')$$

and it holds that

$$0 \leqslant \iint_{\Omega\times\Omega} \Gamma(x-x')\big(1-\tilde{\varphi}(x')\big)u^0(x)u^0(x')\,dx\,dx' = \big\langle \Gamma * u^0, (1-\tilde{\varphi})u^0\big\rangle.$$

This term is $o(1)$ because $\|(1-\tilde{\varphi})u^0\|_\infty \to 0$ and $\|\Gamma * u^0\|_1 = O(1)$ by $\|u\|_1 = O(1)$. Thus, we obtain

$$\frac{1}{2}\iint_{\Omega\times\Omega} \rho_\psi^0(x,x')u^0(x)u^0(x')\,dx\,dx'$$

$$= -\frac{m-2}{2} \iint_{\Omega \times \Omega} \Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x')\,dx\,dx' + o(1),$$

and (2.70) is reduced to

$$\begin{aligned}
&\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi - \frac{m-2}{2}\iint_{\Omega\times\Omega}\Gamma(x-x')\tilde{u}^0(x)\tilde{u}^0(x')\,dx\,dx' \\
&\quad + m(x_0)(x_0-a)\cdot \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') \\
&\quad + m(x_0)^2(x_0-a)\cdot\nabla_x K(x_0,x_0) = o(1).
\end{aligned}$$

Since

$$\frac{m}{q+1}\int_\Omega w_+^{q+1}\varphi = \frac{m-2}{\gamma}\int_\Omega u^\gamma\varphi = \frac{m-2}{\gamma}\int_\Omega \left(\tilde{u}^0\right)^\gamma + o(1)$$

for $\gamma = 1 + \frac{1}{q} = 2 - \frac{2}{m}$, it holds that

$$\begin{aligned}
&(m-2)\mathcal{F}_0(\tilde{\varphi}u) \\
&\quad + m(x_0)(x_0-a)\cdot\left\{\sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') + m(x_0)\nabla_x K(x_0,x_0)\right\} \\
&\qquad = o(1),
\end{aligned} \tag{2.71}$$

for any $a \in \mathbf{R}^m$. Here and henceforth,

$$\mathcal{F}_0(u) = \frac{1}{\gamma}\int_{\mathbf{R}^m} u^\gamma - \frac{1}{2}\langle \Gamma * u, u\rangle$$

and 0-extension is taken to u where it is not defined. Since a is arbitrary, this implies

$$\frac{m(x_0)}{2}\nabla R(x_0) + \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0,x_0') = 0 \tag{2.72}$$

and also

$$\mathcal{F}_0(\tilde{\varphi}u) = o(1). \tag{2.73}$$

Using $q = \frac{m}{m-2}$, we obtain

$$\mathcal{F}_0(u_\mu) = \mu^{m-2}\mathcal{F}_0(u),$$

where $\mu > 0$ and $u_\mu(x) = \mu^m u(\mu x + x_0)$.

Beginning the blowup analysis, now we prescribe the suffix k again. First, there is a maximum point $x = x_k^0$ of $u_k = u_k(x)$ in $B(x_0, 2R)$ such that $x_k^0 \to x_0$ from the proof of Theorem 2.13. Then, the rescaled $\tilde{u}_k(x) = \mu_k^m u_k(\mu_k x + x_k^0)$ is associated with $\tilde{w}_k(x) = \mu_k^{m-2} w_k(\mu_k x + x_k^0)$, and passing to a subsequence, $\tilde{w}_k \to \tilde{w}$ locally uniformly in $\mathbf{R}^m$, where $\mu_k = u_k(x_k^0)^{-1/m}$ and

$$-\Delta \tilde{w} = \tilde{w}_+^q, \quad \tilde{w} \leqslant \tilde{w}(0) = 1 \quad \text{in } \mathbf{R}^m, \qquad \int_{\mathbf{R}^m} \tilde{w}_+^q < +\infty.$$

This entire solution $\tilde{w}$ of (2.68) is radially symmetric, compactly supported on $\overline{B}$ for some $B = B(0, L)$, and

$$\int_{\mathbf{R}^m} \tilde{w}_+^q = m_*.$$

Here, we reformulate $\tilde{u}_k = \tilde{u}_k(x)$ by $\tilde{u}_k(x) = \mu_k^m (\tilde{\varphi} u_k)(\mu_k x + x_k^0)$ and obtain

$$\mathcal{F}_0(\tilde{u}_k) = \mu_k^{m-2} \mathcal{F}_0(\tilde{\varphi} u_k) \to 0 \tag{2.74}$$

by (2.73). It still holds that $\tilde{u}_k \to \tilde{u} \equiv \tilde{w}_+^q$ locally uniformly in $\mathbf{R}^m$, and therefore, $\nabla \tilde{w} = \nabla \Gamma * \tilde{u}$. This implies

$$\mathcal{F}_0(\tilde{u}) = \frac{1}{\gamma} \int_{\mathbf{R}^m} \tilde{u}^\gamma - \frac{1}{2} \langle \Gamma * \tilde{u}, \tilde{u} \rangle = 0 \tag{2.75}$$

similarly to (2.70), i.e.,

$$\frac{m}{q+1} \int_{\mathbf{R}^m} \tilde{w}_+^q + \iint_{\mathbf{R}^m \times \mathbf{R}^m} x \cdot \nabla \Gamma(x - x') \tilde{u}(x) \tilde{u}(x')\, dx\, dx' = 0.$$

We have, on the other hand,

$$\left\langle \Gamma * \tilde{u}_k, \tilde{u}_k \right\rangle \to \langle \Gamma * \tilde{u}, \tilde{u} \rangle \tag{2.76}$$

passing to a subsequence, because $\{\tilde{u}_k\}$ is bounded in $(L^1 \cap L^\infty)(\mathbf{R}^m)$. Thus, $\int_{\mathbf{R}^m} \tilde{u}_k^\gamma \to \int_{\mathbf{R}^m} \tilde{u}^\gamma$ by (2.74)–(2.76), and therefore,

$$\tilde{u}_k \to \tilde{u} \quad \text{in } L^\gamma(\mathbf{R}^m). \tag{2.77}$$

From the proof of Theorem 2.13, if $m(x_0) > m_*$, then there exist a local maximum point $x = x_k^1$ of $u_k = u_k(x)$ and $r_k^0, r_k^1 \to 0$ such that $x_k^1 \neq x_k^0$, $x_k^1 \to x_0$,

$$\int_{B(x_k^0, r_k^0)} u_k \to m_*, \qquad \int_{B(x_k^1, r_k^1)} u_k \to m_*,$$

and $B(x_k^0, 2r_k^0) \cap B(x_k^1, 2r_k^1) = \emptyset$. Furthermore, the connected components of the support of u_k containing x_k^0 and x_k^1 are contained in $B(x_k^0, 2r_k^0)$ and $B(x_k^1, 2r_k^1)$, respectively, for k large. In the rescaled variables, this means

$$\int_{B(0,L')} \tilde{u}_k \to m_*, \qquad \int_{B(x_k', r_k')} \tilde{u}_k \to m_* \tag{2.78}$$

and $B(0, 2L') \cap B(x_k', 2r_k') = \emptyset$ for some $L' > L$, x_k', and r_k', where the connected components of the support of $\tilde{u}_k$ containing 0 and x_k' are contained in $B(0, 2L')$ and $B(x_k', 2r_k')$, respectively. Here, it holds that $|x_k'| \to +\infty$, because $\tilde{u}_k \to 0$ locally uniformly in $B(0, L')^c$.

The second rescaling is defined by

$$\tilde{u}_k'(x) = \left(\mu_k'\right)^m \tilde{u}_k\left(\mu_k' x + x_k'\right) \quad \text{with } \mu_k' = \tilde{u}_k\left(x_k'\right)^{-1/m} \geqslant 1.$$

Passing to a subsequence, now we shall show $\mu_k' \to +\infty$, which implies also $r_k' \to +\infty$ by (2.78).

In fact, if this is not the case, then it holds that $\mu_k' = \tilde{u}_k(x_k')^{-1/m} \approx 1$. We obtain $\tilde{u}_k'' = \tilde{u}_k(\cdot + x_k') \to \tilde{u}'' = a\tilde{w}_+''^q$ locally uniformly in $\mathbf{R}^n$, where $a > 0$ is a constant and

$$-\Delta \tilde{w}'' = a\tilde{w}_+''^q, \quad 0 < \tilde{w}''(0) = \max_{\mathbf{R}^m} \tilde{w}'', \quad \text{in } \mathbf{R}^m, \qquad \int_{\mathbf{R}^m} \tilde{w}_+''^q < +\infty.$$

This implies $r_k' \approx 1$ with

$$\int_{\mathbf{R}^m} \tilde{u}'' = \int_{\mathbf{R}^m} a\tilde{w}_+''^q = m_*,$$

and therefore, it holds that

$$\lim_k \int_{B(0,2L)^c} \tilde{u}_k^\gamma \geqslant \lim_k \int_{B(x_k', 2r_k')} (\tilde{u}_k')^\gamma > 0.$$

Since this contradicts to (2.77), we obtain $\mu_k' \to +\infty$, or equivalently, $\frac{u_k(x_k^0)}{u_k(x_k^1)} \to +\infty$.

Now, we replace the roles of x_k^0 and x_k^1, and repeat the above argument. Changing notations, this means $\frac{u_k(x_k^0)}{u_k(x_k^1)} \to 0$ and therefore, $\{\tilde{u}_k\}$ concentrates around $x_k' \in B(0, L')^c$. We obtain

$$\int_{B(x_k',1)} \tilde{u}_k \to m_*$$

by $|x_k'| \to +\infty$, while $\int_{B(x_k',1)} \tilde{u}_k^\gamma \to 0$ follows because (2.77) is obtained similarly. This is a contradiction again, and thus, $m(x_0) = m_*$ for each $x_0 \in \mathcal{S}$. Then, (2.51) follows from (2.72). □

3. Eigenvalue problem

Total mass of the stationary closed system is prescribed by the eigenvalue, and this section is devoted to the detailed description concerning eigenvalues and eigenfunctions of nonlinear problems. Thus, taking preliminaries from the linear theory in the first section, we develop the theory of rearrangement in the next section. In the final section, we show a uniqueness theorem which guarantees the propagation of chaos of many vortex points.

3.1. *Linear theory*

Linear eigenvalue problem arises if one adopts the methods of super-position and separation of variables to several partial differential equations. The wave equation,

$$p(x)u_{tt} = \Delta u \quad \text{in } \Omega \times (0,T), \qquad u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3.1}$$

for example, describes the vibration of membrane, where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $p = p(x) > 0$ is a continuous function indicating density. Method of separation of variables assumes the form $u(x,t) = \varphi(x) f(t)$ of the solution, which results in

$$\frac{f''}{f} = \frac{\Delta\varphi}{p(x)\varphi}.$$

Observing that this quantity is a constant, we write it as $-\lambda$ and obtain the eigenvalue problem

$$-\Delta\varphi = \lambda p(x)\varphi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

and

$$f(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t/\sqrt{\lambda},$$

where A, B are constants under the agreement of $\sin\sqrt{\lambda}t/\sqrt{\lambda} = t$ if $\lambda = 0$. If (3.2) has a nontrivial solution $\varphi = \varphi(x) \not\equiv 0$, then this λ is called an eigenvalue. In this case it describes the frequency of oscillation of $u(x,t) = \varphi(x) f(t)$.

If the eigenvalue is prescribed, the set of eigenfunctions forms a linear space, and its dimension is called the multiplicity. In our notation, eigenvalues are counted according to their multiplicities, and therefore, each eigenvalue takes only one eigenfunction $\varphi = \varphi(x)$ normalized by $\|\varphi\|_2 = 1$. Eigenvalues of (3.2) are discrete and it holds that

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \to +\infty$$

under this agreement. This discrete structure is related to the quantization of the energy level of particles in quantum mechanics.

Each finite dimensional linear space generated by the eigenfunctions with the same eigenvalue is as an object bifurcated from the trivial solution $\varphi = 0$ in (3.2), and in this context, (2.33) is regarded as a nonlinear eigenvalue problem. In this problem, quantization is observed in the blowup mechanism of the solution sequence, while bifurcation can actually occur from the branch of nonminimal solutions [83,28].

3.1.1. *Min–Max principle.* We develop the abstract theory for the moment. Let H be a Hilbert space over $\mathbf{R}$ provided with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$, and $b = b(\cdot,\cdot)$ be a positive-definite symmetric bilinear form on H, i.e., $b : H \times H \to \mathbf{R}$ is symmetric, bi-linear, and satisfies

$$b(u,u) \geqslant \delta |u|^2, \qquad \left| b(u,v) \right| \leqslant M |u||v|$$

for $u, v \in H$, where $\delta, M > 0$ are constants. Thus, this $b = b(\cdot,\cdot)$ is regarded as an inner product in H.

Let V be another Hilbert space, continuously imbedded in H, with the inner product $((\cdot,\cdot))$ and the norm $\|\cdot\|$, and $a = a(\cdot,\cdot)$ be another symmetric bi-linear form on V satisfying

$$a(u,u) \geqslant \delta \|u\|^2 - C|u|^2, \qquad \left| a(u,v) \right| \leqslant M \|u\| \|v\|$$

for $u, v \in V$, where $C > 0$ is a constant. Then, we define the abstract eigenvalue problem by finding $(\lambda, \varphi) \in \mathbf{R} \times V$ such that

$$a(\varphi, \psi) = \lambda b(\varphi, \psi) \quad \text{for any } \psi \in V. \tag{3.3}$$

Any $\lambda \in \mathbf{R}$ admits $\varphi = 0$ as a solution to (3.3), and λ is called an eigenvalue if there is $\varphi \neq 0$ satisfying (3.3). This φ is normalized by $b(\varphi, \varphi) = 1$, and then the Hilbert–Schmidt theory follows, justifying the method of super-position [8,17,109].

THEOREM 3.1. *If the imbedding $V \subset H$ is compact, then we obtain the following.*

(1) *Eigenvalues of* (3.3) *are countably many, denoted by $\{\lambda_k\}_{k=1}^{\infty}$. It accumulates only to $+\infty$.*

(2) *The set of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ forms a complete orthonormal system of H with respect to $b = b(\cdot,\cdot)$. Thus, it holds that $-\infty < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to +\infty$, $b(\varphi_k, \varphi_j) = \delta_{kj}$, and*

$$\lim_{k \to \infty} \left| v - \sum_{j=1}^{k} b(v, \varphi_j)\varphi_j \right| = 0 \tag{3.4}$$

for each $v \in H$.

The Fourier expansion (3.4) guarantees

$$b(v,v)=\sum_{k=1}^{\infty} c_k^2 \quad \text{and} \quad a(v,v)=\sum_{k=1}^{\infty} \lambda_k c_k^2 < +\infty$$

for $v \in H$ and $v \in V$, respectively, where $c_k = b(v, \varphi_k)$. Defining the Rayleigh quotient

$$R[v]=\frac{a(v,v)}{b(v,v)}=\frac{\sum_{k=1}^{\infty}\lambda_k c_k^2}{\sum_{k=1}^{\infty} c_k^2}$$

for $v \in V \setminus \{0\}$, therefore, we obtain

$$\lambda_k = \inf\bigl\{R[v] \mid v \in V \cap H_{k-1} \setminus \{0\}\bigr\}, \tag{3.5}$$

where $H_0 = H$, $H_{k-1} = \{v \in H \mid b(v, \varphi_j) = 0,\ 1 \leqslant j \leqslant k-1\}$ for $k = 1, 2, \ldots$. Then, the min–max principles are indicated by

$$\begin{aligned}\lambda_k &= \min\Bigl\{\max_{v\in L_k\setminus\{0\}} R[v] \Bigm| L_k \subset V,\ \dim L_k = k\Bigr\}\\ &= \max\Bigl\{\min_{v\in V_k\setminus\{0\}} R[v] \Bigm| V_k \subset V,\ \dim V/V_k = k-1\Bigr\}.\end{aligned}$$

See [8].

Given a bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$, a relatively open set $\Gamma_0 \subset \partial\Omega$, and continuous functions $c = c(x)$, $p = p(x) > 0$ on $\overline{\Omega}$, we take $H = L^2(\Omega)$, $V = \{v \in H^1(\Omega) \mid v|_{\Gamma_0} = 0\}$, and

$$\begin{aligned}a(u,v) &= \int_\Omega \bigl(\nabla u \cdot \nabla v + c(x)uv\bigr)\,dx,\\ b(u,v) &= \int_\Omega uvp(x)\,dx,\end{aligned} \tag{3.6}$$

where $\cdot|_{\Gamma_0} : H^1(\Omega) \to H^{1/2}(\Gamma_0)$ is the trace operator. Then, (3.3) describes

$$\begin{aligned}&\bigl(-\Delta + c(x)\bigr)\varphi = \lambda p(x)\varphi \quad \text{in } \Omega,\\ &\varphi = 0 \quad \text{on } \Gamma_0,\\ &\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0\end{aligned} \tag{3.7}$$

and Theorem 3.1 is applicable by Rellich–Kondrachov's theorem, where ν is the outer unit normal vector.

3.1.2. *Hartman–Wintner's theorem.* To study the profile of zero-sets of eigenfunctions, the following theorem is useful.

THEOREM 3.2. (See [12,64].) *Let $\Omega \subset \mathbf{R}^n$ be an open set containing the origin and L be an elliptic operator of order $2m$ with C^∞ coefficients. Suppose that $f = f(x)$ is C^∞ around $x = 0$ and satisfies $Lf = 0$ in Ω,*

$$D^\alpha f(0) = 0 \quad \big(|\alpha| \leqslant N\big), \quad \textit{and} \quad \max_{|\alpha|=N+1} \big|D^\alpha f(0)\big| \neq 0.$$

Then, there is a homogeneous polynomial $p = p(x)$ of degree $N+1$ satisfying

$$D^\alpha f(x) = D^\alpha p(x) + o\big(|x|^{N+1-|\alpha|}\big) \tag{3.8}$$

for $|\alpha| \leqslant \min(2m, N+1)$ as $x \to 0$, where $N = 0, 1, \ldots$.

The following theorem shows that if $n = 2$, $m = 1$, and the principal part of L is Δ, then (3.8) is valid up to $|\alpha| \leqslant N+1$. We describe the proof for completeness, while several applications are found [60,144,35,1,104].

THEOREM 3.3. (See [63].) *If $\Omega \subset \mathbf{R}^2$ is an open set containing the origin and $u = u(x)$ is a C^2 function satisfying*

$$|\Delta u| \leqslant C\big(|\nabla u| + |u|\big) \quad \textit{in } \Omega, \tag{3.9}$$

and

$$u(x) = o\big(|x|^n\big) \tag{3.10}$$

as $x \to 0$, then

$$\lim_{x \to 0} u_z / z^n$$

exists, where $n = 0, 1, \ldots$ and $z = x_1 + \imath x_2$. Thus, we obtain $u_z = az^n + o(|x|^n)$ with some $a \in \mathbf{C}$, and therefore, it holds that

$$u(x) = \operatorname{Re}\left(\frac{az^{n+1}}{n+1}\right) + o\big(|x|^{n+1}\big)$$

as $x \to 0$.

PROOF. Since $u = u(x)$ is C^2, this theorem is obvious for $n = 0, 1$. We suppose $n \geqslant 2$ and show that

$$u_z = o\big(|z|^{k-1}\big) \tag{3.11}$$

implies the existence

$$\lim_{x\to 0} u_z/z^k = a_k \tag{3.12}$$

for $k = 1, 2, \dots, n$.

In fact, first, (3.11) is valid to $k = 1$, because $u = u(x)$ is C^2 and (3.10) with $n \geqslant 2$. Next, if (3.12) holds, then

$$u(x) = \mathrm{Re}\left(\frac{a_k z^{k+1}}{k+1}\right) + o\big(|z|^{k+1}\big)$$

and therefore, $a_k = 0$ follows from (3.10) if $k < n$. This means (3.11) with k replaced by $k+1$. Continuing this procedure, we obtain (3.12) and hence the conclusion.

To prove that (3.11) implies (3.12), we suppose $\Omega = B(0, R_0)$ and take $\omega = B(0, R) \setminus (B(0, \varepsilon) \cup B(\zeta, \varepsilon)) \Subset \Omega$, where $0 < |\zeta| < R < R_0$ and $0 < \varepsilon \ll 1$. Since

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \imath \frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \imath \frac{\partial}{\partial x_2}\right)$$

and

$$dz \wedge d\,\overline{z} = -2\imath\, dx_1 \wedge dx_2$$

for $z = x_1 + \imath x_2$, Green's formula is described by

$$\int_\omega (g u_{z\overline{z}} + u_z g_{\overline{z}})\, dz\, d\overline{z} = \int_{\partial\omega} g u_z\, dz.$$

Applying this to $g(z) = z^{-k}(z-\zeta)^{-1}$, we obtain

$$\int_{\partial\omega} z^{-k}(z-\zeta)^{-1} u_z\, dz = \int_\omega z^{-k}(z-\zeta)^{-1} u_{z\overline{z}}\, dz\, d\overline{z},$$

Here, the left-hand side is equal to

$$\left(\int_{|z|=R} - \int_{|z-\zeta|=\varepsilon} - \int_{|z|=\varepsilon}\right) z^{-k}(z-\zeta)^{-1} u_z\, dz$$
$$= \int_{|z|=R} z^{-k}(z-\zeta)^{-1} u_z\, dz - 2\pi\imath\zeta^{-k} u_z(\zeta) + o(1)$$

as $\varepsilon \downarrow 0$ by (3.11), and therefore,

$$-2\pi\imath u_z(\zeta)\zeta^{-k} + \int_{|z|=R} z^{-k}(z-\zeta)^{-1} u_z\, dz = \int_{|z|<R} z^{-k}(z-\zeta)^{-1} u_{z\overline{z}}\, dz\, d\overline{z}. \tag{3.13}$$

To guarantee the convergence of the right-hand side of this equality, first we apply (3.9) and obtain

$$\begin{aligned} &\int_{|z|<R} \left| z^{-k}(z-\zeta)^{-1} u_{z\overline{z}} \right| |dz\, d\overline{z}| \\ &\quad \leqslant C \int_{|z|<R} \big(|u_z| + |u|\big) |z|^{-k} |z-\zeta|^{-1} |dz\, d\overline{z}| \end{aligned} \tag{3.14}$$

by $|\nabla u| = |u_z|$ because $u = u(x)$ is real-valued. Here, we obtain

$$\left| u/z^k \right| = O(1), \qquad \left| u_z/z^k \right| = O\big(|z|^{-1}\big) \tag{3.15}$$

by (3.10)–(3.11), and therefore, the right-hand side of (3.13) converges. Using (3.13), now we obtain

$$\begin{aligned} 2\pi \left| u_z(\zeta) \right| |\zeta|^{-k} \leqslant &\int_{|z|=R} |u_z| |z|^{-k} |z-\zeta|^{-1} |dz| \\ &+ C \int_{|z|<R} \big(|u_z| + |u|\big) |z|^{-k} |z-\zeta|^{-1} |dz\, d\overline{z}|. \end{aligned} \tag{3.16}$$

For $0 < |z_0| < R$, we operate $\int_{|\zeta|<R} |\zeta - z_0|^{-1} \cdot |d\zeta\, d\overline{\zeta}|$ to both sides;

$$\begin{aligned} &2\pi \int_{|\zeta|<R} \left| u_z(\zeta) \zeta^{-k} (\zeta - z_0)^{-1} \right| |d\zeta\, d\overline{\zeta}| \\ &\quad \leqslant \int_{|z|=R} |u_z| |z|^{-k} |dz| \int_{|\zeta|<R} \left| (z-\zeta)(\zeta - z_0) \right|^{-1} |d\zeta\, d\overline{\zeta}| \\ &\qquad + C \int_{|z|<R} \big(|u_z| + |u|\big) |z|^{-k} |dz\, d\overline{z}| \int_{|\zeta|<R} \left| (z-\zeta)(\zeta - z_0) \right|^{-1} |d\zeta\, d\overline{\zeta}|. \end{aligned}$$

We obtain

$$\begin{aligned} \left| (z-\zeta)(\zeta - z_0) \right|^{-1} &= |z - z_0|^{-1} \left| (z-\zeta)^{-1} + (\zeta - z_0)^{-1} \right| \\ &\leqslant |z - z_0|^{-1} \big(|\zeta - z|^{-1} + |\zeta - z_0|^{-1} \big) \end{aligned}$$

and also

$$\int_{|\zeta|<R} |z-\zeta|^{-1} |d\zeta\, d\overline{\zeta}| \leqslant \int_{|\zeta - z|<2R} |\zeta - z|^{-1} 2\, dx = 8\pi R$$

by $|z| < R$. It follows that

$$\int_{|\zeta|<R} \left| (z-\zeta)(\zeta - z_0) \right|^{-1} |d\zeta\, d\overline{\zeta}| \leqslant 16\pi R |z - z_0|^{-1},$$

and therefore,

$$\int_{|\zeta|<R} \left|u_z(\zeta)\zeta^{-k}(\zeta - z_0)^{-1}\right| |d\zeta\, d\overline{\zeta}|$$
$$\leqslant 8R \int_{|z|=R} |u_z||z|^{-k}|z - z_0|^{-1}|dz|$$
$$+ 8RC \int_{|z|<R} \big(|u_z| + |u|\big)|z|^{-k}|z - z_0|^{-1}|dz\, d\overline{z}|,$$

or equivalently,

$$(1 - 8RC) \int_{|z|<R} \left|u_z z^{-k}(z - z_0)^{-1}\right| |dz\, d\overline{z}|$$
$$\leqslant 8R \int_{|z|=R} \left|u_z z^{-k}(z - z_0)^{-1}\right| |dz| + 8RC \int_{|z|<R} \left|u z^{-k}(z - z_0)^{-1}\right| |dz\, d\overline{z}|.$$

Here, we can take $0 < R < 1/(8C)$, while the right-hand side is bounded as $z_0 \to 0$ by (3.15). This implies

$$\int_{|z|<R} \left|u_z z^{-k}(z - \zeta)^{-1}\right| |dz\, d\overline{z}| = O(1)$$

as $\zeta \to 0$, and therefore, the right-hand side of (3.16) is bounded as $\zeta \to 0$ by (3.15). Then, it holds that

$$u_z/z^k = O(1)$$

as $z \to 0$, and the right-hand side of (3.14) converges as $\zeta \to 0$. This means the summability of the right-hand side of (3.13) for $\zeta = 0$, and hence the existence of (3.12). □

3.1.3. *Kuo's theorem.* Combining Theorems 3.2, 3.3 with the following theorem, we can clarify the profile of zero-sets of eigenfunctions.

THEOREM 3.4. (See [73].) *If $f = f(x)$, $h = h(x)$ are smooth around $x = 0 \in \mathbf{R}^n$, satisfy*

$$f(x) = h(x) + o\big(|x|^{N+1}\big), \qquad Df(x) = Dh(x) + o\big(|x|^N\big) \tag{3.17}$$

as $x \to 0$, and

$$D^\alpha h(0) = 0 \quad \big(|\alpha| \leqslant N\big), \qquad \max_{|\alpha| \leqslant N+1} \left|D^\alpha h(0)\right| \neq 0,$$

then there is a local C^1 diffeomorphism Φ satisfying $\Phi(0) = 0$ and

$$f(x) = h\big(\Phi(x)\big),$$

where $N = 0, 1, \ldots.$

PROOF. Given an open set $U \subset \mathbf{R}^n$ containing the origin and $f, h \in C^\infty(U)$, we define the homotopy function

$$F\begin{pmatrix} x \\ a \end{pmatrix} = (1-a)f(x) + ah(x)$$

for $(x, a) \in U \times [0, 1]$. Since

$$\nabla F\begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} (1-a)Df(x) + aDh(x) \\ h(x) - f(x) \end{pmatrix},$$

we obtain

$$\nabla F\begin{pmatrix} 0 \\ a \end{pmatrix} = 0 \quad (0 \leqslant a \leqslant 1) \tag{3.18}$$

and

$$\begin{aligned} \left|\nabla F\begin{pmatrix} x \\ a \end{pmatrix}\right| &\geqslant |Dh| - (1-a)|Df - Dh| - |f - h| \\ &= |Dh| + o\big(|x|^N\big) \approx |x|^N \end{aligned}$$

as $x \to 0$ uniformly in $a \in [0, 1]$. Thus,

$$X\begin{pmatrix} x \\ a \end{pmatrix} = \begin{cases} |\nabla F|^{-2}(h-f)\nabla F & (x \neq 0), \\ 0 & (x = 0) \end{cases}$$

is a C^1 vector field on $U_1 \times [0, 1]$ by (3.17) again, where $U_1 \subset U$ is an open set containing the origin, and thus, we can define

$$v\begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - X\begin{pmatrix} x \\ a \end{pmatrix}.$$

For the moment, $\cdot$ denotes the inner product in $\mathbf{R}^{n+1}$. This vector field v satisfies

$$v \cdot \nabla F = (h - f) - X \cdot \nabla F = 0 \tag{3.19}$$

and

$$\begin{aligned} v \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 1 - X \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 - |\nabla F|^{-2}(h-f)\nabla F \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 1 - |\nabla F|^{-2}(h-f)^2 = 1 - o\big(|x|^2\big) > 0 \end{aligned} \tag{3.20}$$

for $|x| \ll 1$ uniformly in $a \in [0, 1]$.

We take $(x, a) \in U_1 \times (0, 1)$ and introduce the flow $\phi = \phi\left(\begin{smallmatrix} x \\ a \end{smallmatrix}, t\right)$ by

$$\frac{d\phi}{dt} = v(\phi), \qquad \phi|_{t=0} = \begin{pmatrix} x \\ a \end{pmatrix}.$$

Since (3.20), the a-component of ϕ is increasing as far as its x-component is sufficiently close to zero. Therefore, the orbit

$$\left\{ \phi\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, t \right) \;\middle|\; |t| < T \right\}$$

crosses the hyperplane $a = 1$ at a unique point, denoted by $\Phi(x)$, provided that $|x| \ll 1$. This means

$$\phi\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, t \right) = \begin{pmatrix} \Phi(x) \\ 1 \end{pmatrix} \tag{3.21}$$

for some t.

We obtain, first, $\Phi(0) = 0$ by $v\left(\left(\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Since (3.19), next, it holds that

$$\frac{d}{dt} F\left(\phi\left(\begin{pmatrix} x \\ a \end{pmatrix}, t \right) \right) = 0,$$

and therefore,

$$F\begin{pmatrix} x \\ 0 \end{pmatrix} = F\begin{pmatrix} \Phi(x) \\ 1 \end{pmatrix}$$

by (3.21). This means $f(x) = h(\Phi(x))$, and the proof is complete. □

If M is a d-dimensional Riemannian manifold, L is an elliptic differential operator, and $f = f(x)$ is a smooth function satisfying

$$Lf = 0, \qquad f \not\equiv 0 \quad \text{in } M \setminus \partial M, \tag{3.22}$$

then $N(f) = \overline{\{x \in M \mid f(x) = 0\}}$ is called the nodal set of f. If L is the second order, then each zero of f is of finite order by the unique continuation theorem, and therefore, $N(f)$ is locally C^1 diffeomorphic to that of a polynomial by Theorems 3.2 and 3.4. More detailed analysis, however, guarantees the following theorem.

THEOREM 3.5. (See [62].) *If L is a second order elliptic operator with C^∞ coefficients defined on a d-dimensional C^∞ manifold M and f is a C^∞ function on M satisfying $Lf = 0$, then the nodal set $N(f)$ of f is a $(d-1)$-dimensional C^∞ manifold except for a closed set $f^{-1}\{0\} \cap Df^{-1}\{0\}$ of Hausdorff dimension less than or equal to $(d-2)$.*

In the two-dimensional case, we can take $h(x) = \mathrm{Re}(\frac{az^{n+1}}{n+1})$ with some $a \neq 0$ and $n = 1, 2, \ldots,$ and this complex structure guarantees the following theorem easily.

THEOREM 3.6. (See [38].) *If M is a two-dimensional C^2 manifold, L is an elliptic operator on M with bounded coefficients such that the principal part is the Laplace–Beltrami operator, and $f = f(x)$ is a C^2 function satisfying* (3.22), *then the nodal set $N(f)$ is provided with the following properties.*

(1) *Critical points of f on $N(f) \cap M$ are isolated.*
(2) *For any compact set $K \subset M \setminus \partial M$, $N(f) \cap K$ consists of C^2 curves crossing transversally.*

Consequently, if M is a compact Riemannian surface without boundary, then $N(f)$ is composed of a finite number of C^2 Jordan curves crossing transversally at most finitely many times. If M is a compact Riemannian surface with piecewise C^2 boundary ∂M and $f = f(x)$ is a continuous function on M such that C^2 in $M \setminus \partial M$, satisfies (3.22), and

$$f = 0 \quad \text{on } \partial M,$$

then $N(f) \cup \partial M$ comprises a finite number of C^2 curves crossing transversally at most finitely many times.

3.1.4. *Nodal domains.* Each connected component of $\{x \in M \setminus \partial M \mid f(x) \neq 0\}$ is called the nodal domain. If M is a compact Riemannian surface with or without boundary, then each C^2 curve comprising $N(f) = \overline{\{x \in M \mid f(x) = 0\}}$ is called the nodal line.

For the moment, the eigenvalues and the eigenfunctions of (3.7) are denoted by $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$, respectively, where

$$-\infty < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to +\infty, \qquad \|\varphi_k\|_2 = 1.$$

This problem is associated with $a = a(\cdot,\cdot)$ and $b = b(\cdot,\cdot)$ defined by (3.6), where $\Omega = M \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $H = L^2(\Omega)$ and $V = \{v \in H^1(\Omega) \mid v|_{\Gamma_0} = 0\}$.

THEOREM 3.7. (See [41].) *The nodal domain of the kth eigenfunction φ_k of* (3.7) *has the following properties.*

(1) *The number of nodal domains of φ_k is at most k.*
(2) *Only the first eigenfunction φ_1 has a definite sign in Ω.*
(3) *The second eigenfunction φ_2 has just two nodal domains.*

PROOF. The minimum λ_k of (3.5) is attained by a constant times φ_k. From the elliptic estimate, this φ_k is $C^{1,\theta}$ on $\overline{\Omega}$. We suppose that φ_k has at least $k+1$ nodal domains denoted by $\Omega_1, \ldots, \Omega_{k+1} \subset \Omega$, and define ψ_j as the zero extension of $\varphi_k|_{\Omega_j}$. By Theorem 3.5, it holds that $\psi_j \in V$ and

$$\int_{\partial\Omega_j} \psi_j \frac{\partial\psi_j}{\partial\nu} dH^{n-1} = 0 \tag{3.23}$$

for each j, where dH^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure.

We can take $a_1, \dots, a_k \in \mathbf{R}$ satisfying

$$b(\psi, \psi_j) = 0 \quad \text{for } 1 \leqslant j \leqslant k-1$$

for $\psi = a_1\psi_1 + \cdots + a_k\psi_k \not\equiv 0$. Thus, we obtain $\psi \in V \cap H_{k-1}$, while

$$\begin{aligned} a(\psi, \psi) &= \int_\Omega \bigl(|\nabla\psi|^2 + c(x)\psi^2\bigr) = \sum_{j=1}^k a_j^2 \int_\Omega \bigl(|\nabla\psi_j|^2 + c(x)\psi_j^2\bigr) \\ &= \sum_{j=1}^k a_j^2 \int_{\Omega_j} \bigl(-\Delta\psi_j + c(x)\psi_j\bigr)\psi_j \end{aligned}$$

holds by (3.23). This implies

$$a(\psi, \psi) = \lambda_k \sum_{j=1}^k a_j^2 \int_{\Omega_j} p(x)\psi_j^2 = \lambda_k b(\psi, \psi),$$

and therefore, $\psi \not\equiv 0$ attains the minimum of (3.5). Then, it holds that $\psi = \text{constant} \times \varphi_k$.

However, $\psi = 0$ holds in Ω_{k+1}, and therefore, $\psi \equiv 0$ by the unique continuation theorem to φ_k. This is a contradiction, and the number of nodal domains of φ_k is at most k.

In particular, φ_1 has a definite sign. Without loss of generality, henceforth we suppose $\varphi_1 > 0$ in Ω. Then, φ_k for $k \geqslant 2$ cannot have a definite sign, because

$$b(\varphi_1, \varphi_k) = 0.$$

In particular, φ_2 has two nodal domains. □

The above described profile of the nodal domain provides a proof of $\lambda_1 < \lambda_2$.

THEOREM 3.8. *The first eigenvalue λ_1 is simple.*

PROOF. If this is not the case, we obtain $\lambda_1 = \lambda_2$, and therefore, φ_2 attains

$$\lambda_1 = \inf\bigl\{R[v] \bigm| v \in V \setminus \{0\}\bigr\}.$$

This implies that φ_2 has a definite sign in Ω from the proof of the previous theorem, a contradiction. □

3.2. *Rearrangement*

Based on the considerations in the previous section, given a bounded open set $\Omega \subset \mathbf{R}^n$, we define the first eigenvalue of

$$-\Delta\varphi = \lambda\varphi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{3.24}$$

by

$$\lambda_1(\Omega) = \inf\big\{R[v] \bigm| v \in H_0^1(\Omega) \setminus \{0\}\big\},$$

where $R[v] = \|\nabla v\|_2^2/\|v\|_2^2$. It is actually attained by the first eigenfunction $\varphi_1 \in H_0^1(\Omega)$, smooth and positive in Ω. Then, there is an isoperimetric inequality for $\lambda_1(\Omega)$.

THEOREM 3.9. (See [52,72].) *If $\Omega \subset \mathbf{R}^n$ is a bounded open set, then it holds that*

$$\lambda_1(\Omega) \geqslant \lambda_1(\Omega^*), \tag{3.25}$$

where Ω^* is the ball in $\mathbf{R}^n$ provided with the same volume of that of Ω. Furthermore, the equality holds if and only if Ω is a ball.

The kth eigenvalue of (3.24) is reformulated by the min–max principle. It is attained by the kth eigenfunction, denoted by φ_k, smooth in Ω. Then, each connected component of the open set $\{x \in \Omega \mid \varphi_k(x) \neq 0\}$ is called the nodal domain of φ_k. The above theorem implies the following fact.

THEOREM 3.10. (See [96].) *If $n = 2$, only finitely many φ_k's have k nodal domains.*

PROOF. Since $n = 2$, it holds that

$$\lambda_1(\Omega^*) = \pi\ell^2/|\Omega|,$$

where $\ell > 0$ is the first zero of the Bessel function J_0 of order 0:

$$\ell = 2.4048\ldots.$$

If $\Omega_1, \ldots, \Omega_N$ are the nodal domains of φ_k, then φ_k is regarded as the first eigenfunction of

$$-\Delta\varphi = \lambda\varphi \quad \text{in } \Omega_j, \qquad \varphi = 0 \quad \text{on } \partial\Omega_j.$$

Thus, we obtain $\lambda_k = \lambda_1(\Omega_j) \geqslant \pi\ell^2/|\Omega_j|$, which implies

$$|\Omega| \geqslant \pi\ell^2 N/\lambda_k.$$

If $N = k$ occurs to infinitely many times, therefore, it holds that

$$\limsup_{k\to+\infty} \frac{k}{\lambda_k} \leqslant |\Omega|/\pi\ell^2.$$

However, the left-hand side is equal to $|\Omega|/(4\pi)$ by Weyl's formula [41], and hence $\ell \leqslant 2$, a contradiction. □

The eigenvalues and eigenfunctions of

$$-\Delta\varphi = \mu\varphi \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$

is also defined by the min–max principle and $R[v] = \|\nabla v\|_2^2/\|v\|_2^2$. Its first eigenvalue $\mu_1(\Omega)$ is always zero taking the constant eigenfunction. Thus, the second eigenvalue is defined by

$$\mu_2(\Omega) = \left\{ R[v] \;\middle|\; v \in H^1(\Omega) \setminus \{0\},\ \int_\Omega v\,dx = 0 \right\},$$

and there is also an isoperimetric inequality.

THEOREM 3.11. (See [131,142].) *If $\Omega \subset \mathbf{R}^n$ is a bounded open set then it holds that $\mu_2(\Omega) \leqslant \mu_2(\Omega^*)$, where Ω^* is the ball in $\mathbf{R}^n$ provided with the same volume of that of Ω. Furthermore, the equality holds if and only if Ω is a ball.*

3.2.1. *Schwarz symmetrization.* We use the process of rearrangement to prove (3.25). In fact, the first eigenfunction $\varphi_1 = \varphi_1(x)$ of (3.24) is smooth and positive in Ω, and satisfies $\varphi_1 = 0$ on $\partial\Omega$, and therefore, each level set $\{x \in \Omega \mid \varphi(x) > t\}$ is compactly imbedded in Ω for $t > 0$.

Given a measurable function $u : \Omega \to \mathbf{R}$, we take the distribution function $\mu = \mu(t) : [0, +\infty) \to [0, +\infty]$ by

$$\mu(t) = \left|\left\{x \in \Omega \;\middle|\; \left|u(x)\right| > t\right\}\right|, \tag{3.26}$$

where $|\cdot|$ denotes the n-dimensional volume. This $\mu = \mu(t)$ is right-continuous and non-increasing, and it holds that

$$\int_\Omega \left|u(x)\right|^p dx = \int_0^\infty t^p\, d\bigl(-\mu(t)\bigr) \tag{3.27}$$

for $p > 0$, where the left-hand and the right-hand sides are the Lebesque and the Riemann–Stieltjes integrals, respectively.

First, the decreasing rearrangement $u^\star = u^\star(s) : [0, +\infty) \to [0, +\infty]$ of $u = u(x)$ is defined by

$$u^\star(s) = \inf\bigl\{t \geqslant 0 \;\big|\; \mu(t) < s\bigr\}.$$

Then,

$$\int_0^\infty u^\star(s)^p\,ds = \int_0^\infty t^p\,d\bigl(-\mu(t)\bigr) \tag{3.28}$$

follows similarly. The Schwarz symmetrization $u^*: \mathbf{R}^n \to \mathbf{R}$ of $u = u(x)$ is then defined by

$$u^*(x) = u^\star\bigl(c_n|x|^n\bigr),$$

where $c_n = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the n-dimensional unit ball. It follows that

$$\int_\Omega \bigl|u(x)\bigr|^p = \int_{\mathbf{R}^n} \bigl|u^*(x)\bigr|^p \tag{3.29}$$

from (3.27)–(3.28).

In the case that Ω is bounded, the Schwarz symmetrization may be defined by

$$u^*(x) = \begin{cases} \sup\{t \mid x \in \Omega_t^*\} & (x \in \Omega^*), \\ 0 & (x \notin \Omega^*). \end{cases}$$

Here, given a measurable set ω, ω^* denotes the open ball with the center origin satisfying $|\omega| = |\omega^*|$ and $\Omega_t = \{x \in \Omega \mid |u(x)| > t\}$. In fact, we obtain $x \in \Omega_t^* \Leftrightarrow c_n|x|^n < \mu(t)$, and also

$$\sup\bigl\{t \geqslant 0 \bigm| c_n|x|^n < \mu(t)\bigr\} = \inf\bigl\{t \bigm| \mu(t) < c_n|x|^n\bigr\}.$$

The following properties are obvious.

(1) $u^* = u^*(|x|)$ is nonnegative, decreasing in $r = |x|$, and satisfies $\inf_\Omega |u| = \inf_{\mathbf{R}^n} |u|$, $\sup_\Omega |u| = \sup_{\mathbf{R}^n} u^*$.

(2) If $|u_1| \leqslant |u_2|$ in Ω, then $u_1^* \leqslant u_2^*$ in $\mathbf{R}^n$.

The following relation is called Hardy–Littlewood's inequality:

$$\int_\Omega \bigl|u(x)v(x)\bigr|\,dx \leqslant \int_0^\infty u^\star(s)v^\star(s)\,ds = \int_{\mathbf{R}^n} u^*\bigl(|x|\bigr)v^*\bigl(|x|\bigr)\,dx. \tag{3.30}$$

If $\Omega \subset \mathbf{R}^n$ is a bounded open set, then the following properties hold also.

(1) If u is continuous, then u^* is so.

(2) If u is nonnegative, Lipschitz continuous on $\overline{\Omega}$, and $u|_{\partial\Omega} = 0$, then so is true for u^* on $\overline{\Omega^*}$, and the Lipschitz constant of u^* is less than or equal to that of u.

Inequality (3.25) is a consequence of (3.29),

$$\lambda_1 = \inf\bigl\{R[v] \bigm| v \in H_0^1(\Omega) \setminus \{0\}\bigr\}$$

for $R[v] = \|\nabla v\|_2^2/\|v\|_2^2$, and the following fact.

THEOREM 3.12. (See [94].) *If* $\Omega \subset \mathbf{R}^n$ *is an open bounded set,* $1 < p < \infty$, *and* $u \in W_0^{1,p}(\Omega)$, *then* $u^* \in W_0^{1,p}(\Omega^*)$ *and it holds that*

$$\int_\Omega |\nabla u|^p \, dx \geqslant \int_{\Omega^*} |\nabla u^*|^p \, dx.$$

3.2.2. *Talenti's theorem.* There are several monographs concerning the theory of rearrangement and its applications to partial differential equations [94,8,86,71,47]. Here, we describe a pointwise estimate of the solution to

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.31}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set,

$$L = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + c(x) \tag{3.32}$$

is an elliptic operator of the second order in divergence form with the coefficients $a_{ij}, c \in L^\infty(\Omega)$ satisfying $c \geqslant 0$ and

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant |\xi|^2 \quad \big(x \in \Omega,\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n\big), \tag{3.33}$$

and $f \in L^{\frac{2n}{n+2}}(\Omega)$. Actually, Riesz' representation theorem guarantees the unique existence of the weak solution $u = u(x) \in H_0^1(\Omega)$ to (3.31).

THEOREM 3.13. (See [132].) *If* $v = v(x) \in H_0^1(\Omega^*)$ *is the weak solution to*

$$-\Delta v = f^* \quad \text{in } \Omega^*, \qquad v = 0 \quad \text{on } \partial\Omega^*, \tag{3.34}$$

then it holds that $v \geqslant u^*$ *in* Ω^*, *where* Ω^* *is the open ball satisfying* $|\Omega^*| = |\Omega|$.

In the regular case, we can perform the following proof. First, if a_{ij}, c, f and $\partial\Omega$ are smooth, then so is $u = u(x)$. Replacing f by $|f|$, we reduce the theorem to the case of $u > 0$ in Ω. Then, $\Omega_t = \{x \in \Omega \mid u(x) > t\}$ is compactly imbedded in Ω if $t > 0$.

By Sard's lemma, each component of $\partial\Omega_t$ is a smooth $(n-1)$-dimensional compact manifold contained in $\{x \in \Omega \mid u(x) = t\}$ for a.e. $t \in (0, M)$, where $M = \|u\|_\infty$. For such t, the outer normal vector on $\partial\Omega_t$ is given by $-\nabla u/|\nabla u|$, and we obtain

$$\begin{aligned} -\int_{u>t} \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) dx &= \int_{u=t} \sum_{i,j=1}^n a_{ij}(x) u_{x_j} \frac{u_{x_i}}{|\nabla u|} dH^{n-1} \\ &\geqslant \int_{u=t} |\nabla u| \, dH^{n-1} \end{aligned}$$

and therefore,

$$\int_{u=t} |\nabla u| \, dH^{n-1} \leqslant \int_{u>t} f(x) \, dx$$

by (3.33), (3.31), and $c \geqslant 0$.

For the right-hand side, we use Hardy–Littlewood's inequality (3.30):

$$\int_{u>t} f(x) \, dx = \int_{\Omega} \chi_{u>t}(x) f(x) \, dx \leqslant \int_0^{\infty} \chi_{u>t}^{\star}(s) f^{\star}(s) \, ds = \int_0^{\mu(t)} f^{\star}(s) \, ds,$$

where $\mu = \mu(t)$ is the distribution function of $u = u(x)$ defined by (3.26). The co-area formula, on the other hand, is described by

$$-\mu'(t) = \int_{u=t} \frac{dH^{n-1}}{|\nabla u|}, \quad \text{a.e. } t \in (0, M),$$

and it follows also that

$$H^{n-1}\bigl(\{u = t\}\bigr)^2 \leqslant -\mu'(t) \int_{u>t} f(x) \, dx$$

from Schwarz' inequality. Finally, the isoperimetric inequality to $\{u > t\}$ guarantees

$$H^{n-1}\bigl(\{u = t\}\bigr) \geqslant n c_n^{1/n} \mu(t)^{1-1/n} \quad \text{a.e. } t \in (0, M).$$

These relations imply

$$1 \leqslant \frac{-\mu'(t)\mu(t)^{-2+2/n}}{n^2 c_n^{2/n}} \int_0^{\mu(t)} f^{\star}(s) \, ds = \Phi'(t) \quad \text{a.e. } t \in (0, M)$$

for

$$\Phi(t) = \frac{1}{n^2 c_n^{2/n}} \int_{\mu(t)}^{|\Omega|} r^{-2+2/n} \, dr \int_0^r f^{\star}(s) \, ds. \tag{3.35}$$

Since $\Phi = \Phi(t)$ is nondecreasing and $\mu(0) = |\Omega|$, inequality (3.35) implies

$$\begin{aligned} t &\leqslant \int_0^t \Phi'(s) \, ds \leqslant \Phi(t) - \Phi(0) \\ &= \frac{1}{n^2 c_n^{2/n}} \int_{\mu(t)}^{|\Omega|} r^{-2+2/n} \, dr \int_0^r f^{\star}(s) \, ds \quad (0 \leqslant t \leqslant M), \end{aligned}$$

and hence

$$u^{\star}(s) = \inf\bigl\{t \geqslant 0 \bigm| \mu(t) < s\bigr\} \leqslant \frac{1}{n^2 c_n^{2/n}} \int_s^{|\Omega|} r^{-2+2/n} \, dr \int_0^r f^{\star}(s') \, ds'.$$

Thus, we obtain

$$u^*\big(|x|\big) = u^\star\big(c_n|x|^n\big) \leqslant v\big(|x|\big),$$

where

$$v(|x|) = \frac{1}{n^2 c_n^{2/n}} \int_{c_n|x|^n}^{|\Omega|} r^{-2+2/n}\, dr \int_0^r f^\star(s)\, ds$$

is actually the solution to (3.34).

We can show also

$$\int_{\Omega^*} |\nabla v|^2\, dx \geqslant \int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j}\, dx$$

under the same assumption of the above theorem.

3.2.3. *Singular set.* Theorem 3.13 is thus proven by the isoperimetric inequality and the co-area formula. These inequalities are described by perimeters in the irregular case. First, if $E \subset \mathbf{R}^n$ is a measurable set, then its perimeter relative to Ω is defined by

$$P(E,\Omega) = \sup\left\{ \int_E \nabla \cdot \psi \;\middle|\; \psi \in C_0^\infty(\Omega)^n,\ \|\psi\|_\infty \leqslant 1 \right\},$$

where $\|\psi\|_\infty = \max\{\sum_{k=1}^n \psi_k^2\}^{1/2}$ for $\psi = (\psi_1, \ldots, \psi_n)$. The measurable set $E \subset \mathbf{R}^n$ is called a Caccioppoli set if $P(E) < +\infty$, where $P(E) = P(E, \mathbf{R}^n)$. Then, DeGiorgi's isoperimetric inequality and Fleming–Rishel's co-area formula [53] guarantee

$$P(E) \geqslant n c_n^{1/n} |E|^{1-1/n}$$

and

$$\int_{|u|>t} |Du| = \int_t^\infty P\big(\{x \mid |u(x)| > s\}, \Omega\big)\, ds,$$

respectively, where $u = u(x)$ is a function of bounded variation [59] and $E \subset \mathbf{R}^n$ is a Caccioppoli set. The irregular case of Theorem 3.13 is proven by these inequalities, and this argument is useful to derive a dimension estimate of the singular set.

In more precise, if $\Omega \subset \mathbf{R}^n$ is a bounded open set and $\Sigma \subset \Omega$ is compact, then the s-capacity of Σ $(1 \leqslant s < n)$ is defined by

$$\operatorname{Cap}_s(\Sigma) = \inf\left\{ \int_{\mathbf{R}^n} |\nabla f|^s\, dx \;\middle|\; f \geqslant 0,\ f \in C_0^\infty(\mathbf{R}^n),\ f \geqslant 1 \text{ on } \Sigma \right\}.$$

Given a harmonic function $u = u(x)$ in $\Omega \setminus \Sigma$, next, we say that Σ is removable if there is a harmonic function $\tilde{u} = \tilde{u}(x)$ in Ω such that $\tilde{u}|_{\Omega\setminus\Sigma} = u$. Then, Carleson' theorem says that Σ is removable for any harmonic function in $\Omega \setminus \Sigma$ which belongs to $L^\infty_{\text{loc}}(\Omega)$ if and only if $\text{Cap}_2(\Sigma) = 0$ [26]. In the other criterion of Serrin [110], Σ is removable if $u \in L^q(\Omega \setminus \Sigma)$ and $\text{Cap}_s(\Sigma) = 0$, where $2 < s \leqslant n$ and $q > \frac{s}{s-2}$.

There are several generalizations [111,65], but if $|u| = +\infty$ on Σ, on the contrary, then it follows that $\text{Cap}_2(\Sigma) = 0$ and $u \in L_w^{\frac{n}{n-2}}(\Omega)$. Here, $L^p_w(\Omega)$ denotes the weak L^p space on Ω $(1 < p < \infty)$ defined by

$$L^p_w(\Omega) = \left\{ v \in L^1_{\text{loc}}(\Omega) \;\middle|\; \|v\|_{p,w} < +\infty \right\},$$
$$\|v\|_{p,w} = \sup\left\{ |K|^{-1+1/p} \int_K |v|\, dx \;\middle|\; K \subset \Omega \text{ is a compact set} \right\}.$$

This result also has several generalizations including the parabolic case [105], and here we show the following version. We note that $u = u(x)$ discussed here is Hölder continuous in $\Omega \setminus \Sigma$ by Nash–Moser's theorem [57].

THEOREM 3.14. (See [107].) *Let $\Omega \subset \mathbf{R}^n$ $(n \geqslant 3)$ be a bounded open set, $\Sigma \subset \Omega$ be compact, and L be a differential operator defined by* (3.32) *with the coefficients $a_{ij}, c \in L^\infty_{\text{loc}}(\Omega \setminus \Sigma)$ satisfying $c = c(x) \leqslant 0$ and*

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant |\xi|^2 \quad \bigl(x \in \Omega \setminus \Sigma,\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n\bigr).$$

We assume the existence of $u = u(x) \in H^1_{\text{loc}}(\Omega \setminus \Sigma)$ and $s_0 \geqslant 0$ such that

$$Lu = 0 \quad \text{in } \Omega \setminus \Sigma,$$

$\Omega_{s_0} \Subset \Omega$ with the Lipschitz boundary $\Gamma_0 = \partial\Omega_{s_0}$, and Ω_s is open for any $s \geqslant s_0$, where

$$\Omega_s = \left\{ x \in \Omega \setminus \Sigma \;\middle|\; \bigl|u(x)\bigr| > s \right\} \cup \Sigma.$$

Then, it holds that $\text{Cap}_2(\Sigma) = 0$ *and* $u \in L_w^{\frac{n}{n-2}}(\Omega)$.

PROOF. Putting $\Omega_0 = \Omega_{s_0}$, we obtain $u \in H^1_{\text{loc}}(\overline{\Omega}_0 \setminus \Sigma)$ and

$$Lu = 0, \quad |u| > s_0 \quad \text{in } \Omega_0 \setminus \Sigma, \qquad |u| = s_0 \quad \text{on } \partial\Omega_0. \tag{3.36}$$

For $s > s_0$, it holds that $\Sigma \Subset \Omega_s$ and therefore,

$$\varphi_s = (\operatorname{sgn} u) \cdot \max\bigl\{ s - |u|, 0 \bigr\} \in H^1_{\text{loc}}(\overline{\Omega}_0 \setminus \Sigma)$$

satisfies

$$\varphi_s|_{\partial\Omega_0} = (\operatorname{sgn} u) \cdot (s - s_0), \qquad \varphi_s = 0 \quad (\Omega_s \setminus \Sigma)$$

and

$$\nabla \varphi_s = \begin{cases} -\nabla u & (\Omega_0 \setminus \overline{\Omega}_s), \\ 0 & (\Omega_s \setminus \Sigma). \end{cases}$$

Testing this to (3.36), we obtain

$$\int_{\Omega_0 \setminus \Omega_s} a_{ij} D_j u D_i u \, dx = (s - s_0) K + \int_{\Omega_0 \setminus \Omega_s} c|u|\bigl(s - |u|\bigr) \, dx, \tag{3.37}$$

where $K = -\langle \frac{\partial u}{\partial \nu_L}, \operatorname{sgn} u \rangle_{H^{-1/2}(\Gamma_0), H^{1/2}(\Gamma_0)}$ and $\frac{\partial}{\partial \nu_L} = \sum_{i,j} \nu_i a_{ij} D_j$. This implies

$$\int_{\Omega_0 \setminus \Omega_s} |\nabla u|^2 \, dx \leqslant (s - s_0) K = o\bigl(s^2\bigr)$$

by $c \leqslant 0$.

Here, we take $s_1 > s_0$ and prescribe $\chi = \chi(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \leqslant \chi \leqslant 1$, supported in Ω_0, and $\chi = 1$ on Ω_{s_1}. Then,

$$f_s = \frac{1}{s} \min\bigl\{|u|, s\bigr\} \chi \in H^1(\mathbf{R}^n) = W^{1,2}(\mathbf{R}^n)$$

satisfies $\Sigma \subset \{x \in \Omega_0 \mid f_s(x) = 1\}^\circ$ and

$$\nabla f_s = \begin{cases} \frac{1}{s} \nabla(|u|\chi) & (|u| \leqslant s), \\ \nabla \chi & (|u| > s), \end{cases}$$

and therefore, it holds that

$$\operatorname{Cap}_2(\Sigma) \leqslant \int_{\mathbf{R}^n} |\nabla f_s|^2 \, dx = \frac{1}{s^2} \int_{\Omega_0 \setminus \Omega_s} \bigl|\nabla\bigl(|u|\chi\bigr)\bigr|^2 \, dx = o(1)$$

as $s \to +\infty$.

Now, we show $u \in L_w^{\frac{n}{n-2}}(\Omega)$. In fact, the mapping $s \mapsto \int_{\Omega_0 \setminus \Omega_s} c|u|(s - |u|) \, dx$ is decreasing and it holds that

$$\frac{d}{ds} \int_{\Omega_0 \setminus \Omega_s} a_{ij} D_j u D_i u \, dx = -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} a_{ij} D_i u D_i u \, dx \leqslant K \tag{3.38}$$

by (3.37) for a.e. $s \in (s_0, s')$, where $s' > s_0$. Now, we shall show

$$-\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u| \, dx \leqslant \bigl(-\mu'(s)\bigr)^{1/2} \left\{ -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} a_{ij} D_j u D_i u \, dx \right\}^{1/2} \tag{3.39}$$

a.e. $s \in (s_0, s')$, where $\mu(s) = |\Omega_s|$. In fact, $s \in (s_0, s') \mapsto \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u|\, dx$ is nonincreasing, and it holds that

$$\begin{aligned}
&\frac{1}{h}\left[\int_{\Omega_s \setminus \overline{\Omega}_{s'}} - \int_{\Omega_{s+h} \setminus \overline{\Omega}_{s'}}\right] |\nabla u|\, dx \\
&\quad = \frac{1}{h} \int_{\Omega_s \setminus \Omega_{s+h}} |\nabla u|\, dx \\
&\quad \leqslant \left\{ \frac{\mu(s) - \mu(s+h)}{h} \right\}^{1/2} \left\{ \frac{1}{h} \int_{\Omega_s \setminus \Omega_{s+h}} |\nabla u|^2\, dx \right\}^{1/2} \\
&\quad \leqslant \left\{ \frac{\mu(s) - \mu(s+h)}{h} \right\}^{1/2} \left\{ \frac{1}{h} \int_{\Omega_s \setminus \Omega_{s+h}} a_{ij} D_j u D_i u\, dx \right\}^{1/2} \\
&\quad = \left\{ -\mu'(s) \right\}^{1/2} \left\{ -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} a_{ij} D_j u D_i u\, dx \right\}^{1/2} + o(1)
\end{aligned}$$

as $h \downarrow 0$. Then, (3.39) follows.

It holds that

$$n c_n^{1/n} \mu(s)^{1-1/n} \leqslant P(\Omega_s) = -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u|\, dx \tag{3.40}$$

for a.e. $s \in (s_0, s')$ by DeGiorgi's isoperimetric inequality and Fleming–Rishel's co-area formula. Inequalities (3.38)–(3.40) now guarantee

$$n^2 c_n^{2/n} \leqslant -K \mu(s)^{-2(1-1/n)} \mu'(s),$$

or equivalently,

$$c \equiv n^2 c_n^{2/n} K^{-1} \leqslant \frac{d}{ds} \phi\bigl(\mu(s)\bigr)$$

a.e. $s \in (s_0, s')$, where $\phi(\mu) = \frac{n}{n-2} \mu^{-\frac{n-2}{n}}$. This implies

$$\mu(s) \leqslant \left\{ \frac{n}{(n-2)(c(s-s_0) + \phi(\mu(s_0)))} \right\}^{n/(n-2)}$$

for $s > s_0$, and therefore, $\mu(s) s^{n/(n-2)} = O(1)$ as $s \to +\infty$. Then $u \in L_w^{\frac{n}{n-2}}(\Omega)$ follows [137,149]. □

If $c = 0$, then $u \notin H^1_{\mathrm{loc}}(\Omega)$ by (3.37). The crucial assumption of the above theorem which induces this is $\Sigma_{s_0} \Subset \Omega$. We note also that $H^1(\Omega \setminus \Sigma) = H^1(\Omega)$ holds by $\mathrm{Cap}_2(\Sigma) = 0$ ([65]). See [130] for the parabolic case.

3.2.4. *Bandle's rearrangement.* As is described in Section 2.2, Bol's inequality holds on surfaces with bounded Gaussian curvature, and then Bandle's spherically decreasing rearrangement is defined. In more precise, we obtain

$$\ell(\partial\omega)^2 \geqslant \frac{1}{2} m(\omega)\big(8\pi - m(\omega)\big)$$

for

$$\ell(\partial\omega) = \int_{\partial\omega} p^{1/2}\, ds, \qquad m(\omega) = \int_\omega p\, dx,$$

if $p = p(x) > 0$ is a C^2 function defined on the domain $\Omega \subset \mathbf{R}^2$ of which boundary is composed of a finite number of Jordan curves,

$$-\Delta \log p \leqslant p \quad \text{in } \Omega, \tag{3.41}$$

and $\omega \Subset \Omega$ is a sub-domain with the boundary $\partial\omega$ locally homeomorphic to a line. Both geometric and analytic proofs are known for this fact. Here, we emphasize that the simply-connectedness of Ω is not necessary ([8], p. 38, [Topping]).

Let Ω and $p = p(x)$ be such a domain and a function satisfying $p \in C(\overline{\Omega})$ and

$$\lambda = \int_\Omega p(x)\, dx < 8\pi.$$

Putting $\Omega^* = B(0,1) \subset \mathbf{R}^2$, then we obtain a unique $v^* = v^*(x)$ satisfying

$$-\Delta v^* = \frac{\lambda e^{v^*}}{\int_{\Omega^*} e^{v^*}} \quad \text{in } \Omega^*, \qquad v^* = 0 \quad \text{on } \partial\Omega^*.$$

In this case, it holds that

$$-\Delta \log p^* = p^* \quad \text{in } \Omega^* \text{ for } p^* = \frac{\lambda e^{v^*}}{\int_{\Omega^*} e^{v^*}},$$

and also

$$\ell(\partial\omega^*) = \frac{1}{2} m(\omega^*)\big(8\pi - m(\omega^*)\big)$$

if $\omega^* \subset \Omega^*$ is a concentric disc.

Now, given a measurable function $\varphi : \Omega \to \mathbf{R}$ and $t > 0$, we define the concentric disc Ω_t^* of Ω^* by

$$\int_{\Omega_t^*} p^*\, dx = \int_{\Omega_t} p\, dx \equiv a(t) \quad \text{and} \quad \Omega_t = \big\{x \in \Omega \;\big|\; \big|\varphi(x)\big| > t\big\}, \tag{3.42}$$

and put

$$v^*(x) = \sup\{t \mid x \in \Omega_t^*\}.$$

This operation $v \mapsto v^*$ is an equi-measurable rearrangement, and it holds that

$$\int_\Omega \varphi^2 p\,dx = \int_0^\infty t^2\,d\big(-a(t)\big) = \int_{\Omega^*} \varphi^{*2} p^*\,dx,$$

while $\varphi^* = \varphi^*(x)$ is a nonnegative, radially symmetric, and nonincreasing function of $r = |x|$. Thus, Theorem 2.2 is reduced to the decrease of the Dirichlet integral (2.27). Since

$$\nu_1(p,\Omega) = \inf\left\{\int_\Omega |\nabla v|^2\,dx \;\middle|\; v \in H_0^1(\Omega),\ \int_\Omega v^2 p\,dx = 1\right\}$$

is attained by $\varphi \in H_0^1(\Omega)$ smooth and positive in Ω, the following fact is sufficient for this purpose.

THEOREM 3.15. *Let $\Omega \subset \mathbf{R}^2$ be a bounded open set contained in a bounded domain $\hat{\Omega} \subset \mathbf{R}^2$ of which boundary is composed of finitely many Jordan curves and $p = p(x)$ is a positive C^2 function in $\hat{\Omega}$, continuous on $\overline{\hat{\Omega}}$, and satisfies*

$$-\Delta \log p \leqslant p \quad \text{in } \hat{\Omega}. \tag{3.43}$$

Then it holds that

$$\int_\Omega |\nabla\varphi|^2\,dx \geqslant \int_{\Omega^*} |\nabla\varphi^*|^2\,dx, \tag{3.44}$$

provided that $\varphi \in H_0^1(\Omega)$ is a nonnegative C^2 function in Ω. If the equality holds in (3.44), *then Ω is a disc, $\varphi = \varphi(x)$ and $p = p(x)$ are radially symmetric functions, and $-\Delta \log p = p$ in Ω.*

PROOF. The function $a = a(t)$ defined by (3.42) is right-continuous and increasing in $t > 0$. Since $\varphi \in H_0^1(\Omega)$ is C^2 and nonnegative in Ω, it holds that

$$-a'(t) = \int_{\varphi=t} \frac{p\,ds}{|\nabla\varphi|} \quad \text{and} \quad -\frac{d}{dt}\int_{\Omega_t} |\nabla\varphi|^2\,dx = \int_{\varphi=t} |\nabla\varphi|\,ds$$

for a.e. $t > 0$ by co-area formula and Sard's lemma. We obtain also

$$\begin{aligned} -\frac{d}{dt}\int_{\Omega_t} |\nabla\varphi|^2\,dx &\geqslant \left\{\int_{\varphi=t} p^{1/2}\,ds\right\}^2 \Big/ \int_{\varphi=t} \frac{p\,ds}{|\nabla\varphi|} \\ &= \ell\big(\{\varphi = t\}\big)^2 / -a'(t) \\ &\geqslant \frac{1}{2}\big(8\pi - a(t)\big)a(t) / -a'(t) \end{aligned} \tag{3.45}$$

for a.e. $t \in (0, M)$ by Schwarz' and Bol's inequalities, where $M = \|\varphi\|_\infty$. The nondecreasing function

$$j(t) = -\int_{\Omega_t} |\nabla\varphi|^2\, dx$$

is continuous by

$$j(t) - j(t-0) = \int_{\varphi=t} |\nabla\varphi|^2\, dx = 0.$$

Thus, it is absolutely continuous, and it holds that

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2\, dx &= \int_0^\infty \left(-\frac{d}{dt}\int_{\Omega_t} |\nabla\varphi|^2\, dx\right) dt \\ &\geqslant \frac{1}{2}\int_0^\infty \frac{(8\pi - a(t))a(t)}{-a'(t)}\, dt. \end{aligned} \tag{3.46}$$

Since $\varphi^* = \varphi^*(x)$ is a nonincreasing function of $r = |x|$, we obtain the equality at each step of (3.45) for $\Omega = \Omega^*$, $p = p^*$, and $\varphi = \varphi^*$. This implies

$$\begin{aligned}\int_{\Omega^*} |\nabla\varphi^*|^2\, dx &= \int_0^\infty \left(-\frac{d}{dt}\int_{\Omega^*} |\nabla\varphi^*|^2\, dx\right) dt \\ &= \frac{1}{2}\int_0^\infty \frac{(8\pi - a(t))a(t)}{-a'(t)}\, dt, \end{aligned}$$

and hence (3.44). Here, the final assertion concerning the equality of (3.44) follows also. □

3.3. *Spectral analysis*

As is described in Section 2.2, Bandle's symmetrization is reformulated as the Schwarz symmetrization on sphere. In more precise, if $S^2 \subset \mathbf{R}^3$ denotes a round sphere with total area 8π and the area element dv, and $\omega \subset S^2$ is a geodesic disc, we define $\varphi^* = \varphi^*(x)$ by (2.41), where ω_t denotes the concentric disc of ω satisfying (2.42). If $\varphi \in H_0^1(\Omega)$ is a nonnegative C^2 function in Ω, then it holds that (2.43).

The reference problem (2.34), on the other hand, is equivalent to

$$-\Delta_{S^2}\overline{\varphi} = \nu\overline{\varphi} \quad \text{in } \omega, \qquad \overline{\varphi} = 0 \quad \text{on } \partial\omega \tag{3.47}$$

by the stereographic projection $\hat{\tau} : S^2 \to \mathbf{C} \cup \{\infty\}$, where the north pole of S^2 corresponds to ∞ and the south pole $(0, 0, 0) \in S^2$ coincides with the center of ω. In this case, $\lambda = \int_\Omega p\, dx = \int_{\Omega^*} p^*\, dx$ indicates the area of $\omega \subset S^2$: $\lambda = \int_\omega dv$.

Now, we apply the method of separation of variables to (3.47) using three dimensional polar coordinate because $\omega \subset \mathbf{S}^2$ is a geodesic disc, which results in the associated Legendre equation,

$$\begin{aligned}&\big[\big(1-\xi^2\big)\Phi_\xi\big]_\xi + \big[2\nu - m^2/\big(1-\xi^2\big)\big]\Phi = 0 \quad (\xi_\lambda < \xi < 1),\\ &\Phi(1) = 1, \qquad \Phi(\xi_\lambda) = 0,\end{aligned} \tag{3.48}$$

where $m = 0, 1, \ldots$ describes the argument mode of $\overline{\varphi}$ and $\Phi(\mu_\lambda) = 0$ is derived from the boundary condition $\overline{\varphi}|_{\partial\omega} = 0$. More precisely, $\xi_\lambda \in (-1, 1)$ is a decreasing function of $\lambda \in (0, 8\pi)$, and $\lambda = 0, 4\pi, 8\pi$ correspond to $\xi_\lambda = 1, 0, -1$, respectively. Thus, $\nu_1(p^*, \Omega^*) > 1$ means $\Phi(\xi) > 0$ for $\xi \in (\xi_\lambda, 1)$, where $\Phi = \Phi(\xi)$ denotes the solution to (3.48) for $\nu = 1$ and $m = 0$.

Since $\Phi(\xi) = \xi$ in this case, it is equivalent to $\lambda \in (0, 4\pi)$. Thus, we obtain (2.38) if $\Omega \subset \mathbf{R}^2$ is a bounded open set contained in a bounded domain $\hat{\Omega} \subset \mathbf{R}^2$ of which boundary is composed of finitely many Jordan curves, $p = p(x)$ is a positive C^2 function in $\hat{\Omega}$, continuous on $\overline{\Omega}$, satisfies (3.43), and $\int_\Omega p\,dx = \lambda$.

3.3.1. *Mean field equation.* If $\partial\Omega$ is sufficiently smooth, say C^2, then the L^p estimate is valid to $-\Delta$ with the Dirichlet boundary condition. Then, the solution $v = v(x) \in H_0^1(\Omega)$ to the mean field equation (2.33),

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{3.49}$$

is continuous on $\overline{\Omega}$. In this case, its linearized operator is defined by

$$\mathcal{L}\psi = -\Delta\psi - \lambda\left(\frac{e^v\psi}{\int_\Omega e^v} - \frac{\int_\Omega e^v\psi}{(\int_\Omega e^v)^2}\cdot e^v\right).$$

More precisely, this $\mathcal{L}$ is realized as a self-adjoint operator in $L^2(\Omega)$ with the domain $\mathcal{L} = (H^2 \cap H_0^1)(\Omega)$.

If this $\mathcal{L}$ is degenerate, then

$$-\Delta\varphi = \nu p\varphi \quad \text{in } \Omega, \qquad \varphi = \text{constant on } \partial\Omega, \qquad \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} = 0 \tag{3.50}$$

has the eigenvalue $\nu = 1$ for

$$p = \frac{\lambda e^v}{\int_\Omega e^v}.$$

In fact, if $\psi \in (H^2 \cap H_0^1)(\Omega) \setminus \{0\}$ satisfies $\mathcal{L}\psi = 0$, then it holds that

$$-\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} = -\int_\Omega \Delta\psi = 0$$

and therefore,

$$\varphi = \psi - \frac{\int_\Omega e^v \psi}{\int_\Omega e^v}$$

satisfies (3.50) for $\nu = 1$. If $\varphi = 0$, then $\psi =$ constant, which contradicts to $\psi \in H_0^1(\Omega) \setminus \{0\}$. Thus, (3.50) has the eigenvalue $\nu = 1$. The converse is also true [124,146].

The eigenvalue problem (3.50) is formulated by (3.3), using

$$\begin{aligned}
&X = L^2(\Omega),\\
&V = H_c^1(\Omega) \equiv \{ v \in H^1(\Omega) \mid v = \text{constant on } \partial\Omega \},\\
&a(\varphi, \psi) = (\nabla\varphi, \nabla\psi),\\
&b(\varphi, \psi) = \int_\Omega \varphi(x)\psi(x)p(x)\,dx.
\end{aligned}$$

This $p = p(x) > 0$ is continuous on $\overline{\Omega}$, C^2 in Ω, and satisfies (3.41) because $v = v(x)$ is a solution to (3.49). We obtain $\int_\Omega p\varphi\,dx = 0$ in the above transformation $\psi \in H_0^1(\Omega) \mapsto \varphi \in H_c^1(\Omega)$, and actually, the first eigenvalue of (3.50) is $\nu_1 = 0$ associated with the constant eigenfunction $\varphi_1 = \lambda^{-1/2}$ under the normalization $b(\varphi_i, \varphi_j) = \delta_{ij}$. Thus, the second eigenvalue is described by

$$\nu_2(p, \Omega) = \inf\left\{ \int_\Omega |\nabla v|^2\,dx \;\middle|\; v \in H_c^1(\Omega),\ \int_\Omega vp\,dx = 0,\ \int_\Omega v^2 p\,dx = 1 \right\}. \tag{3.51}$$

This value is attained by the second eigenfunction of (3.50), denoted by $\varphi = \varphi_2$. Then, $N(\varphi) \cap K$ is composed of a finite number of smooth curves for any compact set $K \subset \Omega$, where $N(\varphi) = \overline{\{x \in \Omega \mid \varphi(x) = 0\}}$. If $N(\varphi) \cap \partial\Omega \neq \emptyset$, then $\varphi|_{\partial\Omega} = 0$, and therefore, $N(\varphi)$ itself is composed of a finite number of smooth curves, provided that $\partial\Omega$ is smooth. This fact holds obviously in the other case of $N(\varphi) \cap \partial\Omega = \emptyset$, and we can show the following theorem by the method of [124].

THEOREM 3.16. *If $\Omega \subset \mathbf{R}^2$ is a bounded domain with C^2 boundary $\partial\Omega$ and $p = p(x) \in C(\overline{\Omega})$ is C^2 in Ω and satisfies* (3.41), *then it holds that*

$$0 < \lambda = \int_\Omega p\,dx \leqslant 8\pi \quad \Rightarrow \quad \nu_2(p, \Omega) > 1. \tag{3.52}$$

PROOF. Putting $\Omega_\pm = \{x \in \Omega \mid \pm\varphi(x) > 0\}$, we obtain $\overline{\Omega} = \overline{\Omega}_+ \cup \overline{\Omega}_-$ and $\overline{\Omega}_+ \cap \overline{\Omega}_- = N(\varphi) \cup \partial\Omega$. Obviously, $p = p(x)$ satisfies (3.41) for $\Omega = \Omega_\pm$, and it holds also that $\Omega_\pm \neq \emptyset$ by $\int_\Omega \varphi p\,dx = 0$. We obtain, furthermore, $\lambda = \lambda_+ + \lambda_- \leqslant 8\pi$ for

$$\lambda_\pm = \int_{\Omega_\pm} p\,dx.$$

If $c \equiv \varphi|_{\partial\Omega} = 0$, then $\varphi \in H_0^1(\Omega)$, and therefore

$$\nu_2(p,\Omega) = \nu_1(p,\Omega_\pm)$$

follows, where $\nu_1(p,\Omega)$ is the first eigenvalue of (2.25) formulated by (2.23). Since $\lambda_\pm < 4\pi$ and $\lambda_\pm \leqslant 4\pi$ imply $\nu_1(p,\Omega_\pm) > 1$ and $\nu_1(p,\Omega_\pm) \geqslant 1$, respectively, $\nu_2(p,\Omega) = 1$ can occur only in the case of $\lambda_\pm = 4\pi$ and $\nu_1(p,\Omega_\pm) = 1$. This implies, however, that both $\Omega_\pm$ are discs, a contradiction.

In the other case of $c \equiv \varphi|_{\partial\Omega} \neq 0$, it follows that $N(\varphi) \cap \partial\Omega = \emptyset$. Without loss of generality, we assume $c > 0$, and then it holds that $\partial\Omega \subset \partial\Omega_+$. Here, we note that any nodal domain ω of $\varphi = \varphi(x)$ takes the boundary $\partial\omega$ composed of some portions of $N(\varphi) \cup \partial\Omega$, and zero-extension of $\varphi|_\omega$ belongs to $H_c^1(\Omega)$. We obtain, on the other hand,

$$\int_{\partial\Omega_-^i} \frac{\partial\varphi}{\partial\nu}\varphi\, ds = 0,$$

where Ω_-^i $(i \in I)$ are nodal domains of $\varphi|_{\Omega_-}$ and these structures assure that $\varphi|_{\Omega_-}$ has exactly one nodal domain. In other words, Ω_- is a domain satisfying $\Omega_- \Subset \Omega$, and therefore, it holds that $\nu_2(p,\Omega) = \nu_1(p,\Omega_-)$. Thus, we have only to consider the case of $\lambda_- \geqslant 4\pi$, and then, it holds that $\lambda_+ \leqslant 4\pi$.

The set $\Omega_1 = \{x \in \Omega_+ \mid \varphi(x) > c\}$ may be empty, but otherwise we take Bandle's rearrangement to $\psi_1 = \varphi|_{\Omega_1}$. More precisely, we take a geodesic disc $B \subset S^2$ satisfying

$$\int_B dv = \int_{\Omega_1} p\, dx,$$

and define $\psi_1^* = \psi_1^*(x)$ on B by

$$\psi_1^*(x) = \sup\{t \mid x \in \omega_t\},$$

where ω_t is the open concentric disc of B such that

$$\int_{\omega_t} dv = \int_{\psi_1 > t} p\, dx.$$

From the co-area formula and Bol's inequality, it follows that

$$\int_{\Omega_1} \psi_1^2 p\, dx = \int_B \psi_1^{*2}\, dv \quad \text{and} \quad \int_{\Omega_1} |\nabla\psi_1|^2\, dx \geqslant \int_B \left|\nabla\psi_1^*\right|^2 dv.$$

To treat the main part $\Omega_2 = \Omega_+ \setminus \omega$, we take open concentric discs $B_0 \subset B_1 \subset S^2$ satisfying

$$\int_{B_0} dv = \int_{\Omega_-} p\, dx = \lambda_- \quad \text{and} \quad \int_{B_1} dv = \int_{\Omega_+ \setminus \omega} p\, dx$$

and define $\psi_{2*} = \psi_{2*}(x)$ on $A \equiv B_1 \setminus \overline{B}_0$ by

$$\psi_2^*(x) = \inf\{t \mid x \in A_t\},$$

where A_t is the closed concentric annulus of A such that $A_t \cup B_0 \subset S^2$ is a closed disc and

$$\int_{A_t \cup B_0} dv = \int_{\psi_2 \leqslant t} p\, dx$$

for $t \in (0, c)$. For this equi-measurable rearrangement it holds that

$$\int_{\Omega_2} \psi_2^2 p\, dx = \int_A \psi_{2*}^2\, dv \quad \text{and} \quad \int_{\Omega_2} |\nabla \psi_2|^2\, dx \geqslant \int_A |\nabla \psi_{2*}|^2\, dv$$

similarly. In fact, the methods of co-area formula and isoperimetric inequality are valid by $0 \leqslant \psi_2 \leqslant c$ in Ω_2, $\psi_2|_{\partial\Omega_-} = 0$, and $\psi_2|_{\partial\Omega_1} = c$.

These procedures guarantee

$$\begin{aligned} \nu_2(p, \Omega) &\geqslant \nu_* \\ &= \inf\left\{ \int_{A\cup B} |\nabla\psi|^2 \;\middle|\; \psi \in H^1(B \cup A),\ \psi|_{\partial B_0} = 0, \right. \\ &\qquad \left. \psi|_{\partial B_1} = \psi|_{\partial B} = \text{constant},\ \int_{A\cup B} \psi^2\, dv = 1 \right\}. \end{aligned}$$

Since $\lambda_+ \leqslant 4\pi \leqslant \lambda_-$ and $\lambda_+ + \lambda_- \leqslant 8\pi$, we can put B_0 so that its center is the north pole n, it occupies the closed hemisphere S_+ with the center n, and A, B are concentric with the center south pole and $A \cup B \subset S_- = S^2 \setminus S_+$. Then, $\nu = \nu_*$ is the first eigenvalue of

$$\begin{aligned} &-\Delta_{S^2}\psi = \nu\psi \quad \text{in } A \cup B, && \psi|_{\partial B_0} = 0, \\ &\psi|_{\partial B_1} = \psi|_{\partial B} = \text{constant}, && \int_{\partial B_1} \frac{\partial\psi}{\partial\nu}\, ds + \int_{\partial B} \frac{\partial\psi}{\partial\nu}\, ds = 0 \end{aligned}$$

if $B \neq \emptyset$, and

$$\begin{aligned} &-\Delta_{S^2}\psi = \nu\psi \quad \text{in } A, && \psi|_{\partial B_0} = 0, \\ &\psi|_{\partial B_1} = \text{constant}, && \int_{\partial B_1} \frac{\partial\psi}{\partial\nu}\, ds = 0 \end{aligned}$$

if $B = \emptyset$. In the polar coordinate, this induces the associated Legendre equation (3.48), i.e.,

$$\begin{aligned} &\left[(1-\xi^2)\Phi_\xi\right]_\xi + 2\nu\Phi = 0, \quad \Phi > 0 \quad (b_0 < \xi < b_1,\ b < \xi < 1) \\ &\Phi(b_0) = 0, \qquad \Phi(b_1) = \Phi(b), \qquad \Phi'(b_1) = \Phi'(b), \qquad \Phi(1) = 1 \end{aligned}$$

or

$$\left[(1-\xi^2)\Phi_\xi\right]_\xi + 2\nu\Phi = 0, \quad \Phi > 0 \quad (b_0 < \xi < b_1)$$
$$\Phi(b_0) = 0, \qquad \Phi'(b_1) = 0,$$

where

$$0 \leqslant b_0 < b_1 \leqslant b \leqslant 1.$$

Thus, $\nu = \nu_* > 1$ is equivalent to

$$\hat{\Phi}(\xi) > 0 \quad (b_0 < \xi < b_1,\ b < \xi < 1)$$

in the case of $B \neq \emptyset$, where $\hat{\Phi} = \hat{\Phi}(\xi)$ is the solution to

$$\left[(1-\xi^2)\hat{\Phi}_\xi\right]_\xi + 2\hat{\Phi} = 0 \quad (-1 < \xi < 1) \tag{3.53}$$

satisfying $\hat{\Phi}(1) = 1$, $\hat{\Phi}(b) = \hat{\Phi}(b_1)$, and $\hat{\Phi}'(b) = \hat{\Phi}'(b_1)$, and

$$\hat{\Phi}(\xi) > 0 \quad (b_0 < \xi < b_1)$$

in the case of $B = \emptyset$, where $\hat{\Phi} = \hat{\Phi}(\xi)$ is the solution to (3.53) satisfying $\hat{\Phi}(b_1) = 1$ and $\hat{\Phi}'(b_1) = 0$. These elementary inequalities are assured using a fundamental system of solutions to (3.53), say,

$$P_1(\xi) = \xi \quad \text{and} \quad Q_1(\xi) = -1 + \frac{\xi}{2}\log\frac{1+\xi}{1-\xi}. \qquad \square$$

3.3.2. *Global analysis.* Theorem 2.7 guarantees that each $0 < \varepsilon \ll 1$ takes $C_\varepsilon > 0$ such that any solution $v = v(x)$ to (3.49) with $0 < \lambda < 8\pi - \varepsilon$ satisfies $\|v\|_\infty \leqslant C_\varepsilon$. Using this and Theorem 3.16, we can show that the solution to (3.49) with $0 < \lambda < 8\pi$ is unique, and if it is denoted by $\underline{v}_\lambda = \underline{v}_\lambda(x)$, then $\underline{\mathcal{C}} = \{(\lambda, \underline{v}_\lambda) \mid 0 < \lambda < 8\pi\}$ forms a one-dimensional manifold in $\mathbf{R}_+ \times C(\overline{\Omega})$ [124]. This $\underline{v}_\lambda \in H_0^1(\Omega)$ attains $\inf_{H_0^1(\Omega)} \mathcal{J}_\lambda$, because it is actually attained for $0 < \lambda < 8\pi$, where $\mathcal{J}_\lambda = \mathcal{J}_\lambda(v)$ is the Trudinger–Moser functional defined by (2.45). The solution to $\lambda = 8\pi$ is also unique if it exists, and therefore, $\inf_{H_0^1(\Omega)} \mathcal{J}_{8\pi}$ is attained if and only if $\underline{\mathcal{C}}$ does not blowup as $\lambda \uparrow 8\pi$.

Blowup profile of the solution sequence, on the other hand, is obtained in detail using (2.57) and its refinement [33]. Thus, if $\{(\lambda_k, v_k)\}_k$ is a solution sequence to (3.49) satisfying $\lambda_k \to 8\pi$ and $\|v_k\|_\infty \to +\infty$, then, passing to a subsequence it has a unique blowup point denoted by x_0 (Theorem 2.7). It is a critical point of the Robin function $R = R(x)$, and if Ω is simply-connected, then we take a conformal mapping $g : B(0,1) \to \Omega$ satisfying

$g(0) = x_0$. In this case it holds that $g''(0) = 0$, and according to [129,126] we put

$$D(x_0) = \sum_{k=3}^{\infty} \frac{k^2}{k-2} |a_k|^2 - |a_1|^2, \tag{3.54}$$

where $g(z) = \sum_{k=0}^{\infty} a_k z^k$. Then, it holds that

$$\lambda_k = 8\pi + \pi\big(D(x_0) + o(1)\big)\sigma_k \tag{3.55}$$

for $\sigma_k = \lambda_k / \int_\Omega e^{v_k}$ [33,28].

From (3.55), if there is a critical point x_0 of R such that $D(x_0) < 0$, then $\lim_{\lambda\uparrow 8\pi} \|\underline{v}_\lambda\|_\infty = +\infty$. Since $\underline{v}_\lambda$ is a minimizer of $\mathcal{J}_\lambda$ for $0 < \lambda < 8\pi$, this x_0 must be a maximizer of R. If $D(x_0) > 0$ holds for any critical point x_0 of R, conversely, then $\underline{\mathcal{C}}$ does not blowup as $\lambda \uparrow 8\pi$, and therefore, $\mathcal{J}_{8\pi}$ is attained by a unique solution to (3.49) for $\lambda = 8\pi$.

We obtain $\lim_{\lambda\uparrow 8\pi} \|\underline{v}_\lambda\|_\infty = +\infty$ even if there is a critical point x_0 of R such that $D(x_0) = 0$ [28]. In fact, in this case we can take a family of conformal mappings $\{g_k\}$ satisfying $g_k \to g$ uniformly on $\overline{B(0,1)}$, $g_k''(0) = 0$, and $D_k(x_0) < 0$, where $D_k(x_0)$ is the value defined by (3.54) for $g = g_k$. Putting $\Omega_k = g_k(B(0,1))$, we apply the above result to this domain. Thus, the minimizer $\underline{v}_\lambda^k$ of the Trudinger–Moser functional $\mathcal{J}_\lambda$ defined on $H_0^1(\Omega_k)$ blows-up as $\lambda \uparrow 8\pi$, and therefore, any $c \gg 1$ admits $\lambda_k(c) \in (0, 8\pi)$ such that

$$\big\|\underline{v}^k_{\lambda_k(c)}\big\|_\infty = c.$$

Making $k \to \infty$ and passing through a subsequence, we obtain $\lambda_k(c) \to \lambda(c) \in (0, 8\pi]$ and a solution $v_c = v_c(x)$ to (3.49) in Ω for $\lambda = \lambda(c)$ satisfying $\|v_c\|_\infty = c$. Since the solution to (3.49) is unique for $\lambda = 8\pi$, the case $\lambda(c) = 8\pi$ is admitted at most once as $c \uparrow +\infty$, and therefore, it holds that $\lambda(c) < 8\pi$ for $c \gg 1$. This means that $v_c = \underline{v}_{\lambda(c)}$ and hence $\underline{\mathcal{C}}$ blows-up as $\lambda \uparrow 8\pi$. Furthermore, x_0 is a maximizer of R in this case also because it is a maximizer of the Robin function R_k on Ω_k for $k \gg 1$. Using the uniqueness of the solution to $0 < \lambda \leqslant 8\pi$, thus we obtain the following.

THEOREM 3.17. (See [28].) *If Ω is simply-connected, then $I_{8\pi} = \inf_{H_0^1(\Omega)} \mathcal{J}_{8\pi}$ is attained if and only if there is a maximizer x_0 of R satisfying $D(x_0) > 0$. If this is the case, then $D(x_0) > 0$ for any critical point of x_0 of R. If there is a critical point x_0 of R such that $D(x_0) \leqslant 0$, on the contrary, then it is a unique maximizer of R and $\underline{\mathcal{C}}$ blows-up as $\lambda \uparrow 8\pi$.*

PROOF. Let $\mathcal{S} \subset \Omega$ be the set of critical points of R and $\mathcal{S}_0 \subset \Omega$ be that of maximizers of R. It holds that $\mathcal{S}_0 \neq \emptyset$. From the above consideration, if there is $x_0 \in \mathcal{S}$ such that $D(x_0) \leqslant 0$, then $x_0 \in \mathcal{S}_0$ and it follows that $\lim_{\lambda\uparrow 8\pi} \|\underline{v}_\lambda\|_\infty = +\infty$. Thus, if $D(x_0) > 0$ for some $x_0 \in \mathcal{S}_0$, then $D(x_0) > 0$ for any $x_0 \in \mathcal{S}$, which guarantees

$$\limsup_{\lambda\uparrow 8\pi} \|\underline{v}_\lambda\|_\infty < +\infty \tag{3.56}$$

as is described. Here, (3.56) means that $I_{8\pi}$ is attained, and it is equivalent to $D(x_0) > 0$ for some $x_0 \in \mathcal{S}_0$. Finally, each $x_0 \in \mathcal{S}_0$ with $D(x_0) \leqslant 0$ corresponds to the singular limit $v_0(x) = 8\pi G(\cdot, x_0)$ of $\underline{\mathcal{C}}$ as $\lambda \uparrow 8\pi$, and therefore, $D(x_0), D(x_0') \leqslant 0$ with different $x_0, x_0' \in \mathcal{S}_0$ is impossible. □

An immediate consequence is that if $\sharp\mathcal{S}_0 \geqslant 2$, then $I_{8\pi}$ is attained. It is also shown that $I_{8\pi}$ is attained if and only if the criterion of [24] holds, i.e.,

$$I_{8\pi}(\Omega) > 1 + 4\pi \sup_{\Omega} R + \log \frac{\pi}{|\Omega|}.$$

See [28].

3.3.3. *Neumann problem.* The Neumann problem

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \quad \text{in } \Omega, \qquad \int_\Omega v = 0, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{3.57}$$

arises as a stationary state of the simplified system of chemotaxis obtained by Jäger and Luckhaus [69], and is regarded as a special case of the mean field equation

$$-\Delta v = \lambda\left(\frac{e^v}{\int_M e^v} - \frac{1}{|M|}\right) \quad \text{in } M, \qquad \int_M v = 0, \tag{3.58}$$

where M is a compact Riemannian surface [127]. Blowup of the solution sequence occurs only to $\lambda \in 8\pi\mathbf{N}$ in (3.58), while these quantized values are reduced to $4\pi\mathbf{N}$ in (3.57) [108, 91,92]. See [76,77,31,39,29] for the uniqueness of the solution to (3.58).

If $v = v(x)$ is a solution to (3.57) and its linearized operator is degenerate, then

$$-\Delta\psi = \lambda\left(\frac{e^v\psi}{\int_\Omega e^v} - \frac{\int_\Omega e^v\psi}{(\int_\Omega e^v)^2} \cdot e^v\right) \quad \text{in } \Omega, \qquad \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$

has a nontrivial solution. Using

$$p = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{and} \quad \varphi = \psi - \frac{\int_\Omega e^v\psi}{\int_\Omega e^v}, \tag{3.59}$$

this implies that

$$-\Delta\varphi = \nu p\varphi \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{3.60}$$

has the eigenvalue $\nu = 1$ with the nonconstant eigenfunction. Actually, the first eigenvalue of (3.60) is $\nu = 0$ with the constant eigenfunction, and the other eigenvalue $\nu > 0$ does not take the constant eigenfunction.

Here, we note that the method of symmetrization does not work to evaluate $\mu_2(\Omega)$ in Theorem 3.11. In more precise, [142] adopts a direct calculation, while [131] uses the method of conformal transplantation. The latter is valid only to the case that $\Omega \subset \mathbf{R}^2$ is simply-connected, but is applicable to (3.60). Using Theorem 2.4, we obtain the following.

THEOREM 3.18. (See [8].) *If $\Omega \subset \mathbf{R}^2$ is a bounded simply-connected domain with smooth boundary $\partial\Omega$, $p = p(x) > 0$ is continuous on $\overline{\Omega}$, C^2 in Ω, satisfies* (3.41) *and $\lambda = \int_\Omega p \leqslant 4\pi$, then it holds that*

$$\frac{1}{\nu_2} + \frac{1}{\nu_3} \geqslant \frac{\lambda}{2\pi}, \tag{3.61}$$

where ν_i $(i = 2, 3)$ denotes the ith eigenvalue of (3.60).

Since we obtain $-\Delta \log p < p$ in Ω for $p = p(x)$ defined by (3.59), the equality is excluded in (3.61), and therefore, $\nu_2 = \nu_2(v, \lambda) < 1$ for any solution $v = v(x)$ to (3.57) with $\lambda = 4\pi$. Since $v = 0$ is a trivial solution, an immediate consequence is that $\nu_2(0, \lambda_1) = 0$ for some $\lambda_1 < 4\pi$. If this $\nu_2 = \nu_2(0, \lambda_1)$ is simple, then we obtain the bifurcated branch $\mathcal{C}$ [42,43]. This $\mathcal{C}$ can reach $\lambda = 4\pi$ only if it loses stability caused by secondary bifurcation, bending, and so forth. Thus, in contrast with (3.49), we obtain generic multiple existence of the solution to (3.57) in $0 < \lambda < 4\pi$ if Ω is simply-connected.

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CHAPTER 5

Stationary Problem of Boltzmann Equation

Seiji Ukai and Tong Yang

Department of Mathematics and Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong

E-mail: *mcukai@cityu.edu.hk*; *matyang@cityu.edu.hk*

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1. Introduction

1.1. *Overview*

It is well known that the Boltzmann equation has a special stationary solution called the Maxwellian. In physics, the Maxwellian is a universal distribution function which appears when the gas attains an equilibrium state. However, if an external forcing is exerted on the gas, the non-Maxwellian steady state may persist. This external forcing may be caused through the boundary of the vessel containing the gas, the external force field, the external gas source, and others. The aim of this chapter is to discuss the stationary problem of the Boltzmann equation in these situations.

The mathematical and physical significance of the non-Maxwellian steady states will be better understood on the emphasis of the fundamental roles that the Maxwellian and the Boltzmann equation play in the kinetic theory of the gas. The Maxwellian is a velocity distribution function of the gas in the equilibrium state while the Boltzmann equation is the most fundamental kinetic equation designed to describe the motion of the nonequilibrium gas in the phase space. They are two keystones of the kinetic theory but were established independently and on different physical principles: J.C. Maxwell [47] discovered his distribution function in 1857 based on the statistical argument on the equi-partition of the kinetic energy of gas particles, while it is in 1872 when L. Boltzmann [14] established his equation based on the Newtonian mechanics.

In [14], Boltzmann showed that the Maxwellian is a special stationary solution of his equation. More precisely, he deduced from his equation the celebrated H-theorem which suggests, among others, that any solution of the Boltzmann equation, if it exists globally in time and has some nice properties, converges to a uniform Maxwellian as the time goes on. In other words, the H-theorem implies that the Maxwellian is the only possible asymptotically stable stationary solution of the Boltzmann equation. From the physical view point, this may be rephrased as the equilibrium state of the gas is uniquely described by the Maxwellian, not by any other distribution functions.[1]

Despite of its significant implications to physics, the mathematical justification of this statement had not been known until 1932 when T. Carleman [16,17] proved that the Cauchy problem for the spatially homogeneous Boltzmann equation has solutions globally in time and that any of the solutions converges to a Maxwellian specified by the initial data as the time goes to infinity. Roughly, the Maxwellian is asymptotically stable for any initial perturbation (global asymptotic stability).

The spatially inhomogeneous Boltzmann equation, which is physically more realistic model, was solved much later. The first solution was constructed, locally in time by H. Grad [33] in 1965, and globally in time by one of the authors [58] in 1974. In [58], the Cauchy problem of the Boltzmann equation is solved on the torus, i.e. under the space-periodic boundary condition, globally in time for initial data close to a uniform Maxwellian and

[1] It is a famous episode of the history of science that the H-theorem raised a long and serious controversy between Boltzmann and his contemporaries. However, it is Boltzmann who was endorsed finally, though more than 100 years later, by Lanford [39] who established the convergence of the Newton equation to the Boltzmann equation and by many people mentioned below who proved the existence of the global solutions. See also [19] for a detail of the controversy.

also is established the exponential convergence of the solution to the relevant Maxwellian. Since then, this result has been extended to the Cauchy problem in the whole space and various initial boundary value problems. We mention, Guo [35], Liu and Yu [44], Liu, Yang and Yu [42], Nishida and Imai [49], Palczewski [50], Shizuta [52], Shizuta and Asano [53], Strain and Guo [57], Ukai [59,61], and references therein. They show, among others, the asymptotic stability of the uniform Maxwellian for initial perturbation close to the same Maxwellian (local asymptotic stability).

For general initial data, the renormalized solutions, initiated by Diperna and Lions [24] for the whole space case, have been constructed also on the torus and in a bounded domain under some physical boundary conditions by Hamdache [36], Arkeryd and Cercignani [4], and others. Moreover, it has been shown that there is a time sequence along which the solution converges to a uniform Maxwellian in a weak topology, see [18,22] for details. However, the uniqueness of the renormalized solutions and hence the uniqueness of the limit Maxwellians are still open and the renormalized solutions in a domain with boundary are known only to satisfy the boundary condition in inequality.

A recent result presented by Desvillettes and Villani [23] suggests that the global asymptotic stability of the uniform Maxwellian holds in a much stronger sense: They discussed both the torus case and the boundary value problem in a bounded domain with the specular or reverse (bounce-back) reflection boundary condition and showed that any sufficiently smooth global solution converges almost exponentially to a uniform Maxwellian at time infinity. This is a remarkable result because no smallness conditions are imposed, although the existence of such smooth solutions is a big open problem in the present mathematical theory of the Boltzmann equation.

Thus, all the results mentioned above lead to the conclusion that the uniform Maxwellian has a kind of universality in the theory of the Boltzmann equation. However, it should be stressed that both the Maxwellian and H-theorem were originally developed in the force-free space.

Suppose that the gas is exerted by an external forcing through the boundary of the vessel of the gas, by the force field, gas source or others. Then, the stationary state of the gas, if sustained, may be different from the uniform Maxwellian. Mathematically, this raises the stationary problems for the Boltzmann equation in a domain with boundary, in the external force field, with the inhomogeneous term, or others. Among typical examples are the half-space problem with the Dirichlet boundary condition, the exterior problem for the flow past an obstacle, and the interior problem with forcing. The half-space problem with the Dirichlet boundary condition describes complex behaviors of the gas near the wall such as the development of the boundary layer, the evaporation-condensation phenomenon, and so on. The classical Milne and Kramer problems are also to be mentioned which are the linear half-space problems with the Dirichlet boundary condition supplemented with the flux conditions that arise from the albedo problem for neutrons or photons. The exterior problem arising in the study of the flow past an obstacle is one of the most classical and important subjects in gas dynamics and fluid mechanics. The point of this problem is to assign the bulk velocity at infinity, which is not a trivial driving force on the flow. The interior problem in the external force field or under inhomogeneous or nonisothermal boundary conditions is also of great importance, describing the forced flows including the Couette and Benard flows. Furthermore, the time-periodic flow induced by the time-periodic external forcing can be viewed as another kind of stationary problems, a boundary value problem with the time-periodic boundary condition.

In this chapter, we will discuss the three of these stationary problems, the half-space problem, the exterior problem and the time-periodic flow problem. Our plan is as follows. Section 2 is devoted to the study of the half-space problem under the Dirichlet boundary condition, which arises, as mentioned above, in the analysis of the complex interaction of the gas and the solid wall of the vessel, see, e.g., [12,54] for the physical background. The main feature of this problem is that it is not a unconditionally solvable problem, and the number of the solvability conditions to be imposed on the Dirichlet data varies with the Mach number of the far field. The stability of the stationary solution is also discussed, though only for restricted Mach numbers.

In Section 3, we study the exterior problem for the flow past an obstacle. For the physical background, see, e.g., [41]. Whereas many important mathematical results have been established for fluid dynamical equations including the Euler and Navier–Stokes equations, little is known for the Boltzmann equation. Here, following [64,65], we construct the Boltzmann flow under the smallness assumption on the Mach number of the far field. The Boltzmann shock profile of transonic and supersonic flows induced by the obstacle is one of the challenging open problems in the mathematical theory of the Boltzmann equation. Most of this section is a reproduction of [64] but a part of the proof is renewed by explicitly using the “velocity averaging” argument, resulting in the relaxation to some extent of the restriction introduced in [64] on the boundary conditions on the boundary of the obstacle.

Finally in Section 4, we will study the inhomogeneous Boltzmann equation with a time-periodic source term. This is a simplest model problem for the generation and propagation of the sound wave in the gas, [41]. Under some smallness condition on the source term, the time-periodic solution with the same period as the source term will be constructed and is proved to be asymptotically stable.

There are two important kinds of stationary problems which are not covered by this chapter. One is concerned with the shock profile solution. This is a traveling wave problem and can be formulated as the stationary problem for the one-dimensional wave pattern connecting two different assigned far fields. The typical wave patterns known in fluid mechanics are the shock, rarefaction, and contact discontinuity waves. The shock wave solution of the Boltzmann equation was constructed in [15] and its stability was studied in [44]. For the other two wave patterns, unlike the shock pattern, only the time-asymptotic wave profiles can be constructed for the Boltzmann equation, which is just the same situation as for the compressible Navier–Stokes equations. The stability of such time-asymptotic wave profiles for the Boltzmann equation were also studied in [38,43]. See also [67] for a review on this subject.

The interior problem, i.e. the boundary value problem in a bounded domain with boundary, is another important stationary problem. However, most of works done so far are devoted to the case where the boundary condition preserves a uniform Maxwellian so that the same Maxwellian is the only possible stable stationary solution, see, e.g., [9,50,53]. If this condition is violated, the non-Maxwellian stationary solution may appear. This is indeed the case which was demonstrated by Sone [54] in a subtle experiment for detecting the “ghost effect” in which the steady flow of a highly rarefied gas is realized by the high gradient of the wall temperature. This is not a fluid dynamical phenomenon in the classical sense and can be explained only by the Boltzmann equation with the nonisothermal boundary conditions. This is another challenging open problem of the mathematical theory of the Boltzmann equation.

We also note a series of works in progress by Arkeryd and Nouri in [5,6,8,7] and others on various stationary problems including the slab case and the Couette flow. Their results are remarkable and the methods are promising because no smallness conditions are introduced. At the present, however, they need some artificial cutoff assumptions on the collision operator (see [7], however) and the uniqueness is still uncertain in various contexts.

The rest of this introduction is devoted to some preliminaries needed for the later sections. In the next subsections, after introducing the Boltzmann equation, we summarize some properties and estimates of the collision operator under the celebrated Grad's cutoff assumption which will be adapted throughout this chapter. Finally in Section 1.3, we present some classical examples of the boundary conditions which are widely accepted for the Boltzmann equation.

1.2. *Boltzmann equation*

In this chapter, we discuss the Boltzmann equation for the mono-atomic gas. The reader is referred to [19] for the physical background and derivation of the Boltzmann equation. Thus, the Boltzmann equation we study is

$$\begin{aligned} &\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \frac{1}{m} F \cdot \nabla_\xi f = \frac{1}{\kappa} Q(f, f) + S, \\ &\quad (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is the domain of the vessel containing the gas and $f = f(t, x, \xi)$ is the unknown function representing the probability (mass, number) density of gas particles around position $x = (x_1, \dots, x_n) \in \Omega$ with velocity $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. (1.1) is a balance law. The linear term $-\xi \cdot \nabla_x f - F \cdot \nabla_\xi f$, the *transport term*, gives the rate of change of f due to the motion of gas particles in the external force field $F = F(t, x, \xi) = (F_1, \dots, F_n)$, m being the mass of the gas particle. The last term on the right-hand side is the rate of change of f due to the external source of gas particles of intensity $S = S(t, x, \xi)$. The term $\kappa^{-1} Q$ gives the rate of change of f due to binary elastic collision of gas particles, where Q is the nonlinear *collision operator*

$$Q(f, f) = \int_{\mathbb{R}^n \times S^{n-1}} q(v, \theta)\big(f' f'_* - f f_*\big)\, d\xi_*\, d\omega, \tag{1.2}$$

where

$$f = f(t, x, \xi), \quad f' = f(t, x, \xi'), \quad f'_* = f\big(t, x, \xi'_*\big), \quad f_* = f(t, x, \xi_*),$$

with

$$\xi' = \xi - \big((\xi - \xi_*) \cdot \omega\big)\omega, \quad \xi'_* = \xi_* + \big((\xi - \xi_*) \cdot \omega\big)\omega,$$
$$\omega \in S^{n-1}, \quad v = |\xi - \xi_*|, \quad \cos\theta = \frac{\xi - \xi_*}{|\xi - \xi_*|} \cdot \omega. \tag{1.3}$$

Here, ξ, ξ_* can be thought of as the velocities of gas particles before collision and ξ', ξ'_* as those after collision, respectively, and (1.3) describes the laws of the elastic collision. Thus, the conservation laws of momentum and kinetic energy hold in the course of the collision:

$$\xi + \xi_* = \xi' + \xi'_*, \qquad |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + \big|\xi'_*\big|^2. \tag{1.4}$$

(1.3) may be taken to be a set of solutions of equation (1.4) parameterized by ω.

The *collision cross section* $q(v, \theta)$ is determined by the interaction law between two colliding particles. A classical example is the hard sphere gas for which, [34],

$$q(v, \theta) = q_0 v|\cos\theta|, \quad \cos\theta = (\xi - \xi_*) \cdot \omega/v, \tag{1.5}$$

where q_0 is the surface area of a hard sphere, and another well-known example is the inverse power law potential r^{-s} for which [34]

$$q(v, \theta) = v^\gamma |\cos\theta|^{-\gamma'} q_0(\theta), \quad \gamma = 1 - \frac{2(n-1)}{s}, \ \gamma' = 1 + \frac{n-1}{s}, \tag{1.6}$$

where $q_0(\theta)$ is a bounded nonnegative function which does not vanish near $\theta = \pi/2$. The interaction potential is said hard if $s \geqslant 2(n-1)$ and soft if $0 < s < 2(n-1)$.

Finally, κ is the *Knudsen number* (the ratio of the mean free path of the gas particle and the characteristic length of the domain of the vessel containing the gas) which plays an important roll in the study of asymptotic relations between the Boltzmann equation and the fluid dynamical equations such as the Euler and Navier–Stokes equations. Since this chapter deals only with the stationary problem of the Boltzmann equation, however, we will fix simply $\kappa = 1$ as well as $m = 1$, without loss of generality.

1.2.1. *Collision operator Q.* Roughly speaking, the mathematical study of the Boltzmann equation is aimed at revealing various interplays between the linear transport term $-\xi \cdot \nabla_x f - F \cdot \nabla_\xi f$ and the nonlinear collision term $Q(f, f)$. We summarize here three main properties of the operator Q deduced by Boltzmann himself. The reader is referred to e.g. [19,34] for details.

Define the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(\xi) g(\xi)\, d\xi. \tag{1.7}$$

A function $\phi(\xi)$ is called a *collision invariant* if

$$\big\langle \phi, Q(f, f)\big\rangle = 0 \quad \big(\forall f \in C_0^\infty\big(\mathbb{R}^n_\xi, \mathbb{R}_+\big)\big). \tag{1.8}$$

The first property of Q is

[Q1] *Q has $n+2$ collision invariants,*

$$\phi_0(\xi)=1, \qquad \phi_i(\xi)=\xi_i \quad (i=1,2,\dots,n), \qquad \phi_{n+1}(\xi)=\frac{1}{2}|\xi|^2. \tag{1.9}$$

The proof comes via the integral identity

$$\langle\phi, Q(f,f)\rangle=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n\times S^{n-1}} q(v,\theta) f f_*(\phi'+\phi'_*-\phi-\phi_*)\,d\xi\xi_*\,d\omega, \tag{1.10}$$

which can be obtained by means of several changes of variables [16,19]. Then, this integral is zero independently of the particular function f if and only if

$$\phi'+\phi'_*-\phi-\phi_*=0. \tag{1.11}$$

Boltzmann is the first who solved this equation, showing that within the space of twice differentiable functions, the most general solution is a linear combination of the collision invariants (1.9). Later, this result was extended to more general function spaces including the spaces of continuous and L^1_{loc} functions, see [19].

A significant consequence of [Q1] is the conservation laws of the Boltzmann equation, which connects the microscopic description due to the kinetic theory of the gas and the macroscopic description due to fluid mechanics. Since $f(t,x,\xi)$ is the mass density in the (x,ξ)-space, that is, the *microscopic* mass density in the one-particle phase space, its moments with respect to ξ are *macroscopic* quantities in the usual physical space. The first few moments are

$$\begin{aligned}
&\rho=\langle\phi_0, f(t,x,\cdot)\rangle,\\
&\rho u_i=\langle\phi_i, f(t,x,\cdot)\rangle \quad (i=1,2,\dots,n),\\
&\rho E=\langle\phi_{n+1}, f(t,x,\cdot)\rangle.
\end{aligned} \tag{1.12}$$

Here, ρ is the macroscopic mass density, $u=(u_1,u_2,\dots,u_n)$ is the macroscopic (bulk) velocity, and E is the average energy density per unit mass, of the gas around space position $x\in\mathbb{R}^n$ at time t. The temperature T and the pressure p are related to E by

$$E=\frac{1}{2}|u|^2+\frac{n}{2}T, \qquad p=R\rho T, \tag{1.13}$$

where R is the *gas constant* (the Boltzmann constant divided by the mass of the gas particle). The last equation in (1.13) is called the *equation of state* for the ideal gas.

Consider the case $\Omega=\mathbb{R}^n$ and $F=0$, $S=0$. Let f be a smooth solution to (1.1) which vanishes sufficiently rapidly with (x,ξ). Multiply (1.1) by ϕ_j and integrate it over $\mathbb{R}^n_x\times\mathbb{R}^n_\xi$. By virtue of (1.9) and by integration by parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\langle\phi_i, f(t,x,\cdot)\rangle\,dx=0, \quad i=0,1,\dots,n+1, \tag{1.14}$$

which are, in view of (1.12), the conservation laws of total mass ($i = 0$), total momenta ($i = 1, 2, \dots, n$) and total energy ($i = n + 1$), of the gas.

It is seen that similar conservation laws can hold for the case $\Omega \neq \mathbb{R}^3$ and $F \neq 0$ if some appropriate assumptions are imposed on the boundary conditions on the boundary $\partial\Omega$ and on the external force F.

The second property of Q to be mentioned is

[Q2] $\langle \log f, Q(f, f) \rangle \leqslant 0 \quad (\forall f \in C_0^\infty(\mathbb{R}^n_\xi, \mathbb{R}_+))$.

First, we observe [19] that putting $\phi = \log f$ in (1.10) and a simple change of variable give the identity

$$-\langle Q(f, f), \log f \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} q(|\xi - \xi_*|, \omega)(f' f'_* - f f_*) \times \log \frac{f' f'_*}{f f_*} \, d\xi \, d\xi_* \, d\omega. \tag{1.15}$$

Now [Q2] follows in view of the elementary inequality

$$(a - b)(\log a - \log b) \geqslant 0 \quad (a, b > 0).$$

Notice that here the equality holds if and only if $a = b$, which then implies the following.

[Q3] $Q(f, f) = 0 \quad \Leftrightarrow \quad \langle \log f, Q(f, f) \rangle = 0 \quad \Leftrightarrow \quad f = \mathbf{M}(\xi)$ where

$$\mathbf{M}(\xi) = \mathbf{M}_{[\rho,u,T]}(\xi) = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2RT}\right). \tag{1.16}$$

M is the *Maxwellian*, and according to Maxwell, it represents the velocity distribution of the gas in an equilibrium state with the mass density $\rho > 0$, bulk velocity $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$, and temperature $T > 0$. Maxwell derived **M** on physical arguments but **[Q3]** implies that it is built-in for the Boltzmann equation. Notice that [Q3] is equivalent to solving the equation

$$f' f'_* - f f_* = 0,$$

and this is reduced to equation (1.11) after plugging $\phi = \log f$. If (ρ, u, T) are constants, **M** is called a uniform (global, absolute) Maxwellian while if they are functions of (x, t), **M** is called a local Maxwellian. An immediate consequence of [Q3] is that the uniform Maxwellian is a stationary solution of (1.1) if the external force and source are absent.

We now recall Boltzmann's H theorem mentioned in Section 1.1, which is the most significant consequence of the properties [Q1]–[Q3]. Let f be a density function of a gas. Since it is nonnegative, we may define the *H-function*,

$$H(t) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f \log f \, dx \, d\xi, \tag{1.17}$$

which gives, according to Boltzmann [14], the negative of the entropy of the gas. Consider again the case $\Omega = \mathbb{R}^n$, $F = 0, S = 0$, and let f be a nonnegative smooth solution to (1.1) with rapid decay properties in (x, ξ). Multiply (1.1) by $\log f$ and integrate in (x, y). Integration by parts, together with [Q1], yields,

$$\frac{dH(t)}{dt} + D(t) = 0, \tag{1.18}$$

where

$$D(t) = -\int_{\mathbb{R}^n} \langle Q(f, f), \log f \rangle \, dx \tag{1.19}$$

is called the *entropy dissipation integral*. Since $D(t)$ is nonnegative by virtue of [Q2], we conclude that

$$\frac{dH}{dt} \leqslant 0. \tag{1.20}$$

This implies that the entropy is increasing with time. Moreover, by virtue of [Q3], the equality in (1.20) holds only when f is a Maxwellian, so that f converges to a Maxwellian as t goes on. This is the celebrated *H-theorem*.

The nonnegativity of the integral D in (1.19) indicates that the Boltzmann equation is a dissipative equation. This fact is essentially used in the L^1 theory of the Boltzmann equation, [24]. On the other hand, its linearized version results in the nonpositivity of the linearized operator of Q at the uniform Maxwellian $\mathbf{M}$, and is a key ingredient in the theory of the Boltzmann equation near the Maxwellian.

This nonpositivity is formulated as follows. Introduce the bilinear symmetric operator associated with the quadratic operator Q:

$$Q(f, g) = \frac{1}{2} \int_{\mathbb{R}^n \times S^{n-1}} q(v, \theta) \big(f' g'_* + g' f'_* - f g_* + g' f'_* \big) \, d\xi_* \, d\omega. \tag{1.21}$$

Let f be a function near $\mathbf{M}$ in the form

$$f = \mathbf{M} + \epsilon \mathbf{M}^{1/2} u, \tag{1.22}$$

for small $\epsilon \in \mathbb{R}$ and some function $u = u(\xi)$. Then, by virtue of [Q1,3] and the bilinearity of Q, we get

$$\langle \log f, Q(f, f) \rangle = \langle \log\big(1 + \epsilon \mathbf{M}^{-1/2} u\big), \epsilon \mathbf{M}^{1/2} \big(\mathbf{L} u + \epsilon \Gamma(u, u) \big) \rangle$$

where

$$\mathbf{L} u = 2\mathbf{M}^{-1/2} Q\big(\mathbf{M}, \mathbf{M}^{1/2} u\big), \qquad \Gamma(u, v) = \mathbf{M}^{-1/2} Q\big(\mathbf{M}^{1/2} u, \mathbf{M}^{1/2} v\big). \tag{1.23}$$

Here, $\mathbf{L}$, the *linearized collision operator*, is a linearized operator of Q around $\mathbf{M}$ while Γ, being its remainder, is a bilinear symmetric operator. Now, we see at least formally that

$$\frac{1}{\epsilon^2}\big\langle \log f, Q(f,f)\big\rangle \to \langle u, \mathbf{L}u\rangle \quad (\epsilon \to 0),$$

concluding, by the aid of [Q2], that

$$\langle u, \mathbf{L}u\rangle \leqslant 0. \tag{1.24}$$

1.2.2. *Grad's angular cutoff.* Needless to say that all the properties of Q in the preceding subsection are substantiated only when the relevant integrals are convergent.

The collision kernel q in Q has no singularity for the hard sphere gas (1.5). However, (1.6) for the inverse power law potential has a strong singularity at $\theta = \pi/2$ which corresponds to the grazing collision, and as a consequence, the integral over S^{n-1} in (1.2) does not converge under a mild assumption on f, g such that they are bounded. Actually, it is well known that $Q(f,g)$ is well defined only for sufficiently smooth f, g as a nonlinear pseudo-differential operator, see, e.g., [2,23,60]. However, this is a too strong restriction to solve the Boltzmann equation in full generality. In order to avoid this difficulty, Grad [34] introduced an idea to cut off the singularity at $\theta = \pi/2$ so that $q_0(\theta) \in L^1(S^{n-1})$. This assumption has been widely accepted and is now called Grad's angular cutoff assumption.

In the following discussion, we assume that $q(v,\theta)$ is a nonnegative measurable function satisfying

$$\int_{S^{n-1}} q(v,\theta)\, d\omega \geqslant q_0 v^{\gamma}, \quad q(v,\theta) \leqslant q_1\big(1 + v^{\gamma} + v^{-\eta}\big)|\cos\theta|, \tag{1.25}$$

for some constants $q_0, q_1 > 0$ and $\gamma, \eta \in [0,1]$ Clearly, this is satisfied by the hard sphere gas (1.5) with $\gamma = 1$ and by the inverse power law potential case (1.6) under the Grad's cutoff with $\gamma = 1 - 2(n-1)/s$ for $s \geqslant 2(n-1)$. Thus, (1.25) is a slightly generalized version of Grad's cutoff hard potential.

Under this assumption, Q becomes well defined for integrable functions. Let us show this for a modified collision operator Γ introduced in (1.23). To this end, with the same Maxwellian $\mathbf{M}$ used there, we define the function

$$\nu(\xi) = \int_{\mathbb{R}^n} q\big(|\xi - \xi_*|, \theta\big)\mathbf{M}^{1/2}(\xi_*)\, d\xi_*\, d\omega, \tag{1.26}$$

which satisfies under the assumption (1.25),

$$\nu_0\big(1 + |\xi|\big)^{\gamma} \leqslant \nu(\xi) \leqslant \bar{\nu}_0\big(1 + |\xi|\big)^{\gamma}, \tag{1.27}$$

for some positive constants $\nu_0, \bar{\nu}_0$.

THEOREM 1.1. *For any* $p \in [1,\infty]$ *and* $\delta \in [0,1]$, *there exists a constant* $C > 0$ *such that the operator* Γ *defined by* (1.23) *satisfies*

$$\left\| \nu^{-\delta} \Gamma(u, v) \right\|_p \leqslant C \left(\left\| \nu^{1-\delta} u \right\|_p \|v\|_p + \|u\|_p \left\| \nu^{1-\delta} v \right\|_p \right), \tag{1.28}$$

where $\| \cdot \|_p$ is the norm of the space $L^p(\mathbb{R}^n_\xi)$.

PROOF. Recall (1.21) and write

$$\Gamma(u, v) = \frac{1}{2} \left\{ \Gamma_1(u, v) + \Gamma_1(v, u) - \Gamma_2(u, v) - \Gamma_2(v, u) \right\}, \tag{1.29}$$

with

$$\begin{aligned} \Gamma_1(u, v) &= \int_{\mathbb{R}^n \times S^{n-1}} q\big(|\xi - \xi_*|, \theta\big) u(\xi') v\big(\xi'_*\big) \mathbf{M}(\xi_*)^{1/2} \, d\xi_* \, d\omega, \\ \Gamma_2(u, v) &= \int_{\mathbb{R}^n \times S^{n-1}} q\big(|\xi - \xi_*|, \theta\big) u(\xi) v(\xi_*) \mathbf{M}(\xi_*)^{1/2} \, d\xi_* \, d\omega, \end{aligned} \tag{1.30}$$

where we have used that

$$\mathbf{M}(\xi') \mathbf{M}\big(\xi'_*\big) = \mathbf{M}(\xi) \mathbf{M}(\xi_*)$$

which comes from the conservation laws (1.4).

First, we consider Γ_1. The Hölder inequality gives

$$\begin{aligned} \big|\Gamma_1(u, v)\big| &\leqslant \left(\int_{\mathbb{R}^n} q\big(|\xi - \xi_*|, \theta\big)^q \mathbf{M}(\xi_*)^{q/2} \, d\xi_* \, d\omega \right)^{1/q} \\ &\quad \times \left(\int_{\mathbb{R}^n} \big|u(\xi')\big|^p \big|v\big(\xi'_*\big)\big|^p \, d\xi_* \, d\omega \right)^{1/p} \\ &\leqslant C \nu(\xi) \left(\int_{\mathbb{R}^n} \big|u(\xi')\big|^p \big|v\big(\xi'_*\big)\big|^p \, d\xi_* \, d\omega \right)^{1/p} \end{aligned}$$

with $p \in [1, \infty)$, $1/p + 1/q = 1$. The last line comes from

$$\int_{\mathbb{R}^n} q\big(|\xi - \xi_*|, \theta\big)^q \mathbf{M}(\xi_*)^{q/2} \, d\xi_* \, d\omega \leqslant C \big(1 + |\xi|\big)^{\gamma q} \leqslant C \nu(\xi)^q,$$

which holds by virtue of (1.25) and (1.27). Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^n} \big|\nu(\xi)^{-\delta} \Gamma_1(u, v)(\xi)\big|^p \, d\xi \\ &\quad \leqslant C \int_{\mathbb{R}^n \times \mathbb{R}^n S^{n-1}} \nu(\xi)^{(1-\delta)p} \big|u(\xi')\big|^p \big|v\big(\xi'_*\big)\big|^p \, d\xi \, d\xi_* \, d\omega. \end{aligned}$$

By virtue of (1.3) and (1.27),

$$\begin{aligned}\nu(\xi) &\leqslant C\big(1+|\xi|\big)^{\gamma} = C\big(1+\big|\xi' - \big\{\big(\xi'-\xi'_*\big)\cdot\omega\big\}\omega\big|\big)^{\gamma} \\ &\leqslant C\big(2+|\xi'|+\big|\xi'_*\big|\big)^{\gamma} \leqslant C\big(\nu(\xi')+\nu\big(\xi'_*\big)\big).\end{aligned}$$

Since the Jacobian of the change of variable $(\xi,\xi_*,\omega)\leftrightarrow(\xi',\xi'_*,-\omega)$ is unity, we finally have

$$\begin{aligned}&\int_{\mathbb{R}^n}\big|\nu(\xi)^{-\delta}\Gamma_1(u,v)(\xi)\big|^p\,d\xi \\ &\quad\leqslant C\int_{\mathbb{R}^n\times\mathbb{R}^n\times S^{n-1}}\big(\nu(\xi')^{(1-\delta)p}+\nu\big(\xi'_*\big)^{(1-\delta)p}\big)\big|u(\xi')\big|^p\big|v\big(\xi'_*\big)\big|^p\,d\xi'\,d\xi'_*\,d\omega.\end{aligned}$$

This proves (1.28) for Γ_1 for the case $p\in[1,\infty)$. The case $p=\infty$ can be proved similarly, and the proof for Γ_2 is also similar but much simpler. Now, the proof of the theorem is complete. □

The function $\nu(\xi)$, *the collision frequency*, is bounded if $\gamma=0$ and unbounded of order $O(|\xi|^\gamma)$ if $\gamma>0$. As a consequence, Theorem 1.1 asserts that Q is a bounded operator if $\gamma=0$ whereas it is well defined but unbounded with the weigh loss of order $O(|\xi|^\gamma)$ if $\gamma>0$.

REMARK 1.2. Theorem 1.1 was first proved in [34] for the case $p=\infty,\delta=1$ and will be used in the sequel, while the case $p=2,\delta=1/2$ is due to [31] and was used successfully in conjunction with the L^2 energy method for the Boltzmann equation initiated by [44] and developed in [35,42] and others.

1.2.3. *Linearized collision operator* **L**. Theorem 1.1 ensures also the well-definedness of the linearized collision operator **L** in (1.23). Here, we shall summarize its properties which will be used in essential ways throughout this chapter. Most of them go back to Grad [34]. All the results below hold for any choice of the Maxwellian **M** in (1.23), but they are equivalent to each other by a simple scaling and translation of the velocity variables. This can be seen from the definition of Q in (1.2). In particular, Q is translation invariant: With the translation $\tau_c u=u(\xi-c)$, $c\in\mathbb{R}^n$, it holds that

$$\tau_c Q(f,g)=Q(\tau_c f,\tau_c g). \tag{1.31}$$

Thus, in the sequel, we fix **M** to be the standard Maxwellian $\mathbf{M}_{[1,0,1]}$, without loss of generality.

PROPOSITION 1.3. (See [34].) *Under the assumption* (1.25), *the linearized collision operator* **L** *defined by* (1.23) *has the decomposition*

$$\mathbf{L}u=-\nu(\xi)u+Ku, \tag{1.32}$$

where $\nu(\xi)$ *is just the function in* (1.26) *while* K *is a linear integral operator in* ξ;

$$Ku = \int_{\mathbb{R}^n} K(\xi, \xi_*) u(\xi_*)\, d\xi_*, \tag{1.33}$$

whose kernel is real symmetric and enjoys the estimates

$$\int_{\mathbb{R}^n} \bigl|K(\xi, \xi_*)\bigr|^2\, d\xi_* \leqslant C, \tag{1.34}$$

$$\int_{\mathbb{R}^n} \bigl|K(\xi, \xi_*)\bigr| \bigl(1 + |\xi_*|\bigr)^{-\beta}\, d\xi_* \leqslant C_\beta \bigl(1 + |\xi|\bigr)^{-\beta-1}, \quad \beta \geqslant 0, \tag{1.35}$$

for some constants C, C_β *independent of* ξ.

In fact, the estimate

$$\bigl|K(\xi, \xi_*)\bigr| \leqslant k_0 \biggl\{ \bigl(v + v^{-\eta}\bigr) \exp\biggl(-\frac{1}{4}\bigl(|\xi|^2 + |\xi_*|^2\bigr)\biggr) + \bigl(v^{-1} + v^{-(n-2)}\bigr) \exp\biggl(-\frac{1}{8}\bigl(v^2 + |\zeta|^2\bigr)\biggr)\biggr\} \tag{1.36}$$

holds for $k_0 > 0$ and $\eta \in [0, 1)$, where $v = |\xi - \xi_*|$, $\zeta = (|\xi|^2 - |\xi_*|^2)/v$. This was proved for $n = 3$ in [34] and for $n \geqslant 4$ in [52]. Then, (1.34) for $n = 3$ and (1.35) follow. (1.34) should be modified for the other cases but is assumed here for the sake of simplicity.

For the hard sphere gas (1.5), the explicit expressions for $n = 3$ and for the standard Maxwellian $\mathbf{M} = \mathbf{M}_{[1,0,1]}$ are found in [34,19]:

$$\nu(\xi) = (2\pi)^{1/2} \biggl\{ \bigl(|\xi| + |\xi|^{-1}\bigr) \int_0^{|\xi|} \exp\bigl(-u^2/2\bigr)\, du + \exp\bigl(-|\xi|^2/2\bigr) \biggr\},$$

$$K(\xi, \xi_*) = (2\pi)^{1/2} v^{-1} \exp\biggl(-\frac{1}{8}\bigl(v^2 + |\zeta|^2\bigr)\biggr) - \frac{1}{2} v \exp\biggl(-\frac{1}{4}\bigl(|\xi|^2 + |\xi_*|^2\bigr)\biggr).$$

The operator K is to be considered in various function spaces such as

$$L^p = L^p\bigl(\mathbb{R}^n_\xi\bigr), \quad p \in [1, \infty], \tag{1.37}$$

$$L^\infty_\beta = L^\infty\bigl(\mathbb{R}^n_\xi; \bigl(1 + |\xi|\bigr)^\beta\, d\xi\bigr), \quad \beta \in \mathbb{R}.$$

Note that $L^\infty_0 = L^\infty$. The following lemma is partly from [19,34,61].

LEMMA 1.4. (a) *Let* $2 \leqslant p \leqslant r \leqslant \infty$. *Then, the operator*

$$K : L^p \to L^r \tag{1.38}$$

is bounded.

(b) *Let* $\beta \geqslant 0$. *Then, the operator*

$$K : L^\infty_\beta \to L^\infty_{\beta+1} \tag{1.39}$$

is bounded.

PROOF. We first prove the part (a) for the case $p = r \in [2, \infty]$. The case $p = r = \infty$ comes from (1.35):

$$\left|Ku(\xi)\right| \leqslant \int_{\mathbb{R}^n} \left|K(\xi, \xi_*)\right| d\xi_* \|u\|_{L^\infty}.$$

Let $p < \infty$. Again by (1.35) and the Hölder inequality,

$$\left|Ku(\xi)\right|^p \leqslant \left(\int_{\mathbb{R}^n} \left|K(\xi, \xi_*)\right| d\xi_*\right)^{p-1} \int_{\mathbb{R}^n} \left|K(\xi, \xi_*)\right| \left|u(\xi_*)\right|^p d\xi_*,$$

whence follows

$$\|Ku\|^p_{L^p} \leqslant C_0^{p-1} C_{p-1} \int_{\mathbb{R}^n} \left(1 + |\xi_*|\right)^{-p} \left|u(\xi_*)\right|^p d\xi_*. \tag{1.40}$$

Thus,

$$K : L^p \to L^p, \tag{1.41}$$

is bounded for any $p \in [2, \infty]$.

Next, we prove the lemma for the case with $p = 2$ and $r = \infty$. (1.34) and the Schwarz inequality give

$$\left|Ku(\xi)\right|^2 \leqslant \int_{\mathbb{R}^n} \left|K(\xi, \xi_*)\right|^2 d\xi_* \int_{\mathbb{R}^n} \left|u(\xi_*)\right|^2 d\xi_* \leqslant C \|u\|^2_{L^2},$$

which indicates that

$$K : L^2 \to L^\infty \tag{1.42}$$

is bounded.

Finally, the usual interpolation (see, e.g., [13]) between (1.41) for $p = \infty$ and (1.42) proves (1.38) for the case $p \in [2, \infty]$ and $r = \infty$, and then the interpolation between this result and (1.41) for $p \in [2, \infty]$ proves (1.38) for $p \leqslant r$ for any $r \in [p, \infty]$, proving (a).

For the proof of (b), use again (1.35) to compute

$$\left|Ku(\xi)\right| \leqslant \int_{\mathbb{R}^n} \left|K(\xi, \xi_*)\right| \left(1 + |\xi|\right)^{-\beta} d\xi_* \|u\|_{L^\infty_\beta} \leqslant C_\beta \left(1 + |\xi|\right)^{-\beta-1} \|u\|_{L^\infty_\beta}.$$

Thus the proof of the lemma is complete. □

The following lemma is classical, [34].

LEMMA 1.5. *K is a self-adjoint compact operator on L^2.*

PROOF. K is a bounded operator on L^2 according to (a) of the previous lemma and it is self-adjoint since the integral kernel is real symmetric according to Proposition 1.3. To prove that it is compact, for $R > 0$, let $\chi_R(\xi)$ be a characteristic function for $|\xi| < R$ and put $\bar{\chi}_R = 1 - \chi_R$. By (1.40) for $p = 2$, we get

$$\begin{aligned} \|K\bar{\chi}_R u\|_{L^2}^2 &\leqslant C_0 C_1 \int_{\mathbb{R}^n} \big(1 + |\xi_*|\big)^{-2} \bar{\chi}_R(\xi_*) \big|u(\xi_*)\big|^2 \, d\xi_* \\ &\leqslant C_0 C_1 (1 + R)^{-2} \|u\|_{L^2}^2, \end{aligned}$$

which indicates that

$$\|K\bar{\chi}_R\| \to 0 \quad (R \to \infty),$$

in the operator norm of L^2. The same is true for $\bar{\chi}_R K$ by the direct calculation or via the adjointness argument. On the other hand, the operator $\chi_R K \chi_R$ is a compact operator on L^2, or more precisely, it is of Hilbert–Schmidt type as (1.34) implies that its integral kernel is in $L^2(\mathbb{R}^n_\xi \times \mathbb{R}^n_{\xi_*})$. Now, the proof is completed by [40, Theorem III.4.7]. □

Fundamental properties of the operator $\mathbf{L}$ are now summarized in

PROPOSITION 1.6. *Assume* (1.25) *with $\gamma \in [0, 1]$ and consider the operator $\mathbf{L}$ defined by* (1.23) *with the domain of definition*

$$D(\mathbf{L}) = \{u \in L^2 \mid \nu(\xi)u \in L^2\}.$$

Then, the following holds.

(1) *$\mathbf{L}$ is a linear densely defined closed operator in L^2.*
(2) *$\mathbf{L}$ is self-adjoint and nonpositive in L^2.*
(3) *Its spectrum $\sigma(\mathbf{L})$ satisfies the following.*
   (a) $\sigma(\mathbf{L}) \subset (-\infty, 0]$.
   (b) *The set $\sigma(\mathbf{L}) \cap (-\nu_*, 0]$ consists only of discrete semi-simple eigenvalues and its only possible accumulation point is $-\nu_*$ where*

$$\nu_* \equiv \inf_{\xi \in \mathbb{R}^n} \nu(\xi). \tag{1.43}$$

   Note from (1.27) *that $\nu_* \geqslant \nu_0$.*
   (c) *0 is a semi-simple eigenvalue of $\mathbf{L}$. Its eigenspace, which is the null space of $\mathbf{L}$, denoted by $\mathcal{N}$, is $(n+2)$-dimensional and spanned by collision invariants weighted by $\mathbf{M}^{1/2}$,*

$$\mathcal{N} = \operatorname{span}\left\{\mathbf{M}^{1/2},\ \xi_i \mathbf{M}^{1/2}\ (i = 1, 2, \ldots, n),\ \frac{1}{2}|\xi|^2 \mathbf{M}^{1/2}\right\}. \tag{1.44}$$

(4) *Let*

$$\mathbf{P}: L^2 \to \mathcal{N} \tag{1.45}$$

be the orthogonal projection. It is a bounded operator from L^2 *into* L^∞_β, $\beta \in \mathbb{R}$, *as well as from* L^2 *into itself. Further,*

$$\mathbf{P}\mathbf{L}u = 0, \qquad \mathbf{P}\Gamma(u, v) = 0, \tag{1.46}$$

for any u, v.

PROOF. (i) is true for the multiplication operator ν defined by

$$\nu u = \nu(\xi)u(\xi), \qquad D(\nu) = \{u \in L^2 \mid \nu(\xi)u \in L^2\}, \tag{1.47}$$

where $\nu(\xi)$ is the function in (1.26). Since K is bounded owing to Lemma 1.4(a), $\mathbf{L}$ is a bounded perturbation of $-\nu$, and (i) follows from [40].

It is easy to see that ν is self-adjoint, and consequently, so is $\mathbf{L}$ since K is self-adjoint bounded, owing to Lemma 1.4(a). Now, it is clear that (1.24) is valid for any $u \in D(\mathbf{L})$, hence the nonpositivity. This proves (2) and (3)(a) is a simple consequence of (2).

On the other hand, (3)(b) follows from Weyl's theorem on the compact perturbation, [40, Theorem IV.5.35], since $\sigma(\nu) \subset (-\infty, -\nu_*]$ due to (1.27) and since K is compact. And, (4) is a direct consequence of [Q1], see [19,34]. Thus, the proposition is proved. □

Note that (3)(b,c) in the above proposition imply the spectral gap: There exists a constant $\mu_1 > 0$ such that

$$\langle u, \mathbf{L}u\rangle \leqslant -\mu_1 \|(I - \mathbf{P})u\|^2_{L^2}, \quad \forall u \in D(L) \tag{1.48}$$

holds. Actually, $-\mu_1$ is the first largest nonzero eigenvalue of $\mathbf{L}$ in $(-\nu_*, 0)$ or if it does not exist, $\mu_1 = \nu_*$. This can be strengthened as

LEMMA 1.7. *There is a constant* $\mu_* > 0$ *such that*

$$\langle u, \mathbf{L}u\rangle \leqslant -\mu_* \|\nu^{1/2}(I - \mathbf{P})u\|^2_{L^2}, \quad \forall u \in D(\mathbf{L}),$$

where $\nu^{1/2}$ *is the multiplication operator by the function* $\nu^{1/2}(\xi)$.

This lemma is due to [31] and comes from the decomposition (1.32) and the spectral gap (1.48).

Finally, we note

LEMMA 1.8. *The operator* $K\nu$ *is a bounded operator on* L^2.

PROOF. Use again (1.40) for $p = 2$ and recall (1.27). Then,

$$\|K\nu u\|_{L^2}^2 \leqslant C_0 C_1 \int_{\mathbb{R}^n} \bigl(1 + |\xi_*|\bigr)^{-2} \nu(\xi_*)^2 \bigl|u(\xi_*)\bigr|^2 \, d\xi_* \leqslant C\|u\|_{L^2},$$

whence the lemma follows. □

1.3. *Boundary condition*

Physically plausible boundary conditions for the Boltzmann equation are those which describe the interaction of gas particles with the wall of the vessel. Although the physical investigation of this interaction has a long history, as is seen from an example of the boundary condition given below which has the name after Maxwell, there are still difficulties to write down the correct boundary conditions, due to the lack of knowledge of the fine structure of the solid surface and hence of the effective interaction potential of the gas particle with the wall. Thus, most of the boundary conditions proposed so far for the Boltzmann equation are heuristic models. Here, we present some examples which are now widely accepted for the Boltzmann equation. In this chapter, especially in Section 3, however, we do not specify the boundary condition but work only under rather general and mild restrictions on the boundary condition.

Recall that $\Omega \subset \mathbb{R}^n$ denotes the domain of the vessel. Let $\partial\Omega$ be its boundary and $n(x)$ the outward normal to $\partial\Omega$ at point x. Set

$$\begin{aligned} S^+ &= \bigl\{(x,\xi) \in \partial\Omega \times \mathbb{R}^3 \bigm| n(x)\cdot\xi > 0\bigr\},\\ S^- &= \bigl\{(x,\xi) \in \partial\Omega \times \mathbb{R}^3 \bigm| n(x)\cdot\xi < 0\bigr\}. \end{aligned} \tag{1.49}$$

According to the choice of the direction of $n(x)$ (outward), a gas particle having the position x and velocity ξ in S^+ is a particle just striking the wall and a particle with those in S^- is a particle just emerging from the wall. In the sequel, we will call the striking and emerging particles simply outgoing and incoming particles respectively. Notice that in some literature, the inner normal is used with the reverse usage of the words "outgoing" and "incoming." Here, the outward direction is chosen as in the presentation of Green's formula in usual textbooks of vector analysis. This formula plays a central role when carrying out the energy method for PDE's, and this is the case also in this chapter.

Write the restrictions of the density distribution function $f = f(t,x,\xi)$ to $S^\pm$ as

$$\gamma^\pm f = f|_{S^\pm} \quad \text{(same signs throughout)}. \tag{1.50}$$

They are the boundary values or *traces* of f and $\gamma^\pm$ are called the *trace operators*. Clearly, $\gamma^+ f$ (respectively $\gamma^- f$) represents the density of outgoing (respectively incoming) particles. The well-definedness of the trace operators $\gamma^\pm$ will be discussed in Section 3.2.2.

Note that the incoming particles are those either reflected by the wall or emitted from the source installed on or inside the wall. This means that the natural boundary condition to the Boltzmann equation should take the form

$$\gamma^- f = \mathbb{B}\gamma^+ f + h \quad \text{on } S^-, \tag{1.51}$$

where $\mathbb{B}$, being an operator which maps functions on S^+ to those on S^-, describes the reflection law of gas particles by the wall, and $h = h(x, \xi)$ is a given function on S^- which stands for the intensity of emitted particles from the source.

It is worth noting that, unlike (1.51), the boundary condition of the type

$$\gamma^+ f = \mathbb{B}\gamma^- f + h \quad \text{on } S^+ \tag{1.52}$$

does not make sense for the Boltzmann equation. Indeed, from the mathematical view point, the (initial) boundary value problem can be well-posed with the boundary condition of the type (1.51) but becomes ill-posed with (1.52), as will be seen from the energy estimates established in the following sections. This is plausible also from the physical view point because the incoming density $\gamma^- f$ can be controlled by the external source installed inside or on the wall while the outgoing density $\gamma^+ f$ cannot be controlled by any external force or source.

Now, we present some typical examples of the reflection operators $\mathbb{B}$.

(1) Perfect absorption, Dirichlet boundary condition. Suppose that outgoing particles are all absorbed by the solid wall and never return to the vessel. Thus, $\mathbb{B} = 0$, or

$$\gamma^- f = h \quad \text{on } S^-. \tag{1.53}$$

This is the Dirichlet boundary condition to the Boltzmann equation.

(2) Specular reflection. Suppose that outgoing particles are all reflected specularly by the wall. Let ξ' and ξ be the velocities of a particle at $x \in \partial\Omega$ before and after reflection respectively. Clearly, $(x, \xi') \in S^+$, $(x, \xi) \in S^-$, and $\xi' = \xi - 2(\xi \cdot n(x))n(x)$, and thereby, the reflection law is described by the operator

$$\mathbb{B}\gamma^+ f = (\gamma^+ f)\big(t, x, \xi - 2\big(\xi \cdot n(x)\big)n(x)\big), \quad t \in \mathbb{R},\ (x, \xi) \in S^-. \tag{1.54}$$

(3) Reverse reflection. If outgoing particles are all bounced back in the reverse direction by the wall, the boundary condition is described by the operator

$$\mathbb{B}\gamma^+ f = (\gamma^+ f)(t, x, -\xi), \quad t \in \mathbb{R},\ (x, \xi) \in S^-. \tag{1.55}$$

(4) Diffuse reflection. If the reflection is random, that is, if the velocity of a particle after reflection is not deterministic, the boundary condition may be described by the integral operator

$$\mathbb{B}\gamma^+ f = \int_{n(x)\cdot\xi'>0} m(t,x,\xi,\xi')(\gamma^+ f)(t,x,\xi')\,d\xi',$$
$$t\in\mathbb{R},\ (x,\xi)\in S^-, \tag{1.56}$$

where the integral kernel $m(t,x,\xi,\xi')$ determines the velocity distribution of particles after reflection.

The *diffuse reflection* refers to the case where the distribution is specified by the Maxwellian associated with the wall, which is given, for the wall at rest, by

$$m(x,\xi,\xi') = (2\pi)^{1/2}\mathbf{M}_{[1,0,T_w(x)]}(\xi)\big(n(x)\cdot\xi'\big), \tag{1.57}$$

$T_w(x)$ being a given function on $\partial\Omega$ representing the wall temperature.

Note that all of the above boundary conditions conserve the number of particles during the reflection:

$$\int_{n(x)\cdot\xi<0}(\mathbb{B}\gamma^+ f)(t,x,\xi)\big|n(x)\cdot\xi\big|\,d\xi = \int_{n(x)\cdot\xi>0}(\gamma^+ f)(t,x,\xi)\big|n(x)\cdot\xi\big|\,d\xi.$$

The boundary condition (2)–(3) can be generalized as

$$\mathbb{B}\gamma^+ f = (\gamma^+ f)\big(t, m(x,\xi)\big), \quad t\in\mathbb{R},\ (x,\xi)\in S^-, \tag{1.58}$$

with a suitable map $m: S^-\to S^+$, and (4) as

$$\mathbb{B}\gamma^+ f = \int_{S^+} m(t,x,x',\xi,\xi')(\gamma^+ f)(t,x',\xi')\,d\sigma_{x'}\,d\xi',$$
$$t\in\mathbb{R},\ (x,\xi)\in S^-, \tag{1.59}$$

$d\sigma_{x'}$ being the measure on $\partial\Omega$, and so on. Further, more general boundary conditions are any (convex) linear combination of different reflection laws. For example, write $\mathbb{B}$ for (1)–(4) by $\mathbb{B}_i$, $i=1,2,3,4$, and define

$$\mathbb{B} = \sum_{i=1}^{4}\kappa_i\mathbb{B}_i, \quad \kappa_i\in[0,1],\ \sum_{i=0}^{4}\kappa_i = 1. \tag{1.60}$$

For the case $\kappa_4\neq 0$ with (1.57), this is the (generalized) Maxwell (or accommodation) boundary condition, [19].

2. Half-space problem—nonlinear boundary layer

2.1. *Mathematical formulation*

Most of the physical models have boundaries so that the study on the boundary effects has its importance both in mathematics and physics. For the Boltzmann equation, it is well known that when the Knudsen number tends to zero, the Hilbert expansion and the Chapman–Enskog expansion give the Euler and Navier–Stokes equations respectively. This implies that at least smooth solutions to the Boltzmann equation can be well described by the classical fluid dynamical systems when the Knudsen number is sufficiently small. However, this is not true in the appearance of the boundary layer, shock layer or initial layer. In this section, we will consider the boundary layer for the Boltzmann equation in the simplest setting. The presentation follows closely from the recent works [20,68,69,71] on the existence and stability of the boundary layers to the Boltzmann equation. For the half space problem for the Boltzmann equation, please also refer to the recent survey paper [12]. And the dimension of the velocity space is set to be three in this section.

Recall the Boltzmann equation (1.1) in the form

$$f_t + \xi \cdot \nabla_x f = \frac{1}{\kappa} Q(f, f), \tag{2.1}$$

where κ is the Knudsen number. Assume that the boundary is a flat plane and the particle distribution function depends only on time t, velocity ξ and the distance of the particle from the boundary still denoted by x for simplicity of notations. Let ξ_1 be the component of the velocity in the direction of x, by using the scaling $x \to \frac{x}{\kappa}$, the leading term in equation (2.1) gives the stationary equation

$$\xi_1 F_x = Q(F, F), \tag{2.2}$$

which is the equation for the boundary layer.

The main purpose of this section is then to study the half-space (initial) boundary value problem of the nonlinear Boltzmann equation by assigning the Dirichlet data for the incoming particles at the boundary (see Section 1.3 for the incoming and outgoing particles) and a Maxwellian in the far field. This type of (initial) boundary value problems arise in the analysis of the kinetic boundary layer, the condensation–evaporation problem and other problems related to the kinetic behavior of the gas near the wall, [12,54].

An interesting feature of this problem is that not all Dirichlet data are admissible and the number of admissible conditions changes with the Mach number of the far field Maxwellian. This has been shown for the linear case by many authors [10,18,21,32], mainly in the context of the classical Milne and Kramer problems. A nonlinear admissible condition was derived for the discrete velocity model in [62] and the stability of steady solutions was proven in [48]. The full nonlinear problem was solved on the existence of solutions in [31] for the case of the specular reflection boundary condition, whose proof, however, does not work for the Dirichlet boundary condition, and in [5] for this case, but with the ambiguity that the far field Maxwellian cannot be fixed a priorly.

It should be mentioned that in [3,54,55] and references therein, an extensive numerical computation is made on the nonlinear problem. And the theoretical analysis so far has justified part of their numerical results.

The problem considered can be formulated as follows. Consider the boundary value problem for the stationary Boltzmann equation:

$$\begin{cases} \xi_1 F_x = Q(F,F), & x>0,\ \xi\in\mathbb{R}^3, \\ F|_{x=0} = F_b(\xi), & \xi_1>0,\ (\xi_2,\xi_3)\in\mathbb{R}^2, \\ F\to \mathbf{M}_\infty(\xi) \quad (x\to\infty), & \xi\in\mathbb{R}^3. \end{cases} \tag{2.3}$$

Here, Q is as in (1.2), and throughout this section, we assume Grad's cutoff hard potential (1.25). The second equation in (2.3) is the Dirichlet boundary condition (1.53). The Dirichlet data $F_b(\xi)$ can be assigned only for incoming particles ($\xi_1 > 0$), but not for outgoing ones ($\xi_1 < 0$) because, then, the problem becomes ill-posed as will be seen from the a priori estimate stated in the following discussion. This corresponds to the physical situation that one can control the incoming distribution but not the outgoing one, on the wall.

It is clear that the far field $\mathbf{M}_\infty(\xi)$ in the third equation of (2.3) cannot be assigned arbitrarily but must be a zero of Q, and hence a Maxwellian. Thus, we must take

$$\mathbf{M}_\infty = \mathbf{M}_{[\rho_\infty,u_\infty,T_\infty]}, \tag{2.4}$$

where the constants $\rho_\infty > 0, u_\infty = (u_{\infty,1},u_{\infty,2},u_{\infty,3}) \in \mathbb{R}^3$, and $T_\infty > 0$ are the only quantities that we can control. By a shift of the variable ξ in the direction orthogonal to the x-axis, we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$, and then, the sound speed and Mach number with sign of this equilibrium state are given by

$$c_\infty = \sqrt{\frac{5}{3}T_\infty}, \qquad \mathcal{M}^\infty = \frac{u_{\infty,1}}{c_\infty}, \tag{2.5}$$

respectively, see [19]. Note that the flow at infinity is incoming (respectively outgoing) if $\mathcal{M}^\infty < 0$ (respectively > 0) and supersonic (respectively subsonic) if $|\mathcal{M}^\infty| > 1$ (respectively < 1).

We will see that the Mach number $\mathcal{M}^\infty$ determines the structure of the solvability condition for the problem (2.3). Indeed, since $\mathbf{M}_\infty(\xi)$ is the "Dirichlet data" at $x=\infty$ and is imposed for all $\xi_1\in\mathbb{R}$, it is over-determined (ill-posed), and as a consequence, (2.3) may not be solvable unconditionally. Actually, we will show that the co-dimension of the manifold made of the admissible boundary data changes with the Mach number $\mathcal{M}^\infty$. More precisely, set

$$n^+ = \begin{cases} 0, & \mathcal{M}^\infty \in (-\infty,-1), \\ 1, & \mathcal{M}^\infty \in (-1,0), \\ 4, & \mathcal{M}^\infty \in (0,1), \\ 5, & \mathcal{M}^\infty \in (1,\infty). \end{cases} \tag{2.6}$$

The forthcoming existence theorems show that the problem (2.3) is solvable unconditionally for any F_b sufficiently close to $\mathbf{M}_\infty$ if $\mathcal{M}^\infty < -1$ but otherwise not. A physical explanation of this is that if $\mathcal{M}^\infty < -1$, any information near the boundary cannot affect the far field while if $\mathcal{M}^\infty > -1$, a part of it propagates to infinity and affect the far field.

In the numerical works of [3,54,55] and references therein, the Dirichlet data F_b is fixed to be the standard Maxwellian $\mathbf{M}_{[1,0,1]}(\xi)$ for $\xi_1 > 0$, and values of five parameters $(\rho_\infty, \mathcal{M}^\infty, u_{2,\infty}, u_{3,\infty}, T_\infty)$ of the far field Maxwellian (2.4) are sought numerically which admit "smooth" solutions connecting F_b and $\mathbf{M}_\infty$. The conclusion is that the set of such admissible values is, in the parameter space $\mathbb{R}^5$, a union of a 5-dimensional subdomain in the region $\mathcal{M}^\infty < -1$, a 4-dimensional surface when $-1 < \mathcal{M}^\infty < 0$ and a 1-dimensional curve when $0 < \mathcal{M}^\infty < 1$ whereas no solutions are found if $\mathcal{M}^\infty > 1$. The existence theorems of this section agree with this for the case $\mathcal{M}^\infty < 1$ in the sense that the above mentioned regions of admissible values have the co-dimension just equal to n^+ of (2.6) in $\mathbb{R}^5$. For the case $\mathcal{M}^\infty > 1$, $F_b = \mathbf{M}_{[1,0,1]}$ may not be on the manifold defined by the solvability condition and hence, no solutions.

In the second part of this section, we will study the asymptotic stability of the boundary layer solutions for the case of incoming and supersonic far filed by a combination of the energy method and a bootstrap argument.

Notice that the following analysis which relies on a detailed study on the corresponding linearized problem at $\mathbf{M}_\infty$ provides a new aspect of the linear problems discussed in [10, 18,21,32]. To present this, define a weight function:

$$\boldsymbol{W}_\beta(\xi) = \big(1 + |\xi|\big)^{-\beta}\big(\mathbf{M}_{[1,u_\infty,T_\infty]}(\xi)\big)^{1/2}, \tag{2.7}$$

with $\beta \in \mathbb{R}$. Set

$$F(x,\xi) = \mathbf{M}_\infty(\xi) + \boldsymbol{W}_0(\xi) f(x,\xi), \tag{2.8}$$

where $\boldsymbol{W}_0$ is the weight of (2.7) with $\beta = 0$. Then, the problem (2.3) reduces to

$$\begin{cases} \xi_1 f_x - \boldsymbol{L} f = \Gamma(f), & x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi), & \xi \in \mathbb{R}^3_+ = \mathbb{R}_+ \times \mathbb{R}^2, \\ f \to 0 \quad (x \to \infty), & \xi \in \mathbb{R}^3, \end{cases} \tag{2.9}$$

where $\mathbf{L}$ are Γ are as in (1.23). for the sake of convenience, they are reproduced here:

$$\boldsymbol{L} f = \boldsymbol{W}_0^{-1}\big\{Q(\mathbf{M}_\infty, \boldsymbol{W}_0 f) + Q(\boldsymbol{W}_0 f, \mathbf{M}_\infty)\big\},$$
$$\Gamma(f) = \boldsymbol{W}_0^{-1} Q(\boldsymbol{W}_0 f, \boldsymbol{W}_0 f),$$
$$a_0 = \boldsymbol{W}_0^{-1}(F_b - \mathbf{M}_\infty).$$

The linear operator $\boldsymbol{L}$ and the quadratic operator Γ are those in (1.23) associated with the Maxwellian $\mathbf{M}_\infty$.

Now, the linearized problem of (2.3) at $\mathbf{M}_\infty$ is just (2.9) with the term $\Gamma(f)$ dropped:

$$\begin{cases} \xi_1 f_x - \boldsymbol{L} f = 0, & x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi), & \xi \in \mathbb{R}^3_+, \\ f \to 0 \quad (x \to \infty), & \xi \in \mathbb{R}^3. \end{cases} \tag{2.10}$$

One of the consequences of the existence theorem given in this section implies that when $\mathcal{M}^\infty \neq 0, \pm 1$, there exist n^+ functions r_i, $1 \leqslant i \leqslant n^+$, of $L^2(\mathbb{R}^3_+, \xi_1\, d\xi)$ such that for any $a_0 \in R^\perp$ with

$$R = \operatorname{span}\{r_1, r_2, \ldots, r_{n^+}\},$$

the linearized problem (2.10) has a unique solution.

To compare with the previous existence results for the linearized problem given in [21], let $\mathcal{N} = \ker \boldsymbol{L}$ be as in (1.44). As was stated in Proposition 1.6,

$$\mathcal{N} = \operatorname{span}\{\psi_\alpha\}_{\alpha=0,1,\ldots,4} = \operatorname{span}\big\{\boldsymbol{W}_0(\xi)\phi_i(\xi)\big\}_{i=0,1,\ldots,4}, \tag{2.11}$$

where ϕ_i is a collision invariant as in (1.9) and thus $\mathcal{N}$ is a 5-dimensional subspace of $L^2(\mathbb{R}^3)$. Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ and let

$$\boldsymbol{P}_0 : L^2\big(\mathbb{R}^3\big) \to \mathcal{N}, \qquad \boldsymbol{P}_1 : L^2\big(\mathbb{R}^3\big) \to \mathcal{N}^\perp,$$

be the orthogonal projections. Define the operator

$$A = \boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0,$$

which is the 5-dimensional linear bounded self-adjoint operator. By assuming that the Mach number $\mathcal{M}^\infty \neq \pm 1$, or 0, we know that the matrix for $A = \boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0$ is invertible on the Range$(\boldsymbol{P}_0)$ and is given by

$$\begin{pmatrix} u^1_\infty & \sqrt{\frac{3}{5}} c_\infty & 0 & 0 & 0 \\ \sqrt{\frac{3}{5}} c_\infty & u^1_\infty & 0 & 0 & \sqrt{\frac{2}{5}} c_\infty \\ 0 & 0 & u^1_\infty & 0 & 0 \\ 0 & 0 & 0 & u^1_\infty & 0 \\ 0 & \sqrt{\frac{2}{5}} c_\infty & 0 & 0 & u^1_\infty \end{pmatrix},$$

where the entries are calculated by $(\chi_\alpha, \xi_1 \chi_\beta)$, $\alpha, \beta = 0, \ldots, 4$, with $\{\chi_\alpha, \alpha = 0, \ldots, 4\}$ being an orthonormal basis of $\{\psi_\alpha, \alpha = 0, \ldots, 4\}$. The eigenvalues of the above matrix are

$$\lambda_1 = u_{\infty,1} - c_\infty, \qquad \lambda_i = u_{\infty,1}, \quad i = 2, 3, 4, \qquad \lambda_5 = u_{\infty,1} + c_\infty. \tag{2.12}$$

Define

$$I^+ = \{j \mid \lambda_j > 0\}, \qquad I^- = \{j \mid \lambda_j < 0\}.$$

Note that n^+ of (2.6) is nothing but $\#I^+$. Let χ_j be the eigenfunction corresponding to the eigenvalue λ_j. In [21], it is shown that for any $a_0 \in L^2(\mathbb{R}^3_+, \xi_1\, d\xi)$ and for any constants c_j, $j \in I^-$, there exists a unique L^2 solution f satisfying the first two equations of (2.10) and instead of the last one, the auxiliary condition

$$\langle \chi_j, f(x,\cdot)\rangle = c_j, \quad x > 0,\ j \in I^-. \tag{2.13}$$

Moreover, there exists an element $q_\infty \in \mathcal{N}$ such that

$$f \to q_\infty \quad (x \to \infty) \text{ in } L^2\big(\mathbb{R}^3\big).$$

To determine the limit q_∞, we only need to notice that (2.10) is linear and that $q_\infty \in \mathcal{N} = \ker \boldsymbol{L}$. Thus, $\tilde{f} = f - q_\infty$ solves all of three equations in (2.10) with a_0 replaced by $a_0 - q_\infty$. That is, $a_0 - q_\infty \in R^\perp$ which determines q_∞ uniquely together with (2.13).

Finally in the introduction, we want to point out some interesting mathematical problems on the boundary layers which remain unsolved. Firstly, the existence theorem presented here apply only to the case when the Mach number of the far field Maxwellian is not ± 1 or 0. The case with $\mathcal{M}^\infty = \pm 1$ corresponds to the transonic problem while the case with $\mathcal{M}^\infty = 0$ corresponds to the transition between incoming and outgoing boundary conditions. Therefore, the phenomena of the boundary layer in these cases are richer and the analysis should be more difficult, cf. [56]. Recently, this issue is addressed in [29]. Secondly, the stability given in this section is only for the case when $\mathcal{M}^\infty < -1$. For the case when $\mathcal{M}^\infty > 1$, similar analysis combining the energy method and the Green function can be applied. However, for other cases, more detailed information on the wave propagation is needed, for example, from the study of the Green function given in [45]. Thirdly, for the time evolutionary problem, it is natural to consider the pattern of the superposition of the boundary layer and the macroscopic waves. For this kind of problems, even though there are stability results on the boundary layer and the wave patterns separately, the combination of the analysis on these two topics is not clear because of the different mathematical settings. The last but not least, we only consider the one-dimensional boundary layer problems with Dirichlet type condition. In practice, the shape of the boundary and the form of the boundary condition can be various. Therefore, there are many unsolved problems such as those in multi-dimensional space and those with different boundary conditions. And we expect the analysis in this section shed some light on future investigation on these problems.

2.2. *Existence*

In this subsection, we will present the existence theory for the half-space boundary value problem (2.3). It will be shown that the solvability of the problem changes with the Mach number $\mathcal{M}^\infty$ of the far field Maxwellian. If $\mathcal{M}^\infty < -1$, there exists a unique smooth solution connecting the Dirichlet data and the far field Maxwellian as long as they are sufficiently close. Otherwise, such a solution exists only for the Dirichlet data satisfying certain admissible conditions. The set of admissible Dirichlet data form a smooth manifold of co-dimension 1 for the case $-1 < \mathcal{M}^\infty < 0$, 4 for $0 < \mathcal{M}^\infty < 1$ and 5 for $\mathcal{M}^\infty > 1$, respectively. The same is also true for the linearized problem at the far field Maxwellian, and the manifold is, then, a hyperplane.

There are two ingredients in the proof of the existence theorem. One is to add an artificial "damping" term and the other is to introduce some weight functions which are different for the hard sphere model and the cutoff hard potentials. The artificial "damping" term which vanishes under the solvability condition is used to control the incoming macroscopic information, while the weight function is used to capture the dissipation on the outgoing macroscopic information through the convection term.

To construct the damping term, we decompose the operator A on $\mathcal{N}$ into the positive and negative parts A^+, A^-, and denote the corresponding eigenprojections by $\boldsymbol{P}_0^+$, $\boldsymbol{P}_0^-$. Note that if $\mathcal{M}^\infty \neq 0, \pm 1$, then A has no zero eigenvalues (see (2.12)), so that

$$A = A^+ + A^-, \qquad \boldsymbol{P}_0 = \boldsymbol{P}_0^+ + \boldsymbol{P}_0^-.$$

We modify the equation in (2.9) by adding to its right hand side a damping term

$$-\gamma K(x) \boldsymbol{P}_0^+ \xi_1 f, \quad \gamma > 0,$$

and then put $f = e^{-\epsilon\sigma} g$, to deduce

$$\begin{cases} \xi_1 g_x - \epsilon\sigma_x \xi_1 g - \boldsymbol{L}_\epsilon g = h - \gamma K(x) e^{\epsilon\sigma} \boldsymbol{P}_0^+ \xi_1 e^{-\epsilon\sigma} g, \\ \quad x > 0,\ \xi \in \mathbb{R}^3, \\ g|_{x=0} = b_0(\xi) = a_0(\xi) e^{-\epsilon\sigma}|_{x=0}, \quad \xi \in \mathbb{R}^3_+, \\ g \to 0\ (x \to \infty), \quad \xi \in \mathbb{R}^3, \end{cases} \tag{2.14}$$

where $\epsilon > 0$ is a small constant, σ may be a function of both x and ξ, $K(x)$ is a function of x defined later and

$$\boldsymbol{L}_\epsilon = e^{\epsilon\sigma} \boldsymbol{L} e^{-\epsilon\sigma}, \qquad h = e^{\epsilon\sigma} \Gamma\big(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g\big). \tag{2.15}$$

Note that for the case $\mathcal{M}^\infty < -1$, we have $n^+ = 0$ and $\boldsymbol{P}_0^+ = 0$, and hence no damping term is needed. The problem for the hard sphere model and the hard potentials with angular cutoff will be discussed separately as follows.

2.2.1. *Hard sphere model.* For the hard sphere model, we can choose the weight function $\sigma = x$ and $K(x) = 1$ so that $a_0 = b_0$, cf. [68].

As the first step, we consider the corresponding linear problem (2.14) by viewing h as a given source satisfying

$$\boldsymbol{P}_0 h = 0, \qquad \|h\| = \|h\|_{L^2_{x,\xi}(\mathbb{R}^+ \times \mathbb{R}^3)} < \infty. \tag{2.16}$$

The idea of proving global existence of solutions to the linear problem is to introduce a linear functional on a subspace of $L^2_{x,\xi}$. Then, we will show, using the energy estimates, that the linear functional is bounded. Finally, the existence of the weak solution is obtained by the Riesz representation theory. The linear functional is defined on a linear space $\mathbb{W}$ as follows:

$$\ell(\chi) = (h, \phi) + \langle \xi_1 a_0, \phi^0 \rangle_+, \tag{2.17}$$

where $(\,,\,)$ is the inner product in $L^2_{x,\xi}(\mathbb{R}^+ \times \mathbb{R}^3)$ and $\langle\,,\,\rangle_\pm$ is the inner product in $L^2_\xi(\mathbb{R}^3_\pm)$, and

$$\begin{cases} V \equiv \{\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^3) \mid \phi^0 = \phi|_{x=0} = 0 \text{ for } \xi_1 < 0\}, \\ \mathbb{W} \equiv \{\chi \mid \chi = -\xi_1 \phi_x - \epsilon \xi_1 \phi + \gamma P_0^+ \xi_1 \phi - \boldsymbol{L}\phi,\ \phi \in V\}. \end{cases} \tag{2.18}$$

Here, V is the space of test functions for defining the weak solutions and $\mathbb{W}$ is the base set on which the functional ℓ is defined.

First, we shall show that ℓ is well defined on $\mathbb{W}$. To this end, recall from Lemma 1.7 that for the hard sphere model, the H-theorem implies that

$$-\langle f, \boldsymbol{L} f \rangle \geqslant \nu_1 \langle f, (1 + |\xi|) f \rangle,$$

for some constant $\nu_1 > 0$ and $f \in \mathcal{N}^\perp$.

To prove the linear functional ℓ is well defined and bounded on $\mathbb{W}$ when $\gamma > \epsilon > 0$ are sufficiently small, consider the inner product of $\chi \in \mathbb{W}$ and $\phi \in V$:

$$(\chi, \phi) = \langle \xi_1 \phi^0, \phi^0 \rangle_+ - \epsilon(\xi_1 \phi, \phi) + \gamma (\boldsymbol{P}_0^+ \xi_1 \phi, \phi) - (\boldsymbol{L}\phi, \phi). \tag{2.19}$$

Notice that none of the eigenvalues of A is zero when $\mathcal{M}^\infty \neq \pm 1$ or 0. Thus,

$$\begin{aligned} &-\epsilon(\xi_1 \phi, \phi) + \gamma (\boldsymbol{P}_0^+ \xi_1 \phi, \phi) \\ &\quad = \epsilon \big[(-\boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0 \phi_0, \phi_0) - 2(\boldsymbol{P}_0 \xi_1 \phi_1, \phi_0) + (\xi_1 \phi_1, \phi_1) \big] \\ &\qquad + \gamma \big[(A_+ \phi_0, \phi_0) + (\boldsymbol{P}_0^+ \xi_1 \phi_1, \phi_0) \big] \\ &\quad = \epsilon(-A_- \phi_0, \phi_0) + (\gamma - \epsilon)(A_+ \phi_0, \phi_0) - 2\epsilon(\boldsymbol{P}_0 \xi_1 \phi_1, \phi_0) \\ &\qquad + \gamma (\boldsymbol{P}_0^+ \xi_1 \phi_1, \phi_0) + \epsilon(\xi_1 \phi_1, \phi_1) \\ &\quad \geqslant c\epsilon \|\phi_0\|^2 - c^{-1} \epsilon \|\phi_1\|^2 + \epsilon(\xi_1 \phi_1, \phi_1), \end{aligned} \tag{2.20}$$

where we have used the assumption that $\gamma = 0(1)\epsilon$ for some positive constant $0(1) > 1$. Here, $\phi_i = \boldsymbol{P}_i \phi$, $i = 0, 1$, and from now on c denotes a generic constant.

Since $(\boldsymbol{L}\phi, \phi) = (\boldsymbol{L}\phi_1, \phi_1)$, for sufficiently small $\epsilon > 0$, we have

$$(\chi, \phi) \geqslant c \big(\|\phi\|^2 + \langle \xi_1 \phi^0, \phi^0 \rangle_+ \big).$$

Therefore, we have

$$\|\phi\| + \langle \xi_1 \phi^0, \phi^0 \rangle_+^{\frac{1}{2}} \leqslant c \|\chi\|,$$

which implies, firstly, that for any given $\chi \in \mathbb{W}$, there exists a unique $\phi \in V$ such that

$$\chi = -\epsilon\xi_1\phi + \gamma P_0^+\xi_1\phi - \xi_1\phi_x - \boldsymbol{L}\phi,$$

so that ℓ make sense on $\mathbb{W}$, and, secondly, that

$$\begin{aligned} &\bigl|\ell(\chi)\bigr| \leqslant \bigl|(h,\phi)\bigr| + \bigl\langle\xi_1 a_0, \phi^0\bigr\rangle_+ \leqslant c\bigl(\|h\| + \langle\xi_1 a_0, a_0\rangle_+\bigr)\|\chi\|, \\ &\quad \text{for any } \chi \in \mathbb{W}, \end{aligned} \tag{2.21}$$

showing that ℓ is bounded on $\mathbb{W}$ and the bound depends on the boundary data and the given function h.

Now, the linear functional ℓ can be defined on the $L^2_{x,\xi}$ closure of $\mathbb{W}$ with the same bound, and hence, by the Hahn–Banach theorem, there is an extension $\bar{\ell}$ of ℓ to the space $L^2_{x,\xi}$ such that

$$\bigl|\bar{\ell}(\chi)\bigr| \leqslant c\bigl(\|h\| + \langle\xi_1 a_0, a_0\rangle_+\bigr)\|\chi\|, \quad \text{for any } \chi \in L^2_{x,\xi},$$

with the bound unchanged. For this functional $\bar{\ell}$ on L^2, we can apply the Riesz representation theorem so that there exists a unique $g \in L^2_{x,\xi}$ such that

$$\bar{\ell}(\chi) = (g,\chi), \quad \text{for any } \chi \in L^2_{x,\xi}. \tag{2.22}$$

And this implies that g is a weak solution to the linear equation (2.14). To prove both g_x for almost all (x,ξ) and the trace of g at $x = 0$ are well defined, choose a family of particular test functions:

$$\phi = \int_x^\infty \eta(x',\xi)\,dx',$$

where η satisfies $\int_0^\infty \eta(x',\xi)\,dx' = 0$ for any ξ. Applying this test function to (2.14) yields

$$\left(\int_0^x \bigl(-\epsilon\xi_1 g + \gamma P_0^+\xi_1 g - \boldsymbol{L}g - h\bigr)\,dx' + \xi_1 g, \eta\right) = 0,$$

which gives

$$\int_0^x \bigl(-\epsilon\xi_1 g + \gamma P_0^+\xi_1 g - \boldsymbol{L}g - h\bigr)\,dx' + \xi_1 g = b(\xi),$$

for almost all (x,ξ), where $b(\xi)$ is a function of ξ only. Therefore, the trace of g at $x = 0$ is well defined and we have (2.14) for almost all (x,ξ).

In order to prove the uniqueness of g, we need to show that $|\xi_1|^{\frac{1}{2}} g \in L^2$, cf. [10,21] for similar discussion. So far we know that $g \in L^2_{x,\xi}$. To obtain the uniform boundedness on $(|\xi_1|g, g)$, choose a smooth cut-off function $\theta(\xi)$ satisfying that

$$\theta(\xi) = 0, \quad \text{for } |\xi| > M, \qquad \theta(\xi) = 1, \quad \text{for } |\xi| < M - 1,$$

and is monotone when $M-1 \leqslant |\xi| \leqslant M$.

Applying θg to equation (2.14) and then integrating over (x,ξ) give,

$$-\epsilon(\theta\xi_1 g, g) + \gamma\big(\theta P_0^+\xi_1 g, g\big) - \langle\theta\xi_1 g, g\rangle|_{x=0} - (\theta \boldsymbol{L} g, g) = (\theta h, g). \qquad (2.23)$$

Since

$$-(\theta \boldsymbol{L} g, g) = (\theta\nu g, g) - (\theta K g, g),$$

we have for sufficiently small ϵ,

$$\big(\theta|\xi_1|g, g\big) + \big\langle\theta|\xi_1|g, g\big\rangle_{x=0,\xi_1<0} \leqslant c\big(\|h\|^2 + \|g\|^2 + \langle\xi_1 a_0, a_0\rangle_+\big). \qquad (2.24)$$

By letting $M \to \infty$, the above inequality implies that $(|\xi_1|g, g)$ is uniformly bounded. Therefore, the energy estimate similar to the one given above gives the uniqueness of the solution g to (2.14).

Finally, we will show that the solution obtained indeed satisfies the given boundary condition. In fact, if we take the inner product of equation (2.14) with $\phi \in V$, we have after integration by parts

$$\big(g, -\xi_1\phi_x - \epsilon\xi_1\phi + \gamma P_0^+\xi_1\phi - \boldsymbol{L}\phi\big) = (h,\phi) + \big\langle\xi_1 g_0, \phi^0\big\rangle_+. \qquad (2.25)$$

By the definition of χ, (2.17) and (2.22), we have

$$(h,\phi) + \big\langle\xi_1 g^0, \phi^0\big\rangle_+ = (h,\phi) + \big\langle\xi_1 a_0, \phi^0\big\rangle_+,$$

which gives

$$g^0 = a_0, \quad \text{for } \xi_1 > 0, \text{ a.e.}$$

In summary, we have the following existence theorem for the linear equation with damping (2.14).

THEOREM 2.1. *Consider the linear problem* (2.14), *if the boundary condition a_0 and the source term h satisfy*

$$\langle\xi_1 a_0, a_0\rangle_+^{\frac{1}{2}} + (h,h)^{\frac{1}{2}} < \infty,$$

then there exists a unique solution $g \in L^2_{x,\xi}$ with

$$\big\langle|\xi_1|g^0, g^0\big\rangle_-^{\frac{1}{2}} + \big\|\big(1+|\xi|\big)^{\frac{1}{2}} g\big\| + \big\||\xi_1|\big(1+|\xi|\big)^{-\frac{1}{2}} g_x\big\| \leqslant c_0\big(\big||\xi_1|^{\frac{1}{2}} a_0\big|_+ + \|h\|\big).$$

Now we turn to study the nonlinear Boltzmann equation with damping, i.e. problem (2.14) and (2.15). As for the discussion on the Cauchy problem, we use the weighted norm:

$$\|f\|_\beta = \big\|\big(1+|\xi|\big)^\beta f\big\|_{L^\infty_{x,\xi}} = \sup_{x>0,\,\xi\in\mathbb{R}^3}\big(1+|\xi|\big)^\beta\big|f(x,\xi)\big|.$$

Notice that the norm used here is consistent with weighted function $\boldsymbol{W}_\beta$. In order to obtain the estimate on $\|g\|_\beta$, we will use the following weight $w_{-\alpha}$ in the analysis from the L^2 estimates of the solution for the linearized problem to the $\|\cdot\|_\beta$ estimates of the solution for the nonlinear problem. Set

$$w_{-\alpha} = \begin{cases} |\xi_1|^{-\alpha}, & |\xi_1|<1,\\ 1, & |\xi_1|\geqslant 1.\end{cases} \tag{2.26}$$

Precisely, we will prove the following lemma.

LEMMA 2.2. *For* $0<\alpha<\frac{1}{2}$, $\beta>\frac{3}{2}$, *the solution to the problem* (2.14) *satisfies*

$$\|g\|_\beta \leqslant c\big(\big\|\nu^{-1}h\big\|_\beta + \|h\| + \|w_{-\alpha}h\| + |a_0|_{+,\beta}\big), \tag{2.27}$$

where

$$|a_0|_{+,\beta} = \sup_{\xi\in\mathbb{R}^3,\xi_1>0}\big(1+|\xi|\big)^\beta\big|a_0(\xi)\big|. \tag{2.28}$$

PROOF. Rewrite (2.14) as

$$g_x = \left(\epsilon - \frac{\nu(\xi)}{\xi_1}\right)g + \frac{1}{\xi_1}(\bar{K}g+h),$$

where

$$\bar{K} = K - \gamma \boldsymbol{P}_0^+\xi_1.$$

Notice that K is a compact operator and is bounded from the space L^∞_β to $L^\infty_{\beta+1}$, and from L^2_ξ to L^∞_ξ, see Section 1.2.3. Since the operator $\boldsymbol{P}_0^+\xi$ is a mapping to the macroscopic subspace which is of finite dimension, it has the same properties as K stated above. Let

$$\kappa(x,\xi) = \left(-\epsilon + \frac{\nu(\xi)}{\xi_1}\right)x.$$

We have $\kappa(x,\xi)>0$ both for $x>0,\ \xi_1>0$ and $x<0,\ \xi_1<0$ when $\epsilon>0$ is sufficiently small. Therefore, the solution g can be formally written as

$$g = \tilde{a} + U(\bar{K}g+h), \tag{2.29}$$

where

$$\tilde{a} = \begin{cases} \exp(-\kappa(x,\xi))a_0(\xi), & \xi_1 > 0, \\ 0, & \xi_1 < 0, \end{cases} \tag{2.30}$$

and

$$U(h) = \begin{cases} \int_0^x \exp(-\kappa(x-x',\xi))\frac{1}{\xi_1}h(x',\xi)\,dx', & \xi_1 > 0, \\ -\int_x^\infty \exp(-\kappa(x-x',\xi))\frac{1}{\xi_1}h(x',\xi)\,dx', & \xi_1 < 0. \end{cases} \tag{2.31}$$

We first show that the operator U has the following two properties:

$$\begin{aligned} &\left\|U(h)(\cdot,\xi)\right\|_{L^p_x} \leqslant c\nu(\xi)^{-1}\left\|h(\cdot,\xi)\right\|_{L^p_x}, \\ &\left\|U(h)\right\|_{L^r_\xi(L^p_x)} \leqslant c\left\|\nu^{-1}h\right\|_{L^r_\xi(L^p_x)}, \end{aligned} \tag{2.32}$$

where $1 \leqslant p, r \leqslant \infty$. To prove (2.32), rewrite $U(h)$ as:

$$U(h) = \int_{I_{\xi_1}} S(x-x',\xi)h(x',\xi)\,dx',$$

where

$$I_{\xi_1} = \begin{cases} [0,x], & \text{for } \xi_1 > 0, \\ [x,\infty), & \text{for } \xi_1 < 0, \end{cases}$$

and

$$S(x'',\xi) = \begin{cases} \exp(-(-\epsilon + \frac{\nu(\xi)}{\xi_1})x'')\frac{1}{\xi_1}, & \text{for } x'' > 0,\ \xi_1 > 0, \\ -\exp(-(-\epsilon + \frac{\nu(\xi)}{\xi_1})x'')\frac{1}{\xi_1}, & \text{for } x'' < 0,\ \xi_1 < 0. \end{cases}$$

Thus,

$$\begin{aligned} &\int_{I_{\xi_1}} \left|S(x-x',\xi)\right| dx' \\ &\quad = \begin{cases} \int_0^x \exp(-(-\epsilon + \frac{\nu(\xi)}{\xi_1}x'')\frac{1}{\xi_1}\,dx'' \leqslant \frac{1}{\nu(\xi)-\epsilon\xi_1}, & \xi_1 > 0, \\ \int_0^\infty \exp(-(-\epsilon + \frac{\nu(\xi)}{\xi_1})x'')\frac{1}{|\xi_1|}\,dx'' = \frac{1}{\nu(\xi)+\epsilon|\xi_1|}, & \xi_1 < 0. \end{cases} \end{aligned} \tag{2.33}$$

Hence for sufficiently small ϵ, we have

$$\int_{I_{\xi_1}} \left|S(x-x',\xi)\right| dx' \leqslant \frac{c}{\nu(\xi)}.$$

Therefore, for positive integers $p, q \geqslant 1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}\left|U(h)(x,\xi)\right| \leqslant &\left(\int_{I_{\xi_1}}\left|S(x-x',\xi)\right|dx'\right)^{\frac{1}{q}}\\ &\times\left(\int_{I_{\xi_1}}\left|S(x-x',\xi)\right|\left|h(x',\xi)\right|^p dx'\right)^{\frac{1}{p}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^\infty\left|U(h)(x,\xi)\right|^p dx &\leqslant c\nu(\xi)^{-\frac{p}{q}}\int_0^\infty\int_{I_{\xi_1}}\left|S(x-x',\xi)\right|\left|h(x',\xi)\right|^p dx'\,dx\\ &\leqslant c\nu(\xi)^{-\frac{p}{q}}\int_0^\infty\int_{I'_{\xi_1}}\left|S(x-x',\xi)\right|\left|h(x',\xi)\right|^p dx\,dx'\\ &\leqslant c\nu^{-\frac{p}{q}-1}(\xi)\int_0^\infty\left|h(x',\xi)\right|^p dx',\end{aligned}$$

where

$$I'_{\xi_1}=\begin{cases}[x',\infty), & \xi_1>0,\\ [0,x'], & \xi_1<0.\end{cases} \tag{2.34}$$

And this gives (2.32).

By using the properties of the operators U and $\bar{K}$, we have

$$\begin{aligned}\|g\|_\beta &\leqslant |a_0|_{+,\beta}+c\left(\left\|\nu^{-1}\bar{K}g\right\|_\beta+\left\|\nu^{-1}h\right\|_\beta\right)\\ &\leqslant |a_0|_{+,\beta}+c\left(\|g\|_{\beta-1}+\left\|\nu^{-1}h\right\|_\beta\right).\end{aligned} \tag{2.35}$$

Iterating (2.35) gives for $\beta \geqslant 1$,

$$\|g\|_\beta \leqslant c\left(|a_0|_{+,\beta}+\|g\|_0+\left\|\nu^{-1}h\right\|_\beta\right). \tag{2.36}$$

It remains to estimate $\|g\|_0=\|g\|_{L^\infty_{x,\xi}}$. By the energy estimate applied to the linear equation (2.14), we have

$$\|g\|^2 \leqslant c\left(\|h\|^2+\langle\xi_1 a_0, a_0\rangle_+\right)\equiv E_2^2. \tag{2.37}$$

The estimate on $\|g\|_0$ will be obtained by using the properties of the collision kernel and the properties of the operators U and $\bar{K}$.

Let θ be the cutoff function in ξ used above and $w=w(\xi)\geqslant 0$ be a weight function. Applying $\theta^2 w^2 g$ to (2.14) and integrating it over x and ξ give

$$\left\langle|\xi_1|\theta^2 w^2 g, g\right\rangle_- - \epsilon\left(\xi_1\theta^2 w^2 g, g\right)+\left(\nu\theta^2 w^2 g, g\right)$$

$$= \big(\bar{K}g, \theta^2 w^2 g\big) + \big(h, \theta^2 w^2 g\big) + \big\langle \xi_1 \theta^2 w^2 a_0, a_0 \big\rangle_+,$$

which implies

$$\begin{aligned} &\big||\xi_1|^{\frac{1}{2}}\theta w g\big|_-^2 + \big\|\nu^{\frac{1}{2}}\theta w g\big\|^2 \\ &\quad\leqslant c\Big(\|\theta w \bar{K} g\|\|\theta w g\| + \|\theta w h\|\|\theta w g\| + \big||\xi_1|^{\frac{1}{2}}\theta w a_0\big|_+^2\Big). \end{aligned} \tag{2.38}$$

If $w\bar{K}$ is bounded from $L^2_{x,\xi}$ to itself, $wh \in L^2_{x,\xi}$ and $|\xi_1|^{\frac{1}{2}} w a_0 \in L^2_{\xi,+}$, then by letting $\theta(\xi) \to 1$, we have

$$\big\|\nu^{\frac{1}{2}} w g\big\| \leqslant c\Big(E_2 + \|wh\| + \big\||\xi_1|^{\frac{1}{2}} w a_0\big\|_-\Big). \tag{2.39}$$

Thus, if $w(\xi) \equiv 1$, then

$$\big\|\nu^{\frac{1}{2}} g\big\|_{L^2_{x,\xi}} \leqslant c E_2. \tag{2.40}$$

Notice that for $\alpha < 1$,

$$\int \bar{K}(\xi, \xi') w_{-\alpha}(\xi')\, d\xi' \leqslant c.$$

We have both operators $w_{-\alpha}\bar{K}$ and $\bar{K} w_{-\alpha}$ are bounded from L^2_ξ to L^2_ξ when $\alpha < \frac{1}{2}$. Hence they are bounded from $L^2_\xi(L^\infty_x)$ to $L^2_\xi(L^\infty_x)$. Therefore, if we choose $w = w_{-\alpha}(\xi)$ in (2.39) for $0 < \alpha < \frac{1}{2}$,

$$\|w_{-\alpha} g\| \leqslant c\Big(E_2 + \|w_{-\alpha} h\| + \big\||\xi_1|^{\frac{1}{2}} w_{-\alpha} a_0\big\|\Big) = c E_2'. \tag{2.41}$$

By using equation (2.14) and (2.40), we have

$$\|w_1 g_x\| \leqslant c E_2. \tag{2.42}$$

Let $\frac{1}{4} < \beta < \frac{1}{2}$, i.e., $0 < 1 - 2\beta < \frac{1}{2}$, we have

$$\begin{aligned} \|w_\beta g\|^2_{L^\infty_x(L^2_\xi)} &\leqslant \|w_\beta g\|^2_{L^2_\xi(L^\infty_x)} \\ &\leqslant 2\int \big\{w_\beta^2 w_1^{-1}\big\|g(\cdot,\xi)\big\|_{L^2_x}\big\}\big\{w_1\big\|g(\cdot,\xi)\big\|_{L^2_x}\big\}\, d\xi \leqslant c E_2' E_2. \end{aligned} \tag{2.43}$$

Now by the expression (2.29) of g and using (2.43), we have

$$\begin{aligned} \|g\|_{L^2_\xi(L^\infty_x)} &\leqslant \|a_0\|_+ + c\Big(\big\|\nu^{-1}\bar{K} w_{-\beta} w_\beta g\big\|_{L^2_\xi(L^\infty_x)} + \big\|\nu^{-1} h\big\|_{L^2_\xi(L^\infty_x)}\Big) \\ &\leqslant \|a_0\|_+ + c\Big(\|w_\beta g\|_{L^2_\xi(L^\infty_x)} + \big\|\nu^{-1} h\big\|_{L^2_\xi(L^\infty_x)}\Big) \end{aligned}$$

$$\leqslant \|a_0\|_+ + c\big(E_2 + E_2' + \|\nu^{-1}h\|_{L^2_\xi(L^\infty_x)}\big). \tag{2.44}$$

By using the expression of g again and noticing that $\bar{K}$ is a bounded operator from $L^2_\xi(L^\infty_x)$ to $L^\infty_{x,\xi}$, we have

$$\begin{aligned}
\|g\|_{L^\infty_{x,\xi}} &\leqslant \|a_0\|_{+,L^\infty_\xi} + \big\|U(\bar{K}g + h)\big\|_{L^\infty_{x,\xi}} \\
&\leqslant \|a_0\|_{+,L^\infty_\xi} + c\big(\|\bar{K}g\|_{L^\infty_{x,\xi}} + \|\nu^{-1}h\|_{L^\infty_{x,\xi}}\big) \\
&\leqslant \|a_0\|_{+,L^\infty_\xi} + c\big(\|g\|_{L^2_\xi(L^\infty_x)} + \|\nu^{-1}(\xi)h\|_{L^\infty_{x,\xi}}\big) \\
&\leqslant |a_0|_{+,L^\infty_\xi} + c\big(E_2 + E_2' + \|\nu^{-1}h\|_{L^2_\xi(L^\infty_x)} + \|\nu^{-1}h\|_{L^\infty_{x,\xi}}\big).
\end{aligned} \tag{2.45}$$

For $\beta - 1 > \frac{3}{2}$, by using (2.45), we have

$$\begin{aligned}
\|g\|_\beta &\leqslant c\big(|a_0|_{+,\beta} + E_2 + E_2' + \|\nu^{-1}h\|_\beta + \|\nu^{-1}h\|_{L^2_\xi(L^\infty_x)\cap L^\infty_{x,\xi}}\big) \\
&\leqslant c\big(|a_0|_{+,\beta} + \|h\|_{L^2_{x,\xi}} + \|w_{-\alpha}h\| + \|\nu^{-1}h\|_\beta + \|\nu^{-1}h\|_{L^2_\xi(L^\infty_x)\cap L^\infty_{x,\xi}}\big) \\
&\leqslant c\big(|a_0|_{+,\beta} + \|\nu^{-1}h\|_\beta + \|h\| + \|w_{-\alpha}h\|\big).
\end{aligned} \tag{2.46}$$

This completes the proof of the lemma. □

Now we are ready to prove the existence of solutions to the nonlinear Boltzmann equation with damping (2.14) and (2.15). For this, we need the following property of $\Gamma(g,h)$ which corresponds to Theorem 1.1 when $\delta = 1$ and $p = \infty$.

LEMMA 2.3. *The projection of $\Gamma(g,h)$ on the null space of $\mathbf{L}$ vanishes and there exists a positive constant c such that*

$$\|\nu^{-1}\Gamma(g,h)\|_\beta \leqslant c\|h\|_\beta\|g\|_\beta, \tag{2.47}$$

for any $\beta > 0$.

Therefore, for $\beta > 5/2$, we have

$$\|\exp(-\epsilon x)\nu^{-1}\Gamma(g)\|_\beta \leqslant c\|g\|^2_\beta, \qquad \|\exp(-\epsilon x)\nu^{-1}\Gamma(g)\| \leqslant c\|g\|^2_\beta.$$

Moreover, for $0 < \alpha < \frac{1}{2}$, we have

$$\|w_{-\alpha}\exp(-\epsilon x)\Gamma(g)\| \leqslant c\|g\|^2_\beta.$$

In summary, (2.27) gives

$$\|g\|_\beta \leqslant c\big(\|g\|^2_\beta + |a_0|_{+,\beta}\big). \tag{2.48}$$

When $|a_0|_{+,\beta}$ is sufficiently small, the contraction mapping theorem and (2.48) give the following existence theorem for the problem (2.14) and (2.15).

THEOREM 2.4. *When the boundary data a_0 is sufficiently small in the norm $|\cdot|_{+,\beta}$ given in* (2.28), *the nonlinear Boltzmann equation with damping* (2.14) *and* (2.15) *has a unique solution g which is bounded in the norm $\|\cdot\|_\beta$ when $\beta > \frac{5}{2}$.*

In what follows, we will show that under some solvability condition, the solution to the Boltzmann equation with damping is in fact the solution to the Boltzmann equation itself. That is, we will prove the following theorem.

THEOREM 2.5. *For hard sphere model, let $\mathbf{M}_\infty$ be given as in* (2.4) *with $\mathcal{M}^\infty \neq 0, \pm 1$, and let $\beta > 5/2$. Then, there exist positive numbers $\epsilon_0, \epsilon, C_0$, and a C^1 map*

$$\Psi : L^2\big(\mathbb{R}^3_+, \xi_1\, d\xi\big) \to \mathbb{R}^{n+}, \quad \Psi(0) = 0, \tag{2.49}$$

such that the following holds.

(i) *For any F_b satisfying*

$$\big|F_b(\xi) - \mathbf{M}_\infty(\xi)\big| \leqslant \epsilon_0 \boldsymbol{W}_\beta(\xi), \quad \xi \in \mathbb{R}^3_+, \tag{2.50}$$

and

$$\Psi\big(\boldsymbol{W}_0^{-1}(F_b - \mathbf{M}_\infty)\big) = 0, \tag{2.51}$$

the problem (2.3) *has a unique solution $F(x, \xi)$ satisfying*

$$\begin{aligned} &\big|F(x,\xi) - \mathbf{M}_\infty(\xi)\big| + \big|\xi_1\big(1 + |\xi|\big)^{-1/2} F_x(x,\xi)\big| \\ &\quad \leqslant C_0 e^{-\epsilon x} \boldsymbol{W}_\beta(\xi), \quad x > 0,\ \xi \in \mathbb{R}^3. \end{aligned} \tag{2.52}$$

(ii) *The set of F_b satisfying* (2.50) *and* (2.51) *forms a* (*local*) *C^1 manifold of codimension n^+.*

To prove Theorem 2.5, we denote by $\mathbb{I}^\gamma$ the linear solution operator as follows:

$$\mathbb{I}^\gamma(a_0) \equiv f(0, \cdot), \tag{2.53}$$

where $f(x, \xi)$ is given by

$$\begin{cases} \xi_1 f_x = \boldsymbol{L} f - \gamma \boldsymbol{P}_0^+ \xi_1 f, \\ f(0,\xi) = a_0(\xi), \quad \text{for } \xi_1 > 0. \end{cases}$$

Similarly, let $\mathcal{I}^\gamma$ be a nonlinear solution operator

$$\mathcal{I}^\gamma(a_0) \equiv f(0, \cdot), \tag{2.54}$$

where $f(x,\xi)$ is given by

$$\begin{cases} \xi_1 f_x = \boldsymbol{L} f - \gamma \boldsymbol{P}_0^+ \xi_1 f + \Gamma(f,f), \\ f(0,\xi) = a_0(\xi), \quad \text{for } \xi_1 > 0. \end{cases} \tag{2.55}$$

For these two operators $\mathbb{I}^\gamma$ and $\mathcal{I}^\gamma$, we first have the following lemma.

LEMMA 2.6. *The solution operators $\mathbb{I}^\gamma$ and $\mathcal{I}^\gamma$ have the following property.*

$$\begin{cases} \mathbb{I}^\gamma : a_0 \in L^2_{\xi_1,+} \mapsto \mathbb{I}^\gamma(a_0) \in L^2_{|\xi_1|} \text{ is bounded}, \\ \mathcal{I}^\gamma : a_0 \in L^2_{\xi_1,+} \mapsto \mathcal{I}^\gamma(a_0) \in L^2_{|\xi_1|} \text{ is bounded, when } a_0 \text{ is sufficiently small}. \end{cases}$$

This lemma is a direct consequence of the energy estimates for the linear and nonlinear equation based on the estimates obtained above, cf. Theorem 2.1. Therefore, we omit its proof for brevity.

The following theorem gives an implicit condition on the boundary data which guarantees the solution obtained for the Boltzmann equation with damping is exactly the one without damping. That is, it gives the solvability condition of the boundary layer problem for the Boltzmann equation.

Note that the function Ψ in Theorem 2.5 is defined by

$$\Psi \equiv \boldsymbol{P}_0^+ \xi_1 \mathcal{I}^\gamma.$$

THEOREM 2.7. *For a given $\gamma > 0$, suppose that*

$$\boldsymbol{P}_0^+ \xi_1 \mathcal{I}^\gamma(a_0) \equiv 0. \tag{2.56}$$

Then, the solution of (2.14) *and* (2.15) *is a solution when $\gamma = 0$.*

PROOF. For a given $\gamma > 0$ when $\langle a_0, \xi_1 a_0 \rangle_+$ is sufficiently small, (2.55) has a unique solution $f(x,\xi)$. We project the problem (2.55) to its macroscopic component, then we have

$$\partial_x \boldsymbol{P}_0^+ \boldsymbol{P}_0 \xi_1 f = -\gamma \boldsymbol{P}_0^+ \boldsymbol{P}_0 \boldsymbol{P}_0^+ \xi_1 f = -\gamma \boldsymbol{P}_0^+ \xi_1 f.$$

If the boundary condition satisfies $\boldsymbol{P}_0^+ \xi_1 f|_{x=0} = 0$, then we immediately have

$$\boldsymbol{P}_0^+ \xi_1 f \equiv 0, \quad \text{for } x \geqslant 0.$$

That is, under the condition (2.56), the damping term added in (2.14) vanishes identically. □

Similar to Theorem 2.7, we have the following theorem on the homogeneous linearized equation.

THEOREM 2.8. *For a given $\gamma > 0$, suppose that*

$$\boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) \equiv 0. \tag{2.57}$$

Then, the problem

$$\begin{cases} \xi_1 \partial_x f = \boldsymbol{L} f, \\ f(0,\xi) = a_0(\xi), \quad \text{for } \xi_1 > 0, \\ \lim_{x\to\infty} f(x,\xi) = 0 \end{cases}$$

has a unique solution.

Finally, we will classify the boundary data by analyzing the solvability condition (2.57) for the linearized equation, and (2.56) for the nonlinear equation respectively.

Firstly, consider

$$\boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) = 0.$$

Since $\mathbb{I}^\gamma$ defines a bounded linear operator, the function $\boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0)$ defines a bounded linear map from $L^2_{\xi_1,+}$ to a finite dimensional space. According to the Mach number of the far field Maxwellian, we have the following theorem on the co-dimension of the boundary data satisfying (2.57).

THEOREM 2.9. *The classification of the boundary data satisfying the solvability condition* (2.57) *can be summarized in the following table.*

$\mathcal{M}^\infty < -1$	$\mathrm{Codim}(\{a_0 \in L^2_{\xi_1,+} \mid \boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) = 0\}) = 0$
$-1 < \mathcal{M}^\infty < 0$	$\mathrm{Codim}(\{a_0 \in L^2_{\xi_1,+} \mid \boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) = 0\}) = 1$
$0 < \mathcal{M}^\infty < 1$	$\mathrm{Codim}(\{a_0 \in L^2_{\xi_1,+} \mid \boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) = 0\}) = 4$
$\mathcal{M}^\infty > 1$	$\mathrm{Codim}(\{a_0 \in L^2_{\xi_1,+} \mid \boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) = 0\}) = 5$

PROOF. Denote $\boldsymbol{p}_i$ the eigenvector of the operator $A = \boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0$ on $\mathcal{N}$ by

$$\begin{cases} A\boldsymbol{p}_i = \lambda_i \boldsymbol{p}_i, \\ \lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < \lambda_5. \end{cases}$$

We want to show that dimension of the nontrivial solution a_0 to

$$\boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma(a_0) \neq 0, \tag{2.58}$$

is exactly n^+. Since $\boldsymbol{P}_0^+ \xi_1 \mathbb{I}^\gamma$ is a bounded linear operator from $L^2_{\xi_1,+}$ to $\mathbb{R}^{n^+}$, the dimension of the nontrivial solutions is at most n^+. It suffices to find n^+ linearly independent nontrivial solutions to (2.58).

For this, we introduce auxiliary functions

$$\boldsymbol{j}_j^{\gamma}(x,\xi) \equiv e^{-\gamma x}\boldsymbol{p}_j(\xi), \quad j = 1,2,3,4,5.$$

The case for $\mathcal{M}^\infty < -1$ is obvious. When $\mathcal{M}^\infty \in (-1,0)$, the range $\boldsymbol{P}_0^+$ is spanned by $\boldsymbol{p}_5$. Therefore,

$$\dim\bigl(\boldsymbol{P}_0^+\bigr) = n^+ = 1.$$

It is straightforward to check that the function $\boldsymbol{j}_5^\gamma$ satisfies

$$\xi_1\partial_x \boldsymbol{j}_5^\gamma - \boldsymbol{L}\boldsymbol{j}_5^\gamma + \gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{j}_5^\gamma = -\gamma e^{-\gamma x}\boldsymbol{P}_1\xi_1\boldsymbol{p}_5(\xi).$$

Let $\boldsymbol{J}_5^\gamma \equiv \boldsymbol{j}_5^\gamma + \boldsymbol{k}_5^\gamma(x,\xi)$ be a solution of

$$\begin{cases} \xi_1\partial_x \boldsymbol{J}_5^\gamma - \boldsymbol{L}\boldsymbol{J}_5^\gamma + \gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{J}_5^\gamma = 0, \\ \boldsymbol{J}_5^\gamma(0,\xi) = \boldsymbol{j}_5^\gamma(0,\xi), \quad \text{for } \xi_1 > 0. \end{cases}$$

The equation for $\boldsymbol{k}_5^\gamma$ is

$$\begin{cases} \xi_1\partial_x \boldsymbol{k}_5^\gamma - \boldsymbol{L}\boldsymbol{k}_5^\gamma + \gamma\boldsymbol{P}_0^\gamma\xi_1\boldsymbol{k}_5^\gamma = \gamma e^{-\gamma x}\boldsymbol{P}_1\xi_1\boldsymbol{p}_5(\xi), \\ \boldsymbol{k}_5^\gamma(0,\xi) = 0, \quad \text{for } \xi_1 > 0. \end{cases} \tag{2.59}$$

Notice that this equation for $\boldsymbol{k}_5^\gamma$ is just the equation for g in (2.14) by letting $h = \gamma e^{-\gamma x}\boldsymbol{P}_1\xi_1\boldsymbol{p}_5(\xi)$.

By choosing ϵ and γ satisfying

$$|\epsilon| \ll 1, \quad \left|\frac{\gamma}{\epsilon}\right| = O(1), \quad \text{and} \quad \epsilon - \gamma < 0,$$

the estimates from Theorem 2.1 with $\boldsymbol{k}_5^\gamma(0,\xi)|_{\xi_1>0} = 0$ yields that

$$\begin{aligned} \bigl\|\boldsymbol{P}_0^+\xi_1\boldsymbol{J}_5^\gamma\bigr\|\big|_{x=0} &= \bigl\|\boldsymbol{P}_0^+\xi_1\bigl(\boldsymbol{j}_5^\gamma + \boldsymbol{k}_5^\gamma\bigr)\bigr\|\big|_{x=0} \\ &= \bigl\|\boldsymbol{P}_0^+\xi_1\boldsymbol{j}_5^\gamma\big|_{x=0}\bigr\| + O(1)(\gamma)^{\frac{1}{2}} \neq 0. \end{aligned} \tag{2.60}$$

Here we have used the fact that γ can be arbitrarily small. Therefore, if we choose $a_0(\xi) = \boldsymbol{p}_5(\xi)$ for $\xi_1 > 0$, then (2.60) implies that for sufficient small γ,

$$\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0) \neq 0.$$

Then, $\langle\boldsymbol{p}_5, \boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)\rangle$ defines a nontrivial bounded functional from $L^2_{\xi_1,+}$ to $\mathbb{R}$. Then, by Riesz representation theorem there exists $\boldsymbol{r}_5 \in L^2_{\xi_1,+}$ such that

$$\bigl\langle\boldsymbol{p}_5, \boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)\bigr\rangle = \langle\boldsymbol{r}_5, \xi_1 a_0\rangle_+ . \tag{2.61}$$

This shows that

$$\operatorname{codim}\big\{a_0 \in L^2_{\xi_1,+}\colon\ \boldsymbol{P}_0^+\xi_1\mathbb{I}(a_0)=0\big\}=1.$$

When $0<\mathcal{M}^\infty<1$, $\dim(\boldsymbol{P}_0^+)=4$. The range of $\boldsymbol{P}_0^+$ is spanned by $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, $\boldsymbol{p}_4$, and $\boldsymbol{p}_5$. The bounded linear functional $\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)$ can be written as

$$\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)=\sum_{i=2}^{5}\frac{\langle \boldsymbol{p}_i,\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)\rangle}{\langle \boldsymbol{p}_i,\boldsymbol{p}_i\rangle}\boldsymbol{p}_i.$$

Similar to the construction of $\boldsymbol{r}_5$ in the case $\mathcal{M}^\infty\in(-1,0)$, we can use $\boldsymbol{j}_i^\gamma$ for $i=2,\dots,5$ to obtain four linearly independent $\boldsymbol{r}_i$, $i=2,\dots,5$, so that

$$\big\langle \boldsymbol{p}_i,\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)\big\rangle=\langle \boldsymbol{r}_i,\xi_1 a_0\rangle_+\quad \text{for } i=2,\dots,5.$$

This gives the theorem for the case $0<\mathcal{M}^\infty<1$. Similar to the above argument, one can show the theorem holds also for $\mathcal{M}^\infty>1$.

To classify $\boldsymbol{P}_0^\gamma\xi_1\mathcal{I}^\gamma(a_0)=0$, based on the classification of $\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)=0$, we can consider the Fréchet derivative of $\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(a_0)$ and show that it is nontrivial in a space of dimension n^+. For this, we normalize the vectors $\boldsymbol{r}_i$ obtained for the linear operator $\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0)$ in the direction $\boldsymbol{p}_i$ such that

$$\langle \boldsymbol{r}_i,\xi_1\boldsymbol{r}_i\rangle_+=1,\qquad \text{for } i=1,\dots,5,$$

if they exist. From the above analysis, the boundary data a_0 can be decomposed as follows:

$-1<\mathcal{M}^\infty<0$	$a_0=\langle \boldsymbol{r}_5,\xi_1 a_0\rangle_+\boldsymbol{r}_5+\boldsymbol{c}$	$\langle \boldsymbol{c},\xi_1\boldsymbol{r}_5\rangle_+=0$
$0<\mathcal{M}^\infty<1$	$a_0=\sum_{i=2}^{5}\langle \boldsymbol{r}_i,\xi_1 a_0\rangle_+\boldsymbol{r}_i+\boldsymbol{c}$	$\langle \boldsymbol{c},\xi_1\boldsymbol{r}_i\rangle_+=0$ for $i=2,\dots,5$
$\mathcal{M}^\infty>1$	$a_0=\sum_{i=1}^{5}\langle \boldsymbol{r}_i,\xi_1 a_0\rangle_+\boldsymbol{r}_i+\boldsymbol{c}$	$\langle \boldsymbol{c},\xi_1\boldsymbol{r}_i\rangle_+=0$ for $i=1,\dots,5$

From the decomposition in the above table, we can parameterize $\mathcal{I}^\gamma(a_0)$ as follows.

$-1<\mathcal{M}^\infty<0$	$\mathcal{I}^\gamma(a_0)\equiv\bar{\mathcal{I}}^\gamma(b_5,\boldsymbol{c})$	$b_5\in\mathbb{R},\ \boldsymbol{c}\in\{\boldsymbol{r}_5\}^\perp$
$0<\mathcal{M}^\infty<1$	$\mathcal{I}^\gamma(a_0)\equiv\bar{\mathcal{I}}^\gamma(b_2,\dots,b_5,\boldsymbol{c})$	$b_2,\dots,b_5\in\mathbb{R},\ \boldsymbol{c}\in\bigcap_{i=2}^5\{\boldsymbol{r}_i\}^\perp$
$\mathcal{M}^\infty>1$	$\mathcal{I}^\gamma(a_0)\equiv\bar{\mathcal{I}}^\gamma(b_1,\dots,b_5,\boldsymbol{c})$	$b_1,\dots,b_5\in\mathbb{R},\ \boldsymbol{c}\in\bigcap_{i=1}^5\{\boldsymbol{r}_i\}^\perp$

(2.62)

where the orthogonal relation is with respect to the inner product $\langle\xi_1\cdot,\cdot\rangle_+$.

Under this parameterization, we have the following lemma on the Fréchet derivative of $\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(a_0)$. □

LEMMA 2.10. *Referring to the table* (2.62), *suppose that* $\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(a_0)=0$, *then* a_0 *is a function of* $\boldsymbol{c}$ *when* $\langle a_0,\xi_1 a_0\rangle_+$ *is sufficiently small.*

PROOF. It is obvious that $\mathcal{I}^\gamma(0)=0$, and

$$\frac{d}{d\epsilon}\mathcal{I}^\gamma(\epsilon a_0)\Big|_{\epsilon=0}=\mathbb{I}^\gamma(a_0).$$

From this, we have

$$\frac{d}{d\epsilon}\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(\epsilon a_0)\Big|_{\epsilon=0}=\boldsymbol{P}_0^+\xi_1\mathbb{I}^\gamma(a_0). \tag{2.63}$$

Now, we consider the case of $-1<\mathcal{M}^\infty<0$ for illustration. For this case, from (2.63), we have

$$\frac{d}{d\epsilon}\big\langle \boldsymbol{p}_5,\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(\epsilon a_0)\big\rangle\Big|_{\epsilon=0}=\langle \boldsymbol{r}_5,\xi a_0\rangle_+.$$

Take $a_0=\boldsymbol{r}_5$, then we have

$$\frac{d}{d\epsilon}\big\langle \boldsymbol{p}_5,\boldsymbol{P}_0^+\xi_1\mathcal{I}^\gamma(\epsilon \boldsymbol{r}_5)\big\rangle\Big|_{\epsilon=0}=\partial_{b_5}\big\langle \boldsymbol{p}_5,\boldsymbol{P}_0^+\xi_1\bar{\mathcal{I}}^\gamma(b_5,\boldsymbol{c})\big\rangle\Big|_{(0,0)}=\langle \boldsymbol{r}_5,\xi\boldsymbol{r}_5\rangle=1.$$

Then, by the implicit function theorem $\langle \boldsymbol{p}_5,\boldsymbol{P}_0^+\xi_1\bar{\mathcal{I}}^\gamma(b_5,\boldsymbol{c})\rangle=0$ defines $b_5=b_5(\boldsymbol{c})$. This shows the theorem holds for $-1<\mathcal{M}^\infty<0$.

Similar argument applies to the cases $0<\mathcal{M}^\infty<1$ and $\mathcal{M}^\infty>1$. □

Theorem 2.9 and Lemma 2.10 give (ii) of Theorem 2.5.

2.2.2. *Cutoff hard potentials.* In this subsection, we will study the existence of boundary layer solutions for the cutoff hard potentials. The main difficulty comes from the sub-linear growth of the collision frequency in ξ. This requires a complicated weight function of both space and velocity variables.

Throughout this subsection, the cross section $q(v,\theta)$ is assumed to satisfy (1.21) with $0<\gamma<1$. To distinct the notation from the damping coefficient, we denote the parameter γ in the cross section by γ_0 in the following.

The weight function σ can be defined as follows. Let $\eta:[0,\infty)\to\mathbb{R}$ be a smooth non-increasing function, $\eta(s)=1$ for $s\leqslant 1$, $\eta(s)=0$ for $s\geqslant 2$, and $0\leqslant\eta\leqslant 1$. For $x\geqslant 0$, set

$$\begin{aligned}\sigma(x,\xi)=5(\delta x+l)^{\frac{2}{3-\gamma_0}}&\left(1-\eta\left(\frac{\delta x+l}{(1+|\xi-u_\infty|)^{3-\gamma_0}}\right)\right)\\&+\left(\frac{\delta x+l}{(1+|\xi-u_\infty|)^{1-\gamma_0}}+3|\xi-u_\infty|^2\right)\eta\\&\times\left(\frac{\delta x+l}{(1+|\xi-u_\infty|)^{3-\gamma_0}}\right).\end{aligned} \tag{2.64}$$

Note that this weight function also works for soft potentials which needs more subtle calculations and we will not discuss it in this chapter.

The existence theorem for cutoff hard potentials can be stated as follows.

THEOREM 2.11. *Let* $\mathcal{M}^\infty \neq 0, \pm 1$ *and* $\beta > \max\{\frac{3}{2} + \gamma_0, 3 - 2\gamma_0\}$. *Then, there exist positive numbers* $\epsilon, \epsilon_0, \epsilon_1$, *and a* C^1 *map*

$$\Psi : L^2\big(\mathbb{R}^3_+, \xi_1\, d\xi\big) \to \mathbb{R}^{n+}, \quad \Psi(0) = 0, \tag{2.65}$$

such that the following holds.

(i) *Suppose that the boundary data* F_b *satisfies*

$$\big|F_b(\xi) - \mathbf{M}_\infty(\xi)\big| \leqslant \epsilon_0 \sigma_x^{-\frac{1}{2}}(0,\xi) e^{-\epsilon\sigma(0,\xi)} W_\beta(\xi), \quad \xi \in \mathbb{R}^3_+, \tag{2.66}$$

where $l > 0$ *is a large constant and* $\delta > 0$ *is a small constant. Then, the problem* (2.3) *admits a unique solution* F *satisfying*

$$\begin{aligned} &\big|F(x,\xi) - \mathbf{M}_\infty(\xi)\big| + \big|\xi_1 \nu^{-1} F_x(x,\xi)\big| \leqslant \epsilon_1 \sigma_x^{-\frac{1}{2}}(x,\xi) e^{-\epsilon\sigma(x,\xi)} W_\beta(\xi),\\ &x > 0,\ \xi \in \mathbb{R}^3, \end{aligned} \tag{2.67}$$

if and only if F_b *satisfies*

$$\Psi\big(e^{\epsilon\sigma(0,\xi)} W_0^{-1}(F_b - \mathbf{M}_\infty)\big) = 0. \tag{2.68}$$

(ii) *The set of* F_b *satisfying* (2.66) *and* (2.68) *forms a* (*local*) C_1 *manifold of co-dimension* n^+.

REMARK 2.12. It is straightforward to check that the weight function $\sigma(x,\xi)$ satisfies

$$\begin{aligned} &\sigma(x,\xi) \geqslant c(\delta x + l)^{\frac{2}{3-\gamma_0}},\\ &c_1 \min\Big\{(\delta x + l)^{-\frac{1-\gamma_0}{3-\gamma_0}}, \big(1 + |\xi - u_\infty|\big)^{-1+\gamma_0}\Big\} \leqslant \sigma_x \leqslant c_2(\delta x + l)^{-\frac{1-\gamma_0}{3-\gamma_0}}, \end{aligned}$$

for some constants $c_1, c_2 > 0$ when the constant l is large and $\delta > 0$ is small. Hence, $F(x,\xi)$ tends to $\mathbf{M}_\infty(\xi)$ as x tends to infinity by (2.67) exponentially like $e^{-cx^{\frac{2}{3-\gamma_0}}}$, while the convergence rate for the hard sphere model is like e^{-cx} corresponding to $\gamma_0 = 1$.

To apply the weight function $\sigma(x,\xi)$ in the existence analysis, we need the following lemmas. Recall that $\langle\cdot,\cdot\rangle$ is the inner product in L^2_ξ. For simplicity of notations, we denote $\tilde{\xi} = \xi - u_\infty$.

LEMMA 2.13. *There are constants $\epsilon_2 > 0$ and $\nu_2 > 0$ such that if $0 \leqslant \epsilon \leqslant \epsilon_2$ and $g \in \mathcal{N}^{\perp}$, then*

$$-\left\langle g, e^{\epsilon|\tilde{\xi}|^2} \boldsymbol{L} e^{-\epsilon|\tilde{\xi}|^2} g\right\rangle \geqslant \nu_2 \left\langle \nu(\xi) g, g\right\rangle,$$

where $\nu_2 > 0$ depends on ϵ_2.

PROOF. Since $\nu(\xi)$ is commutative with $e^{\epsilon|\tilde{\xi}|^2}$ and $e^{-\epsilon|\tilde{\xi}|^2}$, it suffices to show

$$\left\| e^{\epsilon|\tilde{\xi}|^2} K e^{-\epsilon|\tilde{\xi}|^2} - K \right\|_{L^2_\xi} \to 0, \tag{2.69}$$

as $\epsilon \to 0$. Notice that here K is given by (1.29) with a slight modification because the background Maxwellian is now $\mathbf{M}_\infty$.

By the Schwarz inequality, we have

$$\frac{(|\tilde{\xi}_*|^2 - |\tilde{\xi}|^2)^2}{|\xi_* - \xi|^2} + |\xi_* - \xi|^2 \geqslant 2\left||\tilde{\xi}_*|^2 - |\tilde{\xi}|^2\right|. \tag{2.70}$$

Rewrite

$$\begin{aligned}
&e^{\epsilon|\tilde{\xi}|^2} K(\xi, \xi_*) e^{-\epsilon|\tilde{\xi}_*|^2} - K(\xi, \xi_*) \\
&\quad = \left\{ K(\xi, \xi_*) \exp\left(\frac{1}{16 T_\infty} \left[\frac{(|\tilde{\xi}_*|^2 - |\tilde{\xi}|^2)^2}{|\xi_* - \xi|^2} + |\xi_* - \xi|^2 \right] \right) \right\} \\
&\qquad \times \left\{ \exp\left(\frac{-1}{16 T_\infty} \left[\frac{(|\tilde{\xi}_*|^2 - |\tilde{\xi}|^2)^2}{|\xi_* - \xi|^2} + |\xi_* - \xi|^2 \right] \right) \right. \\
&\qquad \times \left. \left(\exp\left\{ \epsilon\left(|\tilde{\xi}|^2 - |\tilde{\xi}_*|^2\right) \right\} - 1 \right) \right\} \\
&\quad \equiv p(\xi, \xi_*) \times s(\epsilon, \xi, \xi_*).
\end{aligned} \tag{2.71}$$

By (2.70),

$$\sup_{\xi, \xi_*} \left| s(\epsilon, \xi, \xi_*) \right| \to 0, \quad \text{as } \epsilon \to 0.$$

Since $p(\xi, \xi_*)$ is also a kernel of a bounded operator on L^2_ξ by Proposition 1.3 and Lemma 1.4, we have (2.69) and this completes the proof of the lemma. □

Notice that we choose $l > 0$ as a large constant and δ as a small constant to have

$$\partial_x \sigma \sim \partial_x (\delta x + l)^{\frac{2}{3 - \gamma_0}},$$

for $|\xi|$ being large. The factors 5 and 3 in the definition of $\sigma(x, \xi)$ are for $\sigma_x(x, \xi) > 0$. The following lemma shows that the linearized collision operator with some variation still keeps the dissipative effect on the microscopic components.

LEMMA 2.14. *There are constants* $\epsilon_3 > 0$ *and* $\nu_3 > 0$ *such that if* $0 \leqslant \epsilon \leqslant \epsilon_3$ *and* $g \in \mathcal{N}^{\perp}$, *then*

$$-\langle g, e^{\epsilon\sigma(x,\xi)} \boldsymbol{L} e^{-\epsilon\sigma(x,\xi)} g\rangle \geqslant \nu_3 \langle \nu(\xi) g, g\rangle.$$

PROOF. The proof is similar to the one for Lemma 2.13. Let

$$\bar{s}(\epsilon, \xi, \xi_*) \equiv \exp\left(\frac{-1}{16T_\infty}\left[\frac{(|\tilde{\xi}_*|^2 - |\tilde{\xi}|^2)^2}{|\xi_* - \xi|^2} + |\xi_* - \xi|^2\right]\right) \times \left(\exp\left\{\epsilon\left(\sigma(x,\xi) - \sigma(x,\xi_*)\right)\right\} - 1\right).$$

We only need to show

$$\sup_{\xi,\xi_*} \left|\bar{s}(\epsilon, \xi, \xi_*)\right| \to 0, \quad \text{as } \epsilon \to 0, \tag{2.72}$$

by discussing cases according to the values of $|\tilde{\xi}|$, $|\tilde{\xi}_*|$ and x. For illustration, we check only two of them as follows.

- When $(1 + |\tilde{\xi}|)^{3-\gamma_0} \geqslant (1 + |\tilde{\xi}_*|)^{3-\gamma_0} \geqslant 2(\delta x + l)$:

$$\begin{aligned} \left|\sigma(x,\xi) - \sigma(x,\xi_*)\right| &\leqslant \left|\frac{\delta x + l}{(1 + |\tilde{\xi}|)^{1-\gamma_0}} - \frac{\delta x + l}{(1 + |\tilde{\xi}_*|)^{1-\gamma_0}}\right| + 3\left||\tilde{\xi}|^2 - |\tilde{\xi}_*|^2\right| \\ &\leqslant c_1 \frac{\delta x + l}{(1 + |\tilde{\xi}_*|)^{2-\gamma_0}} \left||\tilde{\xi}| - |\tilde{\xi}_*|\right| + 3\left||\tilde{\xi}|^2 - |\tilde{\xi}_*|^2\right| \\ &\leqslant c\left||\tilde{\xi}|^2 - |\tilde{\xi}_*|^2\right|. \end{aligned} \tag{2.73}$$

  This is dominated by the left hand side of (2.70) when ϵ is sufficiently small.
- When $(1 + |\tilde{\xi}|)^{3-\gamma_0} \geqslant 2(\delta x + l)$, $\delta x + l \geqslant (1 + |\tilde{\xi}_*|)^{3-\gamma_0}$:

$$\begin{aligned} \left|\sigma(x,\xi) - \sigma(x,\xi_*)\right| &\leqslant c_1 \frac{\delta x + l}{(1 + |\tilde{\xi}|)^{1-\gamma_0}} + (\delta x + l)^{\frac{2}{3-\gamma_0}} + |\tilde{\xi}|^2 \\ &\leqslant c\left||\tilde{\xi}|^2 - |\tilde{\xi}_*|^2\right|. \end{aligned} \tag{2.74}$$

  Again, this term is dominated by the left hand side of (2.70). All the other cases can be checked similarly to have (2.72) and this completes the proof of the lemma. □

REMARK 2.15. Let the cutoff function in (2.64) be in the form of $\eta(\frac{\delta x + l}{(1+|\tilde{\xi}|)^{\bar{\gamma}}})$. The reason to choose $\bar{\gamma} = 3 - \gamma_0$ is that (2.74) requires that $\bar{\gamma} \leqslant 3 - \gamma_0$, and

$$|\xi_1|\left|\partial_x \sigma(x,\xi)\right| \leqslant c\left(1 + |\xi|\right)^{\gamma_0}, \tag{2.75}$$

is needed for the following energy estimate which requires $\bar{\gamma} \geqslant 3 - \gamma_0$. Therefore, the decay rate of the boundary layer solution at $x = \infty$ stated in Remark 2.12 is canonical for the cutoff hard potentials.

Similar to the hard sphere model, the existence to the linearized problem with an artificial damping term can be proved by energy estimate. Precisely, by using $\sigma(x,\xi)$ defined in (2.64) and choosing $K(x)=(\delta x+l)^{-\Theta}$ with $\Theta=\frac{1-\gamma_0}{3-\gamma_0}$, (2.14) becomes

$$\begin{cases} \xi_1 g_x-\epsilon\sigma_x\xi_1 g-\boldsymbol{L}_\epsilon g=h-D_\epsilon g, & x>0,\ \xi\in\mathbb{R}^3,\\ g|_{x=0}=b_0(\xi)=a_0(\xi)e^{\epsilon\sigma(0,\xi)}, & \xi\in\mathbb{R}^3_+,\\ g\to 0(x\to\infty), & \xi\in\mathbb{R}^3, \end{cases} \tag{2.76}$$

where

$$D_\epsilon=\gamma(\delta x+l)^{-\Theta}e^{\epsilon\sigma}\boldsymbol{P}_0^+\xi_1 e^{-\epsilon\sigma}.$$

Let D_ϵ^* be the adjoint operator of D_ϵ, and set

$$V=\{\chi\mid \chi=-\xi_1\phi_x-\epsilon\sigma_x\xi_1\phi-\boldsymbol{L}_{-\epsilon}\phi+D_\epsilon^*\phi,\ \phi\in C_0^\infty,\ \phi^0\equiv\phi|_{x=0}=0 \text{ for } \xi_1<0\}.$$

Consider

$$(\chi,\phi)=\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left[-\epsilon\xi_1\sigma_x\phi^2+\phi\boldsymbol{L}_\epsilon\phi+\phi D_\epsilon\phi\right]d\xi\,dx + \frac{1}{2}\int_{\xi_1>0}\xi_1\left(\phi^0\right)^2 d\xi. \tag{2.77}$$

LEMMA 2.16. *Let l in* (2.64) *be large enough. Then there is $\nu_4>0$ such that when $\gamma=O(1)\epsilon\ll 1$,*

$$\int_{|\tilde\xi|^{3-\gamma_0}\leqslant\frac{l}{2}}\phi\left(\gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{P}_0-\epsilon\boldsymbol{P}_0\xi_1\boldsymbol{P}_0\right)\phi\,d\xi\geqslant\nu_4\epsilon\int_{\mathbb{R}^3}\phi_0^2\,d\xi, \tag{2.78}$$

where $\phi_0=\boldsymbol{P}_0\phi$.

PROOF. Notice that the macroscopic component ϕ is exponential decay in ξ like $e^{-c|\xi|^2}$. Then for l large enough, the integral $\int_{|\tilde\xi|^{3-\gamma_0}\leqslant\frac{l}{2}}\phi(\gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{P}_0-\epsilon\boldsymbol{P}_0\xi_1\boldsymbol{P}_0)\phi\,d\xi$ has the same order as $\int_{\mathbb{R}^3}\phi(\gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{P}_0-\epsilon\boldsymbol{P}_0\xi_1\boldsymbol{P}_0)\phi\,d\xi$. On the other hand, we know that the operator $\gamma\boldsymbol{P}_0^+\xi_1\boldsymbol{P}_0-\epsilon\boldsymbol{P}_0\xi_1\boldsymbol{P}_0$ is positive definite on the space $\mathcal{N}$ when $\gamma>\epsilon$. Hence, we have the estimate (2.78). □

We now estimate the terms in (2.77) as follows. Recall the notations $\phi_1=\boldsymbol{P}_1\phi$ and $\phi_0=\boldsymbol{P}_0\phi=\sum_{\alpha=0}^4 b_\alpha\psi_\alpha$. By using Lemma 2.16 and the fact ψ_α decays like $e^{-c|\xi|^2}$, we have

$$\int_{\mathbb{R}^3}\left(\phi D_\epsilon\phi-\epsilon\xi_1\sigma_x\phi^2\right)d\xi\geqslant\frac{c\nu_4\epsilon}{4}(\delta x+l)^{-\Theta}\|\phi_0\|^2_{L^2_\xi}$$

$$- c\epsilon \int_{\mathbb{R}^3} \nu(\xi)\phi_1^2 \, d\xi, \tag{2.79}$$

which gives the dissipation estimate on the macroscopic component. Here we have used

$$|\sigma_x \xi_1| \leqslant c\big(1 + |\xi|\big)^{\gamma_0} \leqslant c\nu(\xi).$$

The dissipation on the microscopic component comes from $\phi \boldsymbol{L}_\epsilon \phi$ which can be estimated as follows. By using the exponential decay of ψ_α, we have

$$\begin{aligned} \big|\boldsymbol{L}_\epsilon \phi_0(\xi)\big| &\leqslant |\boldsymbol{L}\phi_0| + \big|(K_\epsilon - K)\phi_0(\xi)\big| \\ &\leqslant c_5 \exp\big[-c(\delta x + l)^{\frac{2}{3-\gamma_0}}\big]\left(\sum_{j=0}^4 b_j^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $K_\epsilon = e^{\sigma} K e^{-\sigma}$.

Similarly, we have

$$|\boldsymbol{L}_{-\epsilon}\phi_0| \leqslant c \exp\big[-c(\delta x + l)^{\frac{2}{3-\gamma_0}}\big]\left(\sum_{j=0}^4 b_j^2\right)^{\frac{1}{2}}. \tag{2.80}$$

It then follows

$$\left| \int_{\mathbb{R}^3} \phi_0 \boldsymbol{L}_\epsilon \phi_0 \, d\xi \right| \leqslant c \exp\big[-c(\delta x + l)^{\frac{2}{3-\gamma_0}}\big] \sum_{j=0}^4 b_j^2. \tag{2.81}$$

Lemma 2.14 and (2.79)–(2.81) imply

$$\begin{aligned} -\int_{\mathbb{R}^3} \phi \boldsymbol{L}_\epsilon \phi \, d\xi &= -\int_{\mathbb{R}^3} \phi_0 \boldsymbol{L}_\epsilon \phi_0 - \int_{\mathbb{R}^3} \phi_1 \boldsymbol{L}_\epsilon \phi_0 - \int_{\mathbb{R}^3} \phi_0 \boldsymbol{L}_\epsilon \phi_1 - \int_{\mathbb{R}^3} \phi_1 \boldsymbol{L}_\epsilon \phi_1 \\ &\geqslant -c\epsilon^2 (\delta x + l)^{-\Theta} \|\phi_0\|_{L^2_\xi}^2 + c^{-1}\langle \nu\phi_1, \phi_1\rangle. \end{aligned} \tag{2.82}$$

By choosing l large enough, say $l \gg \epsilon^{-1}$ when ϵ is sufficiently small, we have

$$(\chi, \phi) \geqslant c_6\Big\{\epsilon \big\|(\delta x + l)^{-\frac{\Theta}{2}}\phi_0\big\|^2 + \big\|\nu^{\frac{1}{2}}\phi_1\big\|^2 + \big\langle \xi^1 \phi^0, \phi^0\big\rangle_+\Big\}. \tag{2.83}$$

Denote

$$\big|[h]\big| = \big\|(\delta x + l)^{\frac{\Theta}{2}} h_0\big\| + \|h_1\|. \tag{2.84}$$

Thus, (2.83) implies that

$$\big|[\chi]\big| \geqslant c\epsilon\Big(\big\|(\delta x + l)^{-\frac{\Theta}{2}}\phi_0\big\| + \|\phi_1\|\Big). \tag{2.85}$$

In summary, an argument similar to the one for the hard sphere model gives the following theorem.

THEOREM 2.17. *Consider the linearized Boltzmann equation with damping* (2.76). *If* $|[\bar{h}]| < \infty$ *then* (2.76) *has a unique solution satisfying*

$$\begin{aligned}&\left\|(\delta x+l)^{-\frac{\Theta}{2}} g_0\right\|^2+\left\|\nu^{\frac{1}{2}} g_1\right\|^2+\left|\langle\xi^1 g^0, g^0\rangle_-\right| \\ &\quad\leqslant c(\epsilon, b_0)\left(\left||\xi_1|^{\frac{1}{2}} a_0\right|_+ + \left|[h]\right|\right),\end{aligned} \tag{2.86}$$

where $g^0 = g|_{x=0}$.

REMARK 2.18. Since $\sigma_x \sim (\delta x+l)^{-\frac{1-\gamma_0}{3-\gamma_0}}$ for $2(1+|\tilde{\xi}|)^{3-\gamma_0} \leqslant \delta x+l$, when l is large enough, (2.86) implies

$$\left\|\sigma_x^{\frac{1}{2}} g_0\right\|^2+\left\|\nu^{\frac{1}{2}} g_1\right\|^2+\left|\langle\xi^1 g^0, g^0\rangle_-\right| \leqslant c(\epsilon, a_0)\left(\left||\xi_1|^{\frac{1}{2}} a_0\right|_+ + \left|[h]\right|\right). \tag{2.87}$$

Based on the estimates on the solution to the linearized problem (2.76), we now prove the existence for the nonlinear problem again by fixed point theorem and then discuss the solvability condition on the boundary data.

Define a weighted norm:

$$[\![h]\!]_\beta = \left\|\sigma_x^{\frac{1}{2}}\left(1+|\xi|\right)^\beta h\right\|_{L^\infty_{x,\xi}} = \sup_{x>0,\xi\in\mathbb{R}^3} \sigma_x^{\frac{1}{2}}\left(1+|\xi|\right)^\beta \left|h(x,\xi)\right|,$$

for any function $h(x,\xi)$ such that the above norm is finite.

Rewrite (2.76) as

$$g_x = \left(\epsilon\sigma_x - \frac{\nu}{\xi_1}\right) g + \frac{1}{\xi_1}(\bar{K} g + h),$$

where

$$\bar{K} = e^{\epsilon\sigma(x,\xi)} K e^{-\epsilon\sigma(x,\xi)} - \gamma(\delta x+l)^{-\Theta} e^{\epsilon\sigma(x,\xi)} \boldsymbol{P}_0^+ \xi_1 e^{-\epsilon\sigma(x,\xi)}.$$

Notice that K is a compact operator on L^2_ξ, and is bounded from the space L^∞_β to $L^\infty_{\beta+1}$, and from L^2_ξ to L^∞_ξ, see Lemmas 1.4 and 1.5. Following the proofs for Lemmas 2.13 and 2.14, we now prove that $\bar{K}$ has the same properties as K in the following lemma.

LEMMA 2.19. *When* ϵ *is sufficiently small,* $\bar{K}$ *satisfies*:

(1) $\sigma_x^{\frac{1}{2}} \bar{K} \sigma_x^{-\frac{1}{2}}$ *is a bounded operator from* L^2_ξ *to itself.*

(2) $\sigma_x^{\frac{1}{2}} \bar{K} \sigma_x^{-\frac{1}{2}}$ *is a bounded operator from* L^2_ξ *to* L^∞_ξ.

(3) $[\![\bar{K} h]\!]_\beta \leqslant c[\![h]\!]_{\beta-1}$, *for* $\beta \in \mathbb{R}$.

PROOF. By the definition of $\sigma(x,\xi)$, we have the following expression for σ_x.

$$\sigma_x = \begin{cases} \delta(1+|\tilde{\xi}|)^{-1+\gamma_0}, & (x,\xi)\in\Omega_1,\\ \delta(c_1(1+|\tilde{\xi}|)^{-1+\gamma_0}+c_2(x+l)^{-\Theta}), & (x,\xi)\in\Omega_2,\\ \frac{10\delta}{3-\gamma_0}(\delta x+l)^{-\Theta}, & (x,\xi)\in\Omega_3, \end{cases} \tag{2.88}$$

where c_1 and c_2 are positive functions depending on (x,ξ) and η with c_1+c_2 having a uniform positive lower bound. Here

$$\begin{aligned} \Omega_1 &= \{(x,\xi):\ \delta x+l \leqslant \big(1+|\tilde{\xi}|\big)^{3-\gamma_0}\},\\ \Omega_2 &= \{(x,\xi):\ \big(1+|\tilde{\xi}|\big)^{3-\gamma_0} < \delta x+l < 2\big(1+|\tilde{\xi}|\big)^{3-\gamma_0}\},\\ \Omega_3 &= \{(x,\xi):\ \delta x+l \geqslant 2\big(1+|\tilde{\xi}|\big)^{3-\gamma_0}\}. \end{aligned} \tag{2.89}$$

Notice that $\bar{K}$ has two parts, and the one coming from the artificial damping is simpler because the operator $\boldsymbol{P}_0^+\xi_1 e^{-\epsilon\sigma}$ maps a function to the macroscopic subspace which is of finite dimension and has exponential decay like $e^{-c|\xi|^2}$. Consider the operator $B_1 = \gamma\sigma_x^{\frac{1}{2}} e^{\epsilon\sigma}\boldsymbol{P}_0^+\xi_1 e^{-\epsilon\sigma}\sigma_x^{-\frac{1}{2}}$ and $\tilde{B}_1 = \gamma e^{\epsilon\sigma}\boldsymbol{P}_0^+\xi_1 e^{-\epsilon\sigma}$. By noticing that $\sigma(x,\xi)$ is bounded by $c|\xi|^2$, it is straightforward to check that $\tilde{B}_1$ has the following three properties:

$$\begin{aligned} &\int_{\mathbb{R}^3} |\tilde{B}_1 h|^2\, d\xi \leqslant c\int_{\mathbb{R}^3} h^2\, d\xi, \qquad \|\tilde{B}_1 h\|_{L^\infty_\xi} \leqslant c\|h\|_{L^2_\xi},\\ &\big\|\tilde{B}_1\big(1+|\xi|\big)^{-\beta}\big\|_{L^\infty_\xi} \leqslant ce^{-c_1|\xi|^2} \leqslant c\big(1+|\xi|\big)^{-\beta-1}. \end{aligned} \tag{2.90}$$

With (2.90), the property of the operator B_1 can be discussed as follows. If the $\sigma_x^{-\frac{1}{2}}(x,\xi)$ in $\boldsymbol{P}_0^+$ is of the order $(1+|\tilde{\xi}|)^{1-\gamma_0}$, then it can be absorbed by the exponential decay factor from the projection on macroscopic component. Otherwise, it is of the order $(\delta x+l)^{\Theta}$. In this case, if the $\sigma_x^{\frac{1}{2}}(x,\xi)$ has the same order, then they cancel. If not, then we have a factor of the order $\frac{(\delta x+l)^{\frac{\Theta}{2}}}{(1+|\tilde{\xi}|)^{\frac{1-\gamma_0}{2}}}$ when $(\delta x+l)\leqslant 2(1+|\tilde{\xi}|)^{3-\gamma_0}$ which is bounded. Therefore, B_1 has the same properties as $\tilde{B}_1$ stated in (2.90).

Denote the first part in $\sigma_x^{\frac{1}{2}}\bar{K}\sigma_x^{-\frac{1}{2}}$ by B_2. Its kernel is given by

$$b_2(\xi,\xi_*) = \sigma_x^{\frac{1}{2}}(x,\xi)e^{\epsilon\sigma}(x,\xi)K(\xi,\xi_*)e^{-\epsilon\sigma}(x,\xi_*)\sigma_x^{-\frac{1}{2}}(x,\xi_*).$$

We will illustrate the following three cases to show that the operator B_2 keeps the main structure as K.

CASE 1. When $(x,\xi),\ (x,\xi_*)\in\Omega_3$, we have $b_2(\xi,\xi_*) = k(\xi,\xi_*)$.

CASE 2. When $(x,\xi),\ (x,\xi_*)\in\Omega_1$, we have

$$\begin{aligned} b_2(\xi,\xi_*) &\leqslant K(\xi,\xi_*)\left(\frac{1+|\tilde{\xi}_*|}{1+|\tilde{\xi}|}\right)^{\frac{1-\gamma_0}{2}}\\ &\qquad\times\exp\left\{\epsilon\left(\frac{\delta x+l}{(1+|\tilde{\xi}|)^{1-\gamma_0}}+3|\tilde{\xi}|^2-\frac{\delta x+l}{(1+|\tilde{\xi}_*|)^{1-\gamma_0}}-3|\tilde{\xi}_*|^2\right)\right\}\\ &\leqslant cK(\xi,\xi_*)\exp\left\{c\epsilon\left||\tilde{\xi}|^2-|\tilde{\xi}_*|^2\right|\right\}. \end{aligned}\tag{2.91}$$

By the discussion for Lemmas 2.13 and 2.14, (2.91) implies that b_2 has the same properties as $K(\xi,\xi_*)$ when ϵ is sufficiently small.

CASE 3. When $(x,\xi)\in\Omega_1,\ (x,\xi_*)\in\Omega_3$, we have

$$\begin{aligned} b_2(\xi,\xi_*) &\leqslant cK(\xi,\xi_*)\frac{(\delta x+l)^{\frac{\Theta}{2}}}{(1+|\xi|)^{\frac{1-\gamma_0}{2}}}\\ &\qquad\times\exp\left\{\epsilon\left(-5(\delta x+l)^{\frac{2}{3-\beta}}+\frac{\delta x+l}{(1+|\tilde{\xi}|)^{1-\gamma_0}}+3|\tilde{\xi}|^2\right)\right\}\\ &\leqslant cK(\xi,\xi_*)\exp\left\{c\epsilon\left||\tilde{\xi}|^2-|\tilde{\xi}_*|^2\right|\right\}, \end{aligned}\tag{2.92}$$

where we have used $\delta x+l\leqslant(1+|\tilde{\xi}|)^{3-\gamma_0}$ for $(x,\xi)\in\Omega_1$, and $||\tilde{\xi}|^2-|\tilde{\xi}_*|^2|\geqslant c(\delta x+l)^{\frac{2}{3-\gamma_0}}$ because $(x,\xi_*)\in\Omega_3$.

In summary, $b_2(\xi,\xi_*)$ also has the same properties as $K(\xi,\xi_*)$. This together with the property of B_1 complete the proof of the lemma.

To have a shorter expression for g, let

$$\kappa(x,x';\xi)=-\epsilon\int_{x'}^{x}\sigma_x(y,\xi)\,dy+\frac{\nu}{\xi_1}(x-x').$$

Since $\kappa(x,x';\xi)>0$ both for $x-x'>0,\ \xi_1>0$ and $x-x'<0,\ \xi_1<0$ when $\epsilon>0$ is sufficiently small, the solution g can be formally written as

$$g=\tilde{b}+U(\bar{K}g+h),\tag{2.93}$$

where

$$\tilde{b}=\begin{cases}\exp(-\kappa(x,0;\xi))b_0, & \xi_1>0,\\ 0, & \xi_1<0,\end{cases}\tag{2.94}$$

and

$$U(h)=\begin{cases}\int_0^x\exp(-\kappa(x,x';\xi))\frac{1}{\xi_1}h(x',\xi)\,dx', & \xi_1>0,\\ -\int_x^\infty\exp(-\kappa(x,x';\xi))\frac{1}{\xi_1}h(x',\xi)\,dx', & \xi_1<0.\end{cases}\tag{2.95}$$

Similar to the hard sphere model, the operator U here also has the following properties. The proof is omitted for brevity. □

LEMMA 2.20. *The operator U satisfies, for $1 \leqslant p, r \leqslant \infty$,*

$$\begin{aligned}
&\| U(h)(\cdot,\xi)\|_{L^p_x} \leqslant c\nu^{-1}(\xi)\|h(\cdot,\xi)\|_{L^p_x},\\
&\| U(h)\|_{L^r_\xi(L^p_x)} \leqslant c\|\nu^{-1}h\|_{L^r_\xi(L^p_x)}.
\end{aligned} \tag{2.96}$$

And for $\beta \in \mathbb{R}$,

$$[\![U(h)]\!]_\beta \leqslant c[\![\nu^{-1}h]\!]_\beta. \tag{2.97}$$

Now in order to obtain the weighted $L^\infty_{x,\xi}$ estimate on g, similar to the hard sphere model, we prove the following lemma based on the energy estimate on the linear Boltzmann equation with damping (2.76). Again, the proof is omitted for brevity.

LEMMA 2.21. *The solution to the linear Boltzmann equation with damping* (2.76) *satisfies for $0 \leqslant \alpha < \frac{1}{2}$,*

$$\|\sigma_x^{\frac{1}{2}} w_{-\alpha}\nu^{\frac{1}{2}} g\| + \|\sigma_x^{\frac{1}{2}} w_1 \nu^{-\frac{1}{2}} g_x\| \leqslant cE_0, \tag{2.98}$$

where

$$E_0 = \||\xi_1|^{\frac{1}{2}} w_{-\alpha} b_0\|_+ + |[h]| + \|w_{-\alpha}\sigma_x^{\frac{1}{2}}(x,\xi)h\|.$$

By Lemma 2.20 and 2.21, we have the following lemma on the $[\![\cdot]\!]_\beta$ norm of g.

LEMMA 2.22. *For $0 < \alpha < \frac{1}{2}$, $\beta > \frac{3}{2}$, the solution to the problem* (2.76) *satisfies*

$$[\![g]\!]_\beta \leqslant c\big([\![\nu^{-1}h]\!]_\beta + |[h]| + \|w_{-\alpha}\sigma_x^{\frac{1}{2}}\bar{h}\| + |b_0|_*\big), \tag{2.99}$$

where

$$\begin{aligned}
&|b_0|_* = |b_0|_{+,\beta} + \||\xi|^{\frac{1}{2}} w_{-\alpha} b_0\|_+,\\
&|b_0|_{+,\beta} = \sup_{\xi\in\mathbb{R}^3,\xi_1>0} \big(1+|\xi|\big)^\beta |b_0|.
\end{aligned} \tag{2.100}$$

PROOF. By using the property of the operators U in (2.97) and noticing that the operator $\bar{K}$ is a bounded operator from $[\![\cdot]\!]_{\beta-1}$ to $[\![\cdot]\!]_\beta$, we have

$$\begin{aligned}
[\![g]\!]_\beta &\leqslant c|b_0|_{+,\beta} + c\big([\![\nu^{-1}\bar{K}g]\!]_\beta + [\![\nu^{-1}h]\!]_\beta\big)\\
&\leqslant c|b_0|_{+,\beta} + c\big([\![g]\!]_{\beta-1} + [\![\nu^{-1}\bar{h}]\!]_\beta\big).
\end{aligned} \tag{2.101}$$

Iterating (2.101) gives

$$[\![g]\!]_\beta \leqslant c\big(|b_0|_{+,\beta} + [\![g]\!]_0 + [\![\nu^{-1}h]\!]_\beta\big), \tag{2.102}$$

where we have used for $\beta \geqslant 1$,

$$[\![\nu^{-1}h]\!]_\beta + \cdots + [\![\nu^{-1}h]\!]_2 + [\![\nu^{-1}h]\!]_1 \leqslant c[\![\nu^{-1}h]\!]_\beta. \tag{2.103}$$

In order to obtain (2.99), we need to estimate $\|g\|_0 = \|\sigma_x^{\frac{1}{2}} g\|_{L^\infty_{x,\xi}}$.

By using (2.98), we have for $\frac{1}{4} < \beta < \frac{1}{2}$, i.e., $0 < 1 - 2\beta < \frac{1}{2}$,

$$\begin{aligned} \big\|\sigma_x^{\frac{1}{2}} w_\beta g\big\|^2_{L^\infty_x(L^2_\xi)} &\leqslant \big\|\sigma_x^{\frac{1}{2}} w_\beta g\big\|^2_{L^2_\xi(L^\infty_x)} \\ &\leqslant c\int \big\{w_\beta^2 w_1^{-1}\nu^{\frac{1}{2}}(\xi)\big\|\sigma_x^{\frac{1}{2}} g(\cdot,\xi)\big\|_{L^2_x}\big\}\big\{w_1\nu^{-\frac{1}{2}}(\xi)\big\|\sigma_x^{\frac{1}{2}} g_x(\cdot,\xi)\big\|_{L^2_x}\big\}\,d\xi \\ &\leqslant cE_0^2. \end{aligned} \tag{2.104}$$

Here we have used $|\sigma_{xx}| \leqslant c\delta\sigma_x$ to absorb some terms from differentiation by $\|\sigma_x^{\frac{1}{2}} w_\beta g\|^2_{L^2_\xi(L^\infty_x)}$ when δ is sufficiently small.

By the expression (2.93) of g, we have

$$\sigma_x^{\frac{1}{2}} g = \sigma_x^{\frac{1}{2}}\tilde{b} + \sigma_x^{\frac{1}{2}} U\sigma_x^{-\frac{1}{2}}\big(\sigma_x^{\frac{1}{2}}\bar{K}g + \sigma_x^{\frac{1}{2}} h\big). \tag{2.105}$$

Similar to the proof for (2.96), we can show that $\sigma_x^{\frac{1}{2}} U\sigma_x^{-\frac{1}{2}}$ shares the same property of U in (2.96). Hence,

$$\begin{aligned} &\big\|\sigma_x^{\frac{1}{2}} g\big\|_{L^2_\xi(L^\infty_x)} \\ &\quad\leqslant c\big(\|b_0\|_+ + \big\|\sigma_x^{\frac{1}{2}}\bar{K}\sigma_x^{-\frac{1}{2}} w_{-\beta}\sigma_x^{\frac{1}{2}} w_\beta g\big\|_{L^2_\xi(L^\infty_x)} + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}} h\big\|_{L^2_\xi(L^\infty_x)}\big) \\ &\quad\leqslant c\big(\|b_0\|_+ + \big\|\sigma_x^{\frac{1}{2}} w_\beta g\big\|_{L^2_\xi(L^\infty_x)} + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}} h\big\|_{L^2_\xi(L^\infty_x)}\big) \\ &\quad\leqslant c\big(\|b_0\|_+ + E_0 + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}} h\big\|_{L^2_\xi(L^\infty_x)}\big), \end{aligned} \tag{2.106}$$

because $\sigma_x^{\frac{1}{2}}\bar{K}\sigma_x^{-\frac{1}{2}} w_{-\beta}$ is bounded from L^2_ξ to itself.

By (2.106), and using the expression of g again and noticing that $\sigma_x^{\frac{1}{2}}\bar{K}\sigma_x^{-\frac{1}{2}}$ is a bounded operator from $L^2_\xi(L^\infty_x)$ to $L^\infty_{x,\xi}$, we have

$$[\![g]\!]_0 \leqslant c\|b_0\|_{+,0} + \big\|\sigma_x^{\frac{1}{2}} U(\bar{K}g + h)\big\|_{L^\infty_{x,\xi}}$$

$$\leqslant c\big(\|b_0\|_{+,0} + \big\|\sigma_x^{\frac{1}{2}}\bar{K}\sigma_x^{-\frac{1}{2}}\sigma_x^{\frac{1}{2}}g\big\|_{L^\infty_{x,\xi}} + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\big\|_{L^\infty_{x,\xi}}\big)$$

$$\leqslant c\big(\|b_0\|_{+,0} + \big\|\sigma_x^{\frac{1}{2}}g\big\|_{L^2_\xi(L^\infty_x)} + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\big\|_{L^\infty_{x,\xi}}\big)$$

$$\leqslant c\big(\|b_0\|_{+,0} + E_0 + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\big\|_{L^2_\xi(L^\infty_x)} + [\![\nu^{-1}h]\!]_0\big). \tag{2.107}$$

By using (2.107), when $\beta > \frac{3}{2}$, we have

$$[\![g]\!]_\beta \leqslant c\big(\|b_0\|_{+,\beta} + E_0 + [\![\nu^{-1}h]\!]_\beta + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\big\|_{L^2_\xi(L^\infty_x)\cap L^\infty_{x,\xi}}\big)$$

$$\leqslant c\big(|b_0|_* + \big|[h]\big| + \big\|w_{-\alpha}\sigma_x^{\frac{1}{2}}h\big\| + [\![\nu^{-1}h]\!]_\beta + \big\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\big\|_{L^2_\xi(L^\infty_x)\cap L^\infty_{x,\xi}}\big)$$

$$\leqslant c\big(|b_0|_* + [\![\nu^{-1}h]\!]_\beta + \big|[h]\big| + \big\|w_{-\alpha}\sigma_x^{\frac{1}{2}}h\big\|\big), \tag{2.108}$$

because $\|\nu^{-1}\sigma_x^{\frac{1}{2}}h\|_{L^2_\xi(L^\infty_x)} \leqslant c[\![\nu^{-1}h]\!]_\beta$ when $\beta > \frac{3}{2}$. This completes the proof of the lemma. □

Now we are ready to prove existence of solution to the nonlinear Boltzmann equation with damping (2.14) and (2.15). Before that, we prove the following lemma on the property on $\Gamma(g,g)$ in the norm $[\![\cdot]\!]_\beta$ which is similar to the one in the norm $\|\cdot\|_\beta$ used for the hard sphere model.

LEMMA 2.23. *The projection of $\Gamma(g,g)$ on the null space of $\boldsymbol{L}$ vanishes and there exists a positive constant c such that for all (x,ξ),*

$$\big|\nu^{-1}(\xi)\sigma_x^{\frac{1}{2}}e^{\epsilon\sigma}\big(1+|\xi|\big)^\beta\Gamma\big(e^{-\epsilon\sigma}g, e^{-\epsilon\sigma}h\big)\big|$$
$$\leqslant c(\delta x + l)^{-\frac{2\beta+\gamma_0-1}{2(3-\gamma_0)}}[\![g]\!]_\beta[\![h]\!]_\beta, \tag{2.109}$$

for any $\beta > \frac{1-\gamma_0}{2}$ and sufficiently small constant $\epsilon > 0$.

PROOF. By the definition of $[\![\cdot]\!]_\beta$, we have

$$\big|\nu^{-1}(\xi)\sigma_x^{\frac{1}{2}}\big(1+|\xi|\big)^\beta e^{\epsilon\sigma}\Gamma\big(e^{-\epsilon\sigma}g, e^{-\epsilon\sigma}g\big)\big| \leqslant \sup_{x,\xi}\big|H(x,\xi)\big|[\![g]\!]_\beta[\![h]\!]_\beta,$$

where

$$H(x,\xi) = \nu^{-1}(\xi)\boldsymbol{W}_0^{-1}\big(1+|\xi|\big)^\beta\sigma_x^{\frac{1}{2}}e^{\epsilon\sigma}$$
$$\times Q\big(e^{-\epsilon\sigma}\boldsymbol{W}_0\sigma_x^{-\frac{1}{2}}\big(1+|\xi|\big)^{-\beta}, e^{-\epsilon\sigma}\boldsymbol{W}_0\sigma_x^{-\frac{1}{2}}\big(1+|\xi|\big)^{-\beta}\big).$$

All we need to prove is that $\sup_{x,\xi} H(x,\xi)$ is uniformly bounded by $c(\delta x+l)^{-\frac{2\beta+\gamma_0-1}{2(3-\gamma_0)}}$ when ϵ is sufficiently small.

According to the definition of $Q(\cdot,\cdot)$, there are two terms representing the gain and loss respectively. Hence, we have

$$\begin{aligned} H &= \nu^{-1}(\xi)\boldsymbol{W}_0^{-1}\big(1+|\xi|\big)^{\beta}\sigma_x^{\frac{1}{2}}e^{\epsilon\sigma}\iint \boldsymbol{W}_0\big(\xi'_*\big)\boldsymbol{W}_0(\xi')\big(1+\big|\xi'_*\big|\big)^{-\beta}\big(1+|\xi'|\big)^{-\beta} \\ &\quad\times \sigma_x^{-\frac{1}{2}}\big(x,\xi'_*\big)\sigma_x^{-\frac{1}{2}}(x,\xi')e^{-\epsilon\sigma'_*-\epsilon\sigma'}q(v,\theta)\,d\xi_*\,dw \\ &\quad- \nu^{-1}(\xi)\iint \boldsymbol{W}_0(\xi_*)\big(1+|\xi_*|\big)^{-\beta}\sigma_x^{-\frac{1}{2}}(x,\xi_*)e^{-\epsilon\sigma_*}q(v,\theta)\,d\xi_*\,dw \\ &= H_1-H_2. \end{aligned} \tag{2.110}$$

Notice that $\boldsymbol{W}_0\boldsymbol{W}_0(\xi_*)=\boldsymbol{W}_0(\xi')\boldsymbol{W}_0(\xi'_*)$. By using $\sigma_*\geqslant c(\delta x+l)^{\frac{2}{3-\gamma_0}}$, we have

$$\begin{aligned} H_2 &\leqslant c\nu^{-1}(\xi) \\ &\quad\times\iint_{\mathbb{R}^3}\boldsymbol{W}_0(\xi_*)q(\xi-\xi_*,\theta)e^{-\epsilon\sigma_*}\max\Big\{(\delta x+l)^{\frac{\Theta}{2}},\big(1+|\tilde{\xi}_*|\big)^{\frac{1-\gamma_0}{2}}\Big\}\,d\xi_*\,d\omega \\ &\leqslant ce^{-\epsilon^2(\delta x+l)^{\frac{2}{3-\gamma_0}}}. \end{aligned} \tag{2.111}$$

The estimation on H_1 is more complicated. Firstly, notice that for any positive constant α,

$$\big(1+|\xi'|\big)^{-\alpha}\big(1+|\xi'_*|\big)^{-\alpha}\leqslant c\big(1+|\xi|\big)^{-\alpha}. \tag{2.112}$$

Depending the locations of (x,ξ), (x,ξ') and (x,ξ'_*), there are different cases. For illustration, we consider the following two cases. The other cases can be proved similarly.

CASE 1. When $(x,\xi),(x,\xi'),(x,\xi'_*)\in\Omega_1$, we have when $\beta>\frac{1-\gamma_0}{2}$,

$$\begin{aligned} H_1 &\leqslant c\nu^{-1}(\xi)\iint_{\mathbb{R}^3}\boldsymbol{W}_0(\xi_*)q(\xi-\xi_*,\theta)\frac{(1+|\xi|)^{\beta-\frac{1-\gamma_0}{2}}}{(1+|\xi'|)^{\beta-\frac{1-\gamma_0}{2}}(1+|\xi'_*|)^{\beta-\frac{1-\gamma_0}{2}}} \\ &\quad\times\exp\bigg\{\epsilon\bigg(\frac{\delta x+l}{(1+|\tilde{\xi}|)^{1-\gamma_0}}+3|\tilde{\xi}|^2-\frac{\delta x+l}{(1+|\tilde{\xi}'|)^{1-\gamma_0}} \\ &\qquad -3|\tilde{\xi}'|^2-\frac{\delta x+l}{(1+|\tilde{\xi}'_*|)^{1-\gamma_0}}-3\big|\tilde{\xi}'_*\big|^2\bigg)\bigg\}\,d\xi_*\,d\omega \\ &\leqslant c\nu^{-1}(\xi)\iint_{\mathbb{R}^3}\boldsymbol{W}_0(\xi_*)q(\xi-\xi_*,\theta)\exp\big\{\epsilon\Delta_1-3\epsilon|\tilde{\xi}_*|^2\big\}\,d\xi_*\,d\omega, \end{aligned} \tag{2.113}$$

where

$$\varDelta_1 = \frac{\delta x + l}{(1 + |\tilde{\xi}|)^{1-\gamma_0}} - \frac{\delta x + l}{(1 + |\tilde{\xi}'|)^{1-\gamma_0}} - \frac{\delta x + l}{(1 + |\tilde{\xi}'_*|)^{1-\gamma_0}}.$$

If $\varDelta_1 \leqslant 0$, then (2.113) yields

$$\begin{aligned} H_1 &\leqslant c_1 \nu^{-1}(\xi) \iint_{\mathbb{R}^3} \boldsymbol{W}_0(\xi_*) q(\xi - \xi_*, \theta)(\delta x + l)^{-\frac{2\beta+\gamma_0-1}{2(3-\gamma_0)}} \, d\xi_* \, d\omega \\ &\leqslant c(\delta x + l)^{-\frac{2\beta+\gamma_0-1}{2(3-\gamma_0)}}. \end{aligned}$$

Otherwise, we have $|\tilde{\xi}| \leqslant \min\{|\tilde{\xi}'|, |\tilde{\xi}'_*|\}$. In this case, since

$$|\tilde{\xi}|^2 + |\tilde{\xi}_*|^2 = |\tilde{\xi}'|^2 + |\tilde{\xi}'_*|^2,$$

we have $|\tilde{\xi}| \leqslant |\tilde{\xi}_*|$ and thus $\frac{\delta x+l}{(1+|\tilde{\xi}|)^{1-\gamma_0}} \leqslant |\tilde{\xi}|^2 \leqslant |\tilde{\xi}_*|^2$. Hence, when ϵ is sufficiently small, $\exp\{\epsilon \varDelta_1\}$ can be absorbed by $\boldsymbol{W}_0(\xi_*)$ so that H_1 is bounded by $c \exp\{-\epsilon^2 (\delta x + l)^{\frac{2}{3-\gamma_0}}\}$.

CASE 2. When $(x, \xi) \in \varOmega_3$ and (x, ξ'), $(x, \xi'_*) \in \varOmega_1$, H_1 can be estimated as follows

$$\begin{aligned} H_1 &\leqslant c\nu^{-1}(\xi) \iint_{\mathbb{R}^3} \boldsymbol{W}_0(\xi_*) q(\xi - \xi_*, \theta)\big(1 + |\tilde{\xi}'|\big)^{\frac{1-\gamma_0}{2}} \big(1 + |\tilde{\xi}'_*|\big)^{\frac{1-\gamma_0}{2}} \\ &\quad \times (\delta x + l)^{-\frac{\varTheta}{2}} \exp\bigg\{\epsilon\bigg(5(x + l)^{\frac{2}{3-\gamma_0}} - \frac{\delta x + l}{(1 + |\tilde{\xi}'|)^{1-\gamma_0}} \\ &\quad - \frac{\delta x + l}{(1 + |\tilde{\xi}'_*|)^{1-\gamma_0}} - 3|\tilde{\xi}'|^2 - 3|\tilde{\xi}'_*|^2\bigg)\bigg\} \, d\xi_* \, d\omega \\ &\leqslant c\nu^{-1}(\xi) \iint_{\mathbb{R}^3} \boldsymbol{W}_0(\xi_*) q(\xi - \xi_*, \theta) \exp\big\{-\epsilon^2 (\delta x + l)^{\frac{2}{3-\gamma_0}}\big\} \, d\xi_* \, d\omega \\ &\leqslant c \exp\big\{-\epsilon^2 (\delta x + l)^{\frac{2}{3-\gamma_0}}\big\}, \end{aligned} \tag{2.114}$$

where we have used

$$|\tilde{\xi}'|^2 + |\tilde{\xi}'_*|^2 \geqslant \frac{9}{5}(\delta x + l)^{\frac{2}{3-\gamma_0}},$$

when l is sufficiently large because (x, ξ') and $(x, \xi'_*) \in \varOmega_1$.

Therefore, combining the estimates on H_1 and H_2 gives the estimate (2.109). □

We are now ready to prove the global existence for the nonlinear Boltzmann equation with damping. For $\beta > \max\{\frac{3}{2}+\gamma_0, 3-2\gamma_0\}$, by (2.109), straightforward calculation yields

$$\begin{aligned}
&\big[\!\big[\nu^{-1} e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big]\!\big]_\beta \leqslant c[\![g]\!]_\beta^2, \\
&\big\|\sigma_x^{-\frac{1}{2}} e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big\| \leqslant c[\![g]\!]_\beta^2, \qquad (2.115)\\
&\big|\big[e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big]\big| \\
&\quad \leqslant c\big(\big\|\sigma_x^{-\frac{1}{2}} e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big\| + \big\|e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big\|\big) \leqslant c[\![g]\!]_\beta^2,
\end{aligned}$$

and for $0 < \alpha < \frac{1}{2}$, we have

$$\big\|w_{-\alpha} \sigma_x^{\frac{1}{2}} e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)\big\| \leqslant c[\![g]\!]_\beta^2.$$

By noticing that $h = e^{\epsilon\sigma} \Gamma(e^{-\epsilon\sigma} g, e^{-\epsilon\sigma} g)$, in summary, (2.99) gives

$$[\![g]\!]_\beta \leqslant c\big([\![g]\!]_\beta^2 + |b_0|_*\big). \qquad (2.116)$$

When $|b_0|_*$ is sufficiently small, the contraction mapping theorem and (2.116) give the following existence theorem for the problem (2.14) and (2.15).

THEOREM 2.24. *When the boundary data F_b satisfies that $e^{\epsilon\sigma(0,\xi)} \mathbf{W}_0^{-1}[F_b - \mathbf{M}_\infty]$ is sufficiently small in the norm $|\cdot|_*$ given in* (2.100), *the nonlinear Boltzmann equation with damping* (2.14) *and* (2.15) *has a unique solution F such that $e^{\epsilon\sigma} \mathbf{W}_0^{-1}[F - \mathbf{M}_\infty]$ is bounded in the norm $[\![\cdot]\!]_\beta$ when $\beta > \max\{\frac{3}{2} + \gamma_0, 3 - 2\gamma_0\}$.*

Similar to the argument for the hard sphere model, the solvability condition on the boundary data can be obtained by considering the corresponding solution operators to ensure the vanishing of the artificial damping. And the co-dimension on the boundary data follows from the same argument.

2.3. *Stability for $\mathcal{M}^\infty < -1$*

In this section, we will apply the energy method and a bootstrap argument to prove the stability of the boundary layer obtained in the previous subsections. Besides its mathematical importance, the stability yields that the nonnegativity of the boundary layer solution obtained in the previous subsections. Again, because of the intrinsic difference between the hard sphere model and the cutoff hard potentials, the convergence rates of the solutions for the initial boundary value problems to the boundary layers are different. Precisely, it is of exponential order for the hard sphere model while it is of only algebraic order for the cutoff hard potentials. Here we only consider the case when the far field is incoming and supersonic. This case is easier than other cases because all the information from infinity goes into the boundary layer and in macroscopic level, no wave propagates to infinity as time tends to infinity.

2.3.1. *Hard sphere model.* Let $\bar{F} = \bar{F}(x, \xi)$ be the stationary solution to the problem (2.3). Consider the initial boundary value problem,

$$\begin{cases} F_t + \xi_1 F_x = Q(F, F), & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ F|_{t=0} = F_0(x, \xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_b(\xi), & t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ F \to \mathbf{M}_\infty(\xi) \quad (x \to \infty), & t > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.117}$$

The stability of the boundary layer when the Mach number of the far field is less than -1 can be stated as follows for the hard sphere model.

THEOREM 2.25. *When $\mathcal{M}^\infty < -1$, under the assumption that*

$$\left|F_b(\xi) - \mathbf{M}_\infty(\xi)\right| \leqslant \epsilon_0 \boldsymbol{W}_\beta(\xi), \quad \xi \in \mathbb{R}^3_+,\ \beta > 5/2,$$

for a sufficiently small positive constant ϵ_0, let $\bar{F}(x, \xi)$ be the boundary layer solution to (2.3) *given in Theorem* 2.5. *For the problem* (2.117), *when the initial data satisfies*

$$[\![\boldsymbol{W}_0^{-1} e^{\epsilon\sigma(0,\xi)} \big(F_0(x, \xi) - \bar{F}(x, \xi)\big)]\!]_\beta < \epsilon_1, \quad \beta > 5/2,$$

where $\epsilon_1 > 0$ is a sufficiently small constant, there exists a unique solution $F(t, x, \xi)$ which decays exponentially in time to $\bar{F}(x, \xi)$. In other words, the boundary layer solution is nonlinearly stable.

To prove the stability result, we will first consider two semi-groups associated with two linearized problems of (2.117) and show that they both have exponential decay property. Then by applying the bootstrap argument and using the smallness assumption on the strength of the boundary layer, we will have the nonlinear stability result stated in Theorem 2.25. This will be done in two steps. The first step is to consider the corresponding linearized problem by energy method for $L^2_{x,\xi}$ estimate and the bootstrap argument for $L^\infty_{x,\xi}$ estimate. The exponential decay in time estimate obtained in the first step can be used in the second step for the nonlinear stability by using the Grad's estimate on nonlinear Boltzmann collision operator to obtain an a priori estimate on the solution for the application of fixed point theorem.

Again, write the solution of (2.117) in the form

$$F(t, x, \xi) = \mathbf{M}_\infty(\xi) + \boldsymbol{W}_0(\xi) f(t, x, \xi). \tag{2.118}$$

Then, the problem (2.117) reduces to

$$\begin{cases} f_t + \xi_1 f_x - \boldsymbol{L} f = \Gamma(f), & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{t=0} = f_0(x, \xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi) = \boldsymbol{W}_0^{-1}(F_b - \mathbf{M}_\infty), & t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ f \to 0 \quad (x \to \infty), & t > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.119}$$

Let $f = e^{-\epsilon x} g$ in (2.119) to have

$$\begin{cases} g_t + \xi_1 g_x - \epsilon \xi_1 g - \boldsymbol{L} g = e^{-\epsilon x} \Gamma(g), \\ \quad t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ g|_{t=0} = g_0(x, \xi), \qquad x > 0,\ \xi \in \mathbb{R}^3, \\ g|_{x=0} = a_0(\xi), \qquad t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ g \to 0 \quad (x \to \infty), \qquad t > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.120}$$

Now, denote the stationary boundary layer solution to (2.120) by $\bar{g}$ and let the initial data g_0 be a small perturbation of $\bar{g}$. Then the stability problem can be formulated as:

$$\begin{cases} \tilde{g}_t + \xi_1 \tilde{g}_x - \epsilon \xi_1 \tilde{g} - \boldsymbol{L} \tilde{g} = e^{-\epsilon x} \{ \bar{\boldsymbol{L}} \tilde{g} + \Gamma(\tilde{g}) \}, \\ \quad t > 0, x > 0,\ \xi \in \mathbb{R}^3, \\ \tilde{g}|_{t=0} = \tilde{g}_0(x, \xi), \qquad x > 0,\ \xi \in \mathbb{R}^3, \\ \tilde{g}|_{x=0} = 0, \qquad t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ \tilde{g} \to 0 \quad (x \to \infty), \qquad t > 0,\ \xi \in \mathbb{R}^3, \end{cases} \tag{2.121}$$

where $\tilde{g} = g - \bar{g}$, $\tilde{g}_0 = g_0 - \bar{g}$ and $\bar{\boldsymbol{L}} \tilde{g} = 2\Gamma(\bar{g}, \tilde{g})$.

Let $S(t)$ be the solution operator (semi-group) of the linear problem

$$\begin{cases} h_t + \xi_1 h_x - \epsilon \xi_1 h - \boldsymbol{L} h = 0, \qquad t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ h|_{x=0} = 0\ (\xi_1 > 0),\ h \to 0\ (x \to \infty), \qquad t > 0,\ \xi \in \mathbb{R}^3, \\ h|_{t=0} = h_0(x, \xi), \qquad x > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.122}$$

Then we have $h = S(t) h_0$.

For the case $\mathcal{M}^\infty < -1$, L^2 decay estimate for (2.122) is easy to establish. Recall that in this case, the operator $A = \boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0$ is negative definite on $\mathcal{N}$. Now for a small $\epsilon > 0$, a straightforward energy estimation gives

$$\frac{1}{2} \frac{d}{dt} \| h(t) \|^2 + \langle |\xi_1| h^0, h^0 \rangle_- + \frac{\nu_1}{2} \| (1 + |\xi|)^{1/2} h(t) \|^2 \leqslant 0,$$

where $h^0 = h|_{x=0}$. This implies that

$$\frac{d}{dt} \left(e^{\frac{\nu_1}{2} t} \| h(t) \|^2 \right) + e^{\frac{\nu_1}{2} t} \left\{ 2 \langle |\xi_1| h^0(t), h^0(t) \rangle_- + \frac{\nu_1}{2} \| (1 + |\xi|)^{1/2} h(t) \|^2 \right\} \leqslant 0.$$

Thus,

$$e^{\frac{\nu_1}{2} t} \| h(t) \|^2 + \int_0^t e^{\frac{\nu_1}{2} t} \left\{ 2 \langle |\xi_1| h^0(t), h^0(t) \rangle_- + \frac{\nu_1}{2} \| (1 + |\xi|)^{1/2} h(t) \|^2 \right\} dt \\ \leqslant \| h_0 \|^2, \tag{2.123}$$

and

$$\left\|S(t)h_0\right\| \leqslant e^{-\kappa t}\|h_0\|, \quad \kappa = \frac{\nu_1}{4}. \tag{2.124}$$

As for the existence analysis, we want to prove the following estimate which is needed for the application of the fixed point theorem to get the global existence of solution to the nonlinear problem (2.121). For $\beta \geqslant 0$,

$$\left\|S(t)h_0\right\|_\beta \leqslant ce^{-\kappa t}\left\{\|h_0\|_\beta + \|h_0\|_{L^2_{x,\xi}}\right\}. \tag{2.125}$$

In order to prove (2.125), we first consider another simpler linear solution operator. Let $S_0(t)$ be the solution operator (semi-group) of

$$\begin{cases} h_t + \xi_1 h_x - \epsilon\xi_1 h + \nu(\xi)h = 0, & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ h|_{x=0} = 0\ (\xi_1 > 0), \quad h \to 0\ (x \to \infty), & t > 0,\ \xi \in \mathbb{R}^3, \\ h|_{t=0} = h_0(x,\xi), & x > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.126}$$

The solution to the above linear initial boundary value problem has the following explicit expression.

$$h = S_0(t)h_0 = e^{-(\nu(\xi)-\epsilon\xi_1)t}\chi(x-\xi_1 t)h_0(x-\xi_1 t,\xi), \tag{2.127}$$

where $\chi(y)$ is the usual characteristic function for $y > 0$. Based on this expression and with the lower bound $\nu(\xi) \geqslant \nu_0 > 0$, a simple calculation yields the following estimate on S_0.

$$\left\|S_0(t)h_0\right\|_X \leqslant ce^{-(2\kappa-\varepsilon)t}\|h_0\|_X, \tag{2.128}$$

with κ chosen to be $\min(\frac{\nu_0}{2}, \frac{\nu_1}{4})$, for some small constant $\epsilon > 0$. Here, the space X can be either L^∞_β or $L^2_{x,\xi}$.

From (2.122) and (2.126), we have

$$\begin{cases} S(t)h_0 = S_0(t)h_0 + \int_0^t S_0(t-s)KS(s)h_0\,ds \\ \qquad\quad = \sum_{j=0}^{m-1} I_j(t) + J_m(t), \\ I_0(t) = S_0(t)h_0, \\ I_j(t) = \int_0^t S_0(t-s)KI_{j-1}(s)\,ds = (S_0K) * I_{j-1}, \\ J_m(t) = \underbrace{(S_0K) * (S_0K) * \cdots * (S_0K)}_{m} * h, \end{cases} \tag{2.129}$$

with $h = S(t)h_0$. Here and hereafter, "$*$" stands for the convolution in t. By using the estimate (2.128) and the regularizing property of the compact operator K, we have for $\beta \geqslant j \geqslant 0$,

$$\left\|I_j(t)\right\|_\beta \leqslant c_j e^{(-2\kappa+\varepsilon)t}\|h_0\|_{\beta-j}. \tag{2.130}$$

The estimate on J_m is more complicated and can be stated in the following bootstrap lemma.

LEMMA 2.26. *For $\beta \geqslant 0$, we have*

$$\left\| J_{\beta+3}(t) \right\|_{\beta} \leqslant c e^{-\kappa t} \|h_0\|_{L^2_{x,\xi}}.$$

PROOF. Firstly, again by the regularizing property of K, we have

$$\left\| J_{\beta+3}(t) \right\|_{\beta} \leqslant \frac{C}{\beta!} \int_0^t (t-\tau)^{\beta} e^{-(2\kappa-\epsilon)(t-\tau)} \|J_2\|_{L^\infty_x(L^2_\xi)}(\tau)\, d\tau, \tag{2.131}$$

where

$$J_2(t) = (S_0 K) * (S_0 K) * h = S_0 * \bar{J}, \tag{2.132}$$

with

$$\bar{J} = K S_0 K * h = \int_0^t K S_0(t-s) K h(s)\, ds = \int_0^t \bar{J}_0(t-s, s)\, ds. \tag{2.133}$$

We now estimate $\bar{J}_0(t,s)$ as follows by using the integrability property of the compact operator K. By definition, we have

$$\begin{aligned} \bar{J}_0(t,s) &= K S_0(t) K h(s) \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} K(\xi,\xi') K(\xi',\xi'') e^{-(\nu(\xi')-\epsilon\xi_1')t} \chi(y) \\ &\quad \times h(s, y, \xi'')\, d\xi'\, d\xi'', \end{aligned} \tag{2.134}$$

where $y = x - \xi_1' t$. Hence,

$$\left|\bar{J}_0(t,s)\right| \leqslant e^{-(\nu_0-\varepsilon)t} \int_{\mathbb{R}\times\mathbb{R}^3} K_0\big(\xi,\xi_1',\xi''\big) \chi(y) \left|h(s,y,\xi'')\right| d\xi_1'\, d\xi'', \tag{2.135}$$

where

$$K_0\big(\xi,\xi_1',\xi''\big) \equiv \int_{\mathbb{R}^2} \left|K(\xi,\xi')\right| \left|K(\xi',\xi'')\right| d\xi_2'\, d\xi_3',$$

with $\xi' = (\xi_1', \xi_2', \xi_3')$.

Notice that the estimate of the kernel $K(\xi,\xi')$ satisfies

$$\begin{aligned} &\int_{\mathbb{R}^3} \left|K(\xi,\xi')\right| d\xi' = \int_{\mathbb{R}^3} \left|K(\xi',\xi)\right| d\xi' \leqslant C_0, \\ &\int_{\mathbb{R}^2} \left|K(\xi,\xi')\right| d\xi_2'\, d\xi_3' \leqslant C_1, \end{aligned}$$

where C_0 and C_1 are some positive constants depending only on the parameters $\rho_\infty, u_\infty, T_\infty$. Thus, we have

$$\int_{\mathbb{R}\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\,d\xi_1'\,d\xi'' = \int_{\mathbb{R}^3\times\mathbb{R}^3} \big|K(\xi,\xi')\big|\big|K(\xi',\xi'')\big|\,d\xi'\,d\xi'' \leqslant C_0^2,$$
$$\int_{\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\,d\xi \leqslant C_0 \int_{\mathbb{R}^2} \big|K(\xi',\xi'')\big|\,d\xi_2'\,d\xi_3'' \leqslant C_0 C_1.$$

By (2.135) and Schwarz inequality,

$$\begin{aligned}\big|\bar{J}_0(t,s)\big|^2 &\leqslant e^{-2(2\kappa-\epsilon)t}\left[\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\,d\xi_1'\,d\xi''\right]\\ &\quad\times\left[\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\chi(y)\big|h(s,y,\xi'')\big|^2\,d\xi_1'\,d\xi''\right]\\ &\leqslant C_0^2 e^{-2(2\kappa-\epsilon)t}\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\chi(y)\big|h(s,y,\xi'')\big|^2\,d\xi_1'\,d\xi''.\end{aligned} \tag{2.136}$$

Therefore, we have

$$\begin{aligned}\big\|\bar{J}_0(t,s)\big\|^2_{L^\infty_x(L^2_\xi)} &= \sup_{x>0}\int_{\mathbb{R}^3}\big|\bar{J}_0(t,s)\big|^2\,d\xi\\ &\leqslant C_0^2 C_0 C_1 e^{-2(2\kappa-\epsilon)t}\int_{\mathbb{R}\times\mathbb{R}^3}\chi(y)\big|h(s,y,\xi'')\big|^2\,d\xi_1'\,d\xi''\\ &= \frac{c}{t}e^{-2(2k-\varepsilon)t}\int_0^\infty dy\int_{\mathbb{R}^3} d\xi''\,\big|h(s,y,\xi'')\big|^2\\ &\leqslant \frac{c}{t}e^{-2(2k-\varepsilon)t}e^{-2ks}\|h_0\|^2_{L^2_{x,\xi}}.\end{aligned} \tag{2.137}$$

Here, we have used the L^2 decay estimate (2.124). Hence (2.133) and (2.137) give

$$\begin{aligned}\big\|\bar{J}(t)\big\|_{L^\infty_x(L^2_\xi)} &\leqslant \int_0^t \big\|\bar{J}_0(t-s,s)\big\|_{L^\infty_x(L^2_\xi)}\,ds\\ &\leqslant c\int_0^t \frac{e^{-(2\kappa-\epsilon)(t-s)}}{\sqrt{t-s}}e^{-\kappa s}\|h_0\|_{L^2_{x,\xi}}\,ds\\ &\leqslant ce^{-kt}\int_0^t \frac{e^{-(\kappa-\epsilon)(t-s)}}{\sqrt{t-s}}\,ds\,\|h_0\| \leqslant ce^{-\kappa t}\|h_0\|.\end{aligned} \tag{2.138}$$

This together with (2.128) and (2.132) give

$$\big\|J_2(t)\big\|_{L^\infty_x(L^2_\xi)} = \|S_0 * \bar{J}\| \leqslant \int_0^t e^{-(2\kappa-\varepsilon)(t-s)}\big\|\bar{J}(s)\big\|_{L^\infty_x(L^2_\xi)}\,ds$$

$$\leqslant c\int_0^t e^{-(2\kappa-\varepsilon)(t-s)}e^{-\kappa s}\,ds\,\|h_0\|_{L^2_{x,\xi}} \leqslant ce^{-\kappa t}\|h_0\|_{L^2_{x,\xi}}. \quad (2.139)$$

Plug this into (2.131) yields

$$\begin{aligned}\big\|J_{\beta+3}(t)\big\|_\beta &\leqslant e^{-\kappa t}\frac{c}{\beta!}\int_0^t (t-\tau)^\beta e^{-(\kappa-\epsilon)(t-\tau)}\,d\tau\,\|h_0\|_{L^2_{x,\xi}}\\ &\leqslant ce^{-\kappa t}\|h_0\|_{L^2_{x,\xi}}.\end{aligned} \quad (2.140)$$

And this completes the proof of the lemma. □

This lemma and (2.130) complete the proof of the L^∞_β decay estimate (2.125).

In order to estimate the nonlinear term $\Gamma(\tilde g)$ and the coupling term $\bar{\boldsymbol{L}}\tilde g$ in (2.121), we need the following lemma.

LEMMA 2.27. *When $\beta \geqslant 0$, for the two semi-groups S_0 and S, we have*

$$\|S_0 * \nu h\|_\beta(t) \leqslant ce^{-\kappa t}\sup_{0\leqslant\tau\leqslant t}\big\{e^{\kappa\tau}\|h\|_\beta(\tau)\big\},$$

$$\|S * \nu h\|_\beta(t) \leqslant ce^{-\kappa/2t}\Big\{\sup_{0\leqslant\tau\leqslant t}\big(e^{\kappa/2\tau}\|h\|_\beta(\tau)\big) + \sup_{0\leqslant\tau\leqslant t}\big(e^{\kappa/2\tau}\|\nu h\|_{L^2_{x,\xi}}(\tau)\big)\Big\},$$

both for every function $h(t,x,\xi)$ with the relevant norm bounded.

PROOF. Firstly, by the expression of the semi-group S_0 and the linear growth rate of $\nu(\xi)$, we have

$$\begin{aligned}\|S_0 * \nu h\|_\beta &\leqslant \sup_{x,\xi}\int_0^t \big(1+|\xi|^\beta\big)e^{-(\nu-\epsilon\xi_1)(t-s)}\chi(x-\xi_1 s)\nu\big|h(s,x-\xi_1 s,\xi)\big|\,ds\\ &\leqslant e^{-\kappa t}\sup_{0\leqslant\tau\leqslant t}\big\{e^{\kappa\tau}\|h\|_\beta(\tau)\big\}\sup_\xi\bigg\{\int_0^t e^{-(\nu(\xi)-\kappa-\epsilon\xi_1)(t-s)}\nu(\xi)\,ds\bigg\}\\ &\leqslant ce^{-\kappa t}\sup_{0\leqslant\tau\leqslant t}\big\{e^{\kappa\tau}\|h\|_\beta(\tau)\big\}.\end{aligned}$$

To give the estimate on S, we use the relation between S and S_0,

$$S = S_0 + S_0 * KS.$$

Write (2.125) as

$$\big\|S(t)h_0\big\|_\beta \leqslant ce^{-\kappa t}\{\{h_0\}\}_\beta, \quad (2.141)$$

with

$$\{\!\{\cdot\}\!\}_\beta = \|\cdot\|_\beta + \|\cdot\|_{L^2_{x,\xi}}. \tag{2.142}$$

We assume $\beta \geqslant 1$ but the proof is similar for other β. By the regularizing property of the operator K again and (2.141), we have

$$\begin{aligned}\|S_0 * KS * \nu h\|_\beta &\leqslant c\int_0^t e^{-\kappa(t-s)}\|S * \nu h\|_{\beta-1}(s)\,ds\\ &\leqslant c\int_0^t e^{-\kappa(t-s)}\int_0^s e^{-\kappa(s-\tau)}\{\!\{\nu h\}\!\}_{\beta-1}(\tau)\,d\tau\,ds\\ &\leqslant c\sup_{0\leqslant\tau\leqslant t}\left\{e^{\kappa/2\tau}\{\!\{\nu h\}\!\}_{\beta-1}(\tau)\right\}\int_0^t e^{-\kappa(t-s)}e^{-\kappa/2s}s\,ds\\ &\leqslant ce^{-\kappa/2t}\sup_{0\leqslant\tau\leqslant t}\left\{e^{\kappa/2\tau}\{\!\{\nu h\}\!\}_{\beta-1}(\tau)\right\}.\end{aligned}$$

Combining this with the estimate for S_0, we have

$$\begin{aligned}&\left\|S * \nu(\xi)h\right\|_\beta(t)\\ &\quad\leqslant ce^{-\kappa/2t}\left\{\sup_{0\leqslant\tau\leqslant t}\left(e^{\kappa/2\tau}\|h\|_\beta(\tau)\right)+\sup_{0\leqslant\tau\leqslant t}\left(e^{\kappa/2\tau}\{\!\{\nu h\}\!\}_{\beta-1}(\tau)\right)\right\}.\end{aligned}$$

Recalling the linear growth of $\nu(\xi)$ and the definition (2.142), this completes the proof of the lemma. □

By using the estimates in the above lemmas and (2.141), we can now construct a global solution to the nonlinear problem (2.121). The definition of the semi-group implies that

$$\tilde g = S(t)\tilde g_0 + S * \left\{e^{-\epsilon x}\left(\bar{\boldsymbol L}\tilde g + \Gamma(\tilde g)\right)\right\}. \tag{2.143}$$

Write the right hand side by $\Phi[\tilde g]$. We have

$$\begin{aligned}\left\|\Phi[\tilde g]\right\|_\beta &\leqslant \left\|S(t)\tilde g_0\right\|_\beta + \left\|S * \left\{\nu\nu^{-1}e^{-\epsilon x}\left(\bar{\boldsymbol L}\tilde g + \Gamma(\tilde g)\right)\right\}\right\|_\beta\\ &\leqslant ce^{-\kappa/2t}\Big\{\{\!\{\tilde g_0\}\!\}_\beta + \sup_{\tau\geqslant 0}\left(e^{\kappa/2\tau}\left\|e^{-\epsilon x}\nu^{-1}\left(\bar{\boldsymbol L}\tilde g+\Gamma(\tilde g)\right)\right\|_\beta(\tau)\right)\\ &\qquad + \sup_{\tau\geqslant 0}\left(e^{\kappa/2\tau}\left\|e^{-\epsilon x}\left(\bar{\boldsymbol L}\tilde g+\Gamma(\tilde g)\right)\right\|_{L^2_{x,\xi}}(\tau)\right)\Big\}\\ &\leqslant ce^{-\kappa/2t}\left\{\{\!\{\tilde g_0\}\!\}_\beta + \|\bar g\|_\beta |\!|\!|\tilde g|\!|\!| + |\!|\!|\tilde g|\!|\!|^2\right\},\end{aligned}$$

where

$$|\!|\!|h|\!|\!| = \sup_{t\geqslant 0}\left\{e^{\kappa/2t}\|h\|_\beta(t)\right\}. \tag{2.144}$$

In the above, we have used for $\beta > \frac{5}{2}$ to obtain

$$\left\| e^{-\epsilon x} \nu h \right\|^2_{L^2_{x,\xi}} \leqslant \left(\int_0^\infty e^{-2\epsilon x}\, dx \int_{\mathbb{R}^3} \nu^2(\xi)\big(1+|\xi|\big)^{-2\beta}\, d\xi \right) \|h\|^2_\beta = c\|h\|^2_\beta.$$

Consequently,

$$\left|\!\left|\!\left| \Phi[\tilde{g}] \right|\!\right|\!\right|_\beta \leqslant c\big(\{\!\{\tilde{g}_0\}\!\}_\beta + \|\bar{g}\|_\beta |\!|\!|\tilde{g}|\!|\!| + |\!|\!|\tilde{g}|\!|\!|^2\big).$$

Similarly,

$$\left|\!\left|\!\left| \Phi[\tilde{g}] - \Phi[\tilde{h}] \right|\!\right|\!\right|_\beta \leqslant c\big(\|\bar{g}\|_\beta |\!|\!|\tilde{g} - \tilde{h}|\!|\!| + |\!|\!|\tilde{g} + \tilde{h}|\!|\!|\, |\!|\!|\tilde{g} - \tilde{h}|\!|\!|\big),$$

with some constant c.

The smallness assumption on $\{\!\{\bar{g}_0\}\!\}_\beta$ and that on $\|\bar{g}\|_\beta$ which follows from the smallness assumption on the boundary data a_0 in (2.120) now guarantee that the nonlinear map Φ is contractive in a small ball in the Banach space defined with the norm (2.144) so that a unique fixed point exists. This implies, taken into account the choice of the norm (2.144), that (2.121) has a unique global in time solution converging exponentially to 0 as $t \to \infty$. Thus Theorem 2.25 follows.

2.3.2. *Cutoff hard potentials.* We now turn to study the stability of the boundary layer to the Boltzmann equation for the cutoff hard potentials. As for the existence theory, the sub-linear growth in the collision frequency creates some difficulty in the analysis. For the stability, an exponential decay in the form of e^{-ct} can not be expected. However, the following analysis gives an algebraic decay rate which may not be optimal.

To formulate the problem on the nonlinear stability of the boundary layers for the cutoff hard potentials, set $\bar{F} = \bar{F}(x,\xi)$ again to be the boundary layer solution. Consider the initial boundary value problem (2.117) and the stability theorem when $\mathcal{M}^\infty < -1$ can be stated as follows.

THEOREM 2.28. *When the Mach number $\mathcal{M}^\infty < -1$, and the boundary data satisfy*

$$\begin{aligned} &\big|F_b(\xi) - \mathbf{M}_\infty(\xi)\big| \leqslant \epsilon_0 \sigma_x^{-\frac{1}{2}} e^{-\epsilon\sigma(0,\,\xi)} \mathbf{W}_\beta(\xi), \\ &\quad \xi \in \mathbb{R}^3_+,\ \beta > \max\left\{\frac{3}{2} + \gamma_0, 3 - 2\gamma_0\right\}, \end{aligned}$$

where ϵ_0 is a sufficiently small positive constant, let $\bar{F}(x,\xi)$ be the boundary layer solution to (2.3) *given in Theorem* 2.11. *For the initial boundary value problem* (2.117), *if the initial data satisfy for $\beta > \max\{3 - \frac{\gamma_0}{2}, \frac{7}{2} - \frac{5}{2}\gamma_0\}$,*

$$\begin{aligned} &\left\| \sigma_x^{-1} \boldsymbol{W}_0^{-1} e^{\epsilon\sigma}\big(F_0(x,\xi) - \bar{F}(x,\xi)\big)\right\|_{L^2_{x,\xi}} \\ &\quad + \big[\!\big[\boldsymbol{W}_0^{-1} e^{\epsilon\sigma}\big(F_0(x,\xi) - \bar{F}(x,\xi)\big)\big]\!\big]_\beta < \epsilon_1, \end{aligned} \tag{2.145}$$

where $\epsilon_1 > 0$ *is a sufficiently small constant, and* $l \geqslant \epsilon^{-3+\gamma_0}$ *in* (2.64), *then there exists a unique solution* $F(t,x,\xi)$ *such that*

$$[\![W_0^{-1} e^{\epsilon\sigma}(F - \bar{F})]\!]_\beta \leqslant C\epsilon_1 (1+t)^{-\frac{3}{2}}.$$

This implies that the perturbation of the boundary layer converges to zero in time with an algebraic rate $(1+t)^{-\frac{3}{2}}$, *that is, the boundary layer solution is nonlinearly stable.*

REMARK 2.29. If we impose faster spatial decay on the initial data, then faster time decay in the perturbation can be obtained. More precisely, for $m \geqslant 2$, if we replace (2.145) by

$$\begin{aligned} &\left\| \sigma_x^{-\frac{m}{2}} W_0^{-1} e^{\epsilon\sigma} \big(F_0(x,\xi) - \bar{F}(x,\xi)\big) \right\|_{L^2_{x,\xi}} \\ &\quad + [\![W_0^{-1} e^{\epsilon\sigma} \big(F_0(x,\xi) - \bar{F}(x,\xi)\big)]\!]_\beta < \epsilon_m, \end{aligned}$$

for $\beta > \max\{\frac{m+4}{2} - \frac{m-1}{2}\gamma_0, \frac{m+5}{2} - \frac{m+3}{2}\gamma_0\}$, then similar analysis gives that the decay rate as $(1+t)^{-\frac{m+1}{2}}$. However, $\epsilon_m \to 0$ as $m \to \infty$.

For later use, notice that the weight function $\sigma(x,\xi)$ satisfies

$$\left|\sigma_{xx}(x,\xi)\right| \leqslant \begin{cases} 0, & (x,\xi) \in \Omega_1, \\ c(\delta x + l)^{-\Theta-1}, & (x,\xi) \in \Omega_2 \cup \Omega_3, \end{cases}$$

where Ω_i, $i = 1,2,3$, are defined in (2.89).

The dissipation of the modified linearized collision operator on $\mathcal{N}^\perp$ will be given as follows. Since the proof is similar to the corresponding lemma for the hard sphere model, we omit it for brevity.

LEMMA 2.30. *There is constant* $\epsilon_2 > 0$ *such that for* $0 < \epsilon < \epsilon_2$ *and* $g \in \mathcal{N}^\perp$

$$\left\langle g, \sigma_x^{-\frac{m}{2}} e^{\epsilon\sigma} L e^{-\epsilon\sigma} \sigma_x^{\frac{m}{2}} g \right\rangle \leqslant -\nu_5 \left\langle \nu\big(|\xi|\big) g, g \right\rangle, \quad m = -1, 0, 1, 2,$$

for some positive constant ν_5.

Moreover, by the dissipation of the convection term on the macroscopic components given in the previous subsection:

$$\int_{|\tilde{\xi}|^{3-\gamma_0} \leqslant \frac{l}{2}} \phi \mathbf{P}_0 \xi_1 \mathbf{P}_0 \phi \, d\xi \leqslant -\nu_6 \int_{\mathbb{R}^3} \phi_0^2 \, d\xi,$$

we have the following lemma for $L_\epsilon = e^{\epsilon\sigma} L e^{-\epsilon\sigma}$.

LEMMA 2.31. *Assume that* $\mathcal{M}^\infty < -1$, ϵ *is sufficiently small and* $l \geqslant \epsilon^{-1}$. *Then there exists a constant* $c > 0$ *such that*

$$-\langle \sigma_x \xi_1 \phi, \phi \rangle \geqslant \frac{c\nu_6}{4} \big\langle (\delta x + l)^{-\frac{\Theta}{2}} \phi_0, (\delta x + l)^{-\frac{\Theta}{2}} \phi_0 \big\rangle - c \big\langle \nu(\xi)\phi_1, \phi_1 \big\rangle,$$

$$-\langle \boldsymbol{L}_\epsilon \phi, \phi \rangle \geqslant -c\epsilon^2 \big\langle (\delta x + l)^{-\frac{\Theta}{2}} \phi_0, (\delta x + l)^{-\frac{\Theta}{2}} \phi_0 \big\rangle + c\nu_6 \big\langle \nu(\xi)\phi_1, \phi_1 \big\rangle.$$

Furthermore,

$$-\varepsilon \langle \sigma_x \xi_1 \phi, \phi \rangle - \langle \boldsymbol{L}_\epsilon \phi, \phi \rangle \geqslant \frac{c\nu_6 \epsilon}{8} \big\langle (\delta x + l)^{-\frac{\Theta}{2}} \phi, (\delta x + l)^{-\frac{\Theta}{2}} \phi \big\rangle.$$

The following lemma comes directly from the definition of $\sigma(x, \xi)$ so that we omit its proof for brevity.

LEMMA 2.32. *When* $\mathcal{M}^\infty < -1$, ϵ *is sufficiently small and* $l \geqslant \epsilon^{-3+\gamma_0}$, *there exists a constant* $c > 0$ *such that for any* ϕ,

$$\left| \int_{\mathbb{R}^3} \frac{\sigma_{xx}}{\sigma_x} \xi_1 \phi^2 \, d\xi \right| \leqslant c\epsilon^2 (\delta x + l)^{-\Theta} \langle \phi_0, \phi_0 \rangle + c\epsilon^{\frac{4}{3}} \big\langle \nu\big(|\xi|\big)\phi_1, \phi_1 \big\rangle.$$

The next lemma is about how to transfer the decay in space to the decay in time through some recursive relations in terms of energy inequalities. This lemma is crucially used to obtain the time convergence of the solution for the initial boundary value problem to the boundary layer.

LEMMA 2.33. *Suppose that* y *and* $\Xi(t, x, \xi) \geqslant 1$ *are functions of* (t, x, ξ). *Define*

$$\mathbf{I}_m(t) = \int_{\mathbf{R} \times \mathbf{R}^3} \Xi(t, x, \xi)^m y(t, x, \xi)^2 \, dx \, d\xi, \quad \textit{for } m \geqslant -1.$$

If there exists a positive constant ϵ *such that*

$$\frac{d}{dt}\mathbf{I}_{-1} \leqslant 0, \ \ldots, \ \frac{d}{dt}\mathbf{I}_{m-1} + \epsilon \mathbf{I}_{m-2} \leqslant 0, \ \frac{d}{dt}\mathbf{I}_m + \epsilon \mathbf{I}_{m-1} \leqslant 0, \tag{2.146}$$

then for $-1 \leqslant n \leqslant m$, *there exists constant* $c_{m,\epsilon}$ *such that*

$$(1 + t)^{m-n} \mathbf{I}_n(t) \leqslant c_{m,\epsilon} \mathbf{I}_m(0). \tag{2.147}$$

PROOF. By definition, notice that $\mathbf{I}_{-1}(t) \leqslant \mathbf{I}_0(t) \leqslant \cdots \leqslant \mathbf{I}_m(t)$ $for\ all\ t \geqslant 0$ because $\Xi \geqslant 1$. Define

$$F(t) = \frac{m!\epsilon}{(m+1)!}(1+t)^{m+1}\mathbf{I}_{-1} + \frac{m!}{m!}(1+t)^m \mathbf{I}_0 + \cdots + \frac{m!}{1!}\epsilon^{-m+1}(1+t)\mathbf{I}_{m-1} + \frac{m!}{0!}\epsilon^{-m}\mathbf{I}_m.$$

From (2.146), it is straightforward to obtain

$$\frac{d}{dt}F(t) = \frac{(1+t)^{m+1}\epsilon}{m+1}\frac{d}{dt}\mathbf{I}_{-1} + (1+t)^m\left[\frac{d}{dt}\mathbf{I}_0 + \epsilon\mathbf{I}_{-1}\right] + \cdots + m!\epsilon^{-m}\left[\frac{d}{dt}\mathbf{I}_m + \epsilon\mathbf{I}_{m-1}\right] \leqslant 0.$$

Thus, $F(t) \leqslant F(0)$ which implies that there exists a constant $c_{m,\epsilon}$ such that

$$(1+t)^{m-n}\mathbf{I}_n(t) \leqslant c_{m,\epsilon}\mathbf{I}_m(0), \quad \text{for } -1 \leqslant n \leqslant m.$$

The proof of the lemma is then completed. □

With the above preparation, we are now ready to prove the stability for incoming and supersonic case. Again, let $f = e^{-\epsilon\sigma(x,\xi)}g$ in (2.14). Then the problem for f is reduced to:

$$\begin{cases} g_t + \xi_1 g_x - \epsilon\sigma_x\xi_1 g - \boldsymbol{L}_\epsilon g \\ \quad = e^{\epsilon\sigma}\Gamma(e^{-\epsilon\sigma}g, e^{-\epsilon\sigma}g), & t>0,\ x>0,\ \xi\in\mathbb{R}^3, \\ g|_{t=0} = g_0(x,\xi) = e^{\epsilon\sigma}\mathbf{M}_\infty^{\frac{1}{2}}(F_0 - \mathbf{M}_\infty), & x>0,\ \xi\in\mathbb{R}^3, \\ g|_{x=0} = b_0(\xi) = e^{\epsilon\sigma(0,\xi)}a_0(\xi) \quad (\xi_1>0), & t>0,\ (\xi_2,\xi_3)\in\mathbb{R}^2, \\ g\to 0 \quad (x\to\infty), & t>0,\ \xi\in\mathbb{R}^3. \end{cases} \tag{2.148}$$

Denote the corresponding stationary boundary layer solution to (2.3) by $\bar{g}$ and let the initial g_0 be a small perturbation of $\bar{g}$. Then the stability problem can be formulated as:

$$\begin{cases} \tilde{g}_t + \xi_1\tilde{g}_x - \epsilon\sigma_x\xi_1\tilde{g} - \boldsymbol{L}_\epsilon\tilde{g} \\ \quad = e^{\epsilon\sigma}\{\bar{\boldsymbol{L}}_\epsilon\tilde{g} + \Gamma(e^{-\epsilon\sigma}\tilde{g}, e^{-\epsilon\sigma}\tilde{g})\}, & t>0,\ x>0,\ \xi\in\mathbb{R}^3, \\ \tilde{g}|_{t=0} = e^{\epsilon\sigma}\mathbf{M}_\infty^{\frac{1}{2}}(F_0 - \bar{F}), & x>0,\ \xi\in\mathbb{R}^3, \\ \tilde{g}|_{x=0} = 0 \quad (\xi_1>0), & t>0,\ (\xi_2,\xi_3)\in\mathbb{R}^2, \\ \tilde{g}\to 0 \quad (x\to\infty) & t>0,\ \xi\in\mathbb{R}^3, \end{cases} \tag{2.149}$$

where $\tilde{g} = g - \bar{g}$, $\tilde{g}_0 = g_0 - \bar{g}$ and $\bar{\boldsymbol{L}}_\epsilon = 2\Gamma(e^{-\epsilon\sigma}\bar{g}, e^{-\epsilon\sigma}\tilde{g})$.

We first derive some energy estimates on the solution to the linearized equation using several weight functions. Let $S(t)$ be the solution operator of the linear problem:

$$\begin{cases} h_t + \xi_1 h_x - \varepsilon\sigma_x\xi_1 h - \boldsymbol{L}_\varepsilon h = 0, & t>0,\ x>0,\ \xi\in\mathbb{R}^3, \\ h|_{x=0} = 0\ (\xi_1>0), & \\ h\to 0\ (x\to\infty), & t>0,\ \xi\in\mathbb{R}^3, \\ h|_{t=0} = h_0(x,\xi), & x>0, \xi\in\mathbb{R}^3. \end{cases} \tag{2.150}$$

For the hard sphere model, the solution operator $S(t)$ has exponential decay property when the Mach number is less then -1. For the cutoff hard potentials, we will use the spatial-temporal estimates in Lemma 2.33 to derive an algebraic decay.

Set

$$\rho^{-1} = \sigma_x^{\frac{1}{2}} h, \qquad \rho^0 = h, \qquad \rho^1 = \sigma_x^{-\frac{1}{2}} h, \qquad \rho^2 = \sigma_x^{-1} h.$$

It is straightforward to derive the following equations for ρ^i, $i = -1, \dots, 2$:

$$\begin{cases} \rho_t^{-1} + \xi_1 \rho_x^{-1} - \frac{\sigma_{xx}}{2\sigma_x} \xi_1 \rho^{-1} - \epsilon \sigma_x \xi_1 \rho^{-1} - \sigma_x^{\frac{1}{2}} \boldsymbol{L}_\epsilon \sigma_x^{-\frac{1}{2}} \rho^{-1} = 0, \\ \rho_t^0 + \xi_1 \rho_x^0 - \epsilon \sigma_x \xi_1 \rho^0 - \boldsymbol{L}_\epsilon \rho^0 = 0, \\ \rho_t^1 + \xi_1 \rho_x^1 + \frac{\sigma_{xx}}{2\sigma_x} \xi_1 \rho^1 - \epsilon \sigma_x \xi_1 \rho^1 - \sigma_x^{-\frac{1}{2}} \boldsymbol{L}_\epsilon \sigma_x^{\frac{1}{2}} \rho^1 = 0, \\ \rho_t^2 + \xi_1 j_x^2 + \frac{\sigma_{xx}}{\sigma_x} \xi_1 \rho^2 - \epsilon \sigma_x \xi_1 \rho^2 - \sigma_x^{-1} \boldsymbol{L}_\epsilon \sigma_x \rho^2 = 0. \end{cases} \tag{2.151}$$

By multiplying $(2.151)_1$ by ρ^{-1} and integrating it over $\mathbb{R}^+ \times \mathbb{R}^3$, when ϵ is sufficiently small, Lemma 2.30 and Lemma 2.31 give

$$\begin{aligned} &\left(\rho^{-1}, \rho^{-1}\right)_t + \left\langle |\xi_1| \rho^{-1}, \rho^{-1} \right\rangle_- \\ &\quad + c\epsilon \left((\delta x + l)^{-\frac{\Theta}{2}} \rho^{-1}, (\delta x + l)^{-\frac{\Theta}{2}} \rho^{-1} \right) \leqslant 0. \end{aligned} \tag{2.152}$$

Similarly, for sufficiently small ϵ, for $i = 0, 1, 2$, we have,

$$\left(\rho^i, \rho^i\right)_t + \left\langle |\xi_1| \rho^i, \rho^i \right\rangle_- + c\epsilon \left((\delta x + l)^{-\frac{\Theta}{2}} \rho^i, (\delta x + l)^{-\frac{\Theta}{2}} \rho^i \right) \leqslant 0.$$

Since $\sigma_x(x, \xi)^{-1} \geqslant c(\delta x + l)^{\Theta}$ and $\langle |\xi_1| \rho^i, \rho^i \rangle_- \geqslant 0$, for $i = -1, \dots, 2$, we have

$$\begin{aligned} &\left(\rho^{-1}, \rho^{-1}\right)_t \leqslant 0, \qquad \left(\rho^0, \rho^0\right)_t + c\epsilon\left(\rho^{-1}, \rho^{-1}\right) \leqslant 0, \\ &\left(\rho^1, \rho^1\right)_t + c\epsilon\left(\rho^0, \rho^0\right) \leqslant 0, \qquad \left(\rho^2, \rho^2\right)_t + c\epsilon\left(\rho^1, \rho^1\right) \leqslant 0. \end{aligned}$$

Corresponding to Lemma 2.33, if we choose $y = \rho^{-1}$, $\Xi = \sigma_x^{-\frac{1}{2}} \geqslant 1$ and $m = 2, n = -1$, (2.147) gives

$$\left(\rho^{-1}, \rho^{-1}\right)(t) \leqslant c\epsilon (1 + t)^{-3} \left(\rho^2, \rho^2\right)(0),$$

that is,

$$\left\| \sigma_x^{\frac{1}{2}} S(t) h_0 \right\|_{L^2_{x,\xi}} \leqslant c\epsilon (1 + t)^{-\frac{3}{2}} \left\| \sigma_x^{-1} h_0 \right\|_{L^2_{x,\xi}}. \tag{2.153}$$

The following decay estimate on the solution operator $S(t)$ is essential to obtain the global existence and convergence rate for the nonlinear problem through the fixed point theorem:

$$\left[\!\left[S(t) h_0 \right]\!\right]_\beta \leqslant c(1 + t)^{-\frac{3}{2}} \left\{ [\![h_0]\!]_\beta + \left\| \sigma_x^{-1} h_0 \right\|_{L^2_{x,\xi}} \right\}, \tag{2.154}$$

for some $\beta > 0$, where $[\![\cdot]\!]_\beta$ is the norm defined in the subsection for existence. To prove (2.154), consider a simpler linear solution operator. Let $S_0(t)$ be the solution operator of

$$\begin{cases} h_t + \xi_1 h_x - \epsilon\sigma_x\xi_1 h + \nu(\xi)h = 0, & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ h|_{t=0} = h_0(x,\xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ h|_{x=0} = 0, & x > 0,\ \xi_1 > 0,\ (\xi_2,\xi_3) \in \mathbb{R}^2, \\ h \to 0 \quad (x \to \infty), & t > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.155}$$

It is straightforward to check that the solution to the above linear initial boundary value problem has the following explicit expression:

$$\begin{aligned} h(t) &= S_0(t)h_0 \\ &= h_0(x-\xi_1 t,\xi)\chi(x-\xi_1 t)e^{-\int_0^t[\nu(\xi)-\epsilon\sigma_x(x-\xi_1(t-s),\xi)\xi_1]\,ds}. \end{aligned}$$

By using this expression and the positive lower bound for $\nu(\xi) \geqslant \nu_0 > 0$, the following lemma gives the exponential decay estimate on the operator $S_0(t)$. We omit its proof for brevity.

LEMMA 2.34. *For $S_0(t)$ defined above, there exist positive constants c and κ such that*

$$\left\|S_0(t)h_0\right\|_X \leqslant Ce^{-\kappa t}\|h_0\|_X, \quad \kappa > 0,\ X = [\![\cdot]\!]_\beta \text{ or } \|\cdot\|_{L^2_{x,\xi}}. \tag{2.156}$$

Similar to the discussion for the hard sphere model, we can rewrite $S(t)$ in terms of $S_0(t)$ and $\tilde{K} = e^{\epsilon\sigma}Ke^{-\epsilon\sigma}$:

$$\begin{aligned} S(t)h_0 &= S_0(t)h_0 + \int_o^t S_0(t-s)\tilde{K}S(s)h_0\,ds \\ &= \sum_{j=0}^{m-1} I_j(t) + J_m(t), \end{aligned}$$

where $h = S(t)h_0$ and I_j and J_m are defined as in (2.129) with K replaced by $\tilde{K}$. By using the estimate (2.156) and the regularizing property of the compact operator $\tilde{K}$ given in Lemma 2.19, we have for $\beta \geqslant j \geqslant 0$,

$$[\![I_j(t)]\!]_\beta \leqslant ce^{-\kappa t}[\![h_0]\!]_\beta.$$

The estimate on J_m can be obtained by a bootstrap argument as for the hard sphere model. The following proof mainly follows from the one of Lemma 2.26 where we replace K by $\sigma_x^{\frac{1}{2}}\tilde{K}\sigma_x^{-\frac{1}{2}}$ and the norm $\|\cdot\|_\beta$ by $[\![\cdot]\!]_\beta$. Moreover, the exponential decay for the hard sphere model now becomes the algebraic decay from the energy estimate given in (2.153). Thus, we omit its proof.

LEMMA 2.35. *For $\beta \geqslant 0$, there exists a constant $c > 0$ such that*

$$\big[\!\big[J_{\beta+3}(t) \big]\!\big]_\beta \leqslant c(1+t)^{-\frac{3}{2}} \big\| \sigma_x^{-1} h_0 \big\|_{L_x^\infty(L_\xi^2)}.$$

In order to estimate the nonlinear term and the coupling term with the boundary layer, the following lemma is needed which is similar to the one for the hard sphere model. The main difference is that here only algebraic decay is involved.

LEMMA 2.36. *When $\beta > 0$, for the two semi-groups S_0 and S, we have*

$$[\![S_0 * \nu h]\!]_\beta(t) \leqslant C(1+t)^{-\frac{3}{2}} \sup_{0 \leqslant \tau \leqslant t} \big\{ (1+\tau)^{\frac{3}{2}} [\![h]\!]_\beta(\tau) \big\},$$

$$[\![S * \nu h]\!]_\beta(t) \leqslant C(1+t)^{-\frac{3}{2}} \sup_{0 \leqslant \tau \leqslant t} \big\{ (1+\tau)^{\frac{3}{2}} [\![h]\!]_\beta(\tau) + (1+\tau)^{\frac{3}{2}} \big\| \nu \sigma_x^{-1} h \big\|_{L^2_{x,\xi}}(\tau) \big\},$$

for any function $h(t,x,\xi)$ with the corresponding norms bounded.

With the above estimates, the global solution to the nonlinear problem (2.149) can be proved as follows. Note that

$$\tilde{g} = S(t)\tilde{g}_0 + S * \big\{ e^{\epsilon\sigma} \big(\bar{\boldsymbol{L}}_\epsilon \tilde{g} + \Gamma(e^{-\epsilon\sigma}\tilde{g}, e^{-\epsilon\sigma}\tilde{g}) \big) \big\}.$$

By denoting the right-hand side of the above equation by $T[\tilde{g}]$, we have

$$\begin{aligned}
\big[\!\big[T[\tilde{g}] \big]\!\big]_\beta &\leqslant \big[\!\big[S(t)\tilde{g}_0 \big]\!\big]_\beta + \big[\!\big[S * \nu\nu^{-1} e^{\epsilon\sigma} \big(\bar{\boldsymbol{L}}_\epsilon \tilde{g} + \Gamma(e^{-\epsilon\sigma}\tilde{g}, e^{-\epsilon\sigma}\tilde{g}) \big) \big]\!\big]_\beta \\
&\leqslant C(1+t)^{-\frac{3}{2}} \Big\{ \{\!\{\!\{ \sigma_x^{-1}\tilde{g}_0 \}\!\}\!\}_\beta + \sup_{0 \leqslant \tau \leqslant t} \Big((1+\tau)^{\frac{3}{2}} \\
&\quad \times \big[\!\big[\nu^{-1} e^{\epsilon\sigma} \big(\bar{\boldsymbol{L}}_\epsilon \tilde{g} + \Gamma(e^{-\epsilon\sigma}\tilde{g}, e^{-\epsilon\sigma}\tilde{g}) \big) \big]\!\big]_\beta \Big) \\
&\quad + \sup_{0 \leqslant \tau \leqslant t} \Big((1+\tau)^{\frac{3}{2}} \big\| \sigma_x^{-1} \nu\nu^{-1} e^{\epsilon\sigma} \big(\bar{\boldsymbol{L}}_\epsilon \tilde{g} + \Gamma(e^{-\epsilon\sigma}\tilde{g}, e^{-\epsilon\sigma}\tilde{g}) \big) \big\|_{L^2_{x,\xi}} \Big) \Big\} \\
&\leqslant c(1+t)^{-\frac{3}{2}} \big\{ \{\!\{\!\{ \tilde{g}_0 \}\!\}\!\}_\beta + [\![\tilde{g}]\!]_\beta |\!|\!| \tilde{g} |\!|\!| + |\!|\!| \tilde{g} |\!|\!|^2 \big\},
\end{aligned}$$

where

$$\{\!\{\!\{ h \}\!\}\!\} = \big\| \sigma_x^{-1} h \big\|_{L^2_{x,\xi}} + [\![h]\!]_\beta, \qquad [\![\![h]\!]\!] = \sup_{t \geqslant 0} \big\{ (1+t)^{\frac{3}{2}} [\![h]\!]_\beta(t) \big\}.$$

Here, we have used for $\beta > \max\{3 - \frac{\gamma_0}{2}, \frac{7}{2} - \frac{5}{2}\gamma_0\}$,

$$
\begin{aligned}
&\left\|\sigma_x^{-1}e^{\epsilon\sigma}\Gamma(e^{-\epsilon\sigma}g,e^{-\epsilon\sigma}h)\right\|_{L^2_{x,\xi}}^2\\
&\quad\leqslant \int_0^\infty dx\int_{\mathbb{R}^3}\left|\sigma_x^{-3}\nu(\xi)^2\right|\left|\nu(\xi)^{-1}\sigma_x^{\frac12}e^{\epsilon\sigma}\Gamma(e^{-\epsilon\sigma}g,e^{-\epsilon\sigma}h)\right|^2 d\xi\\
&\quad\leqslant c\int_0^\infty dx\int_{\mathbb{R}^3}\left|\sigma_x^{-3}\nu(\xi)^2\right|(\delta x+l)^{-\frac{2\beta+\gamma_0-1}{3-\gamma_0}}\big(1+|\xi|\big)^{-2\beta}d\xi\,[\![g]\!]_\beta^2[\![h]\!]_\beta^2\\
&\quad\leqslant c\int_0^\infty dx\int_{\mathbb{R}^3}\Big((\delta x+l)^{3\Theta}+\big(1+|\tilde\xi|\big)^{3(1-\gamma_0)}\Big)(\delta x+l)^{-\frac{2\beta+\gamma_0-1}{3-\gamma_0}}\\
&\qquad\times\big(1+|\xi|\big)^{-2(\beta-\gamma_0)}d\xi\,[\![g]\!]_\beta^2[\![h]\!]_\beta^2\\
&\quad\leqslant c[\![g]\!]_\beta^2[\![h]\!]_\beta^2.
\end{aligned}
$$

Consequently,

$$
[\![\![T[\tilde g]]\!]\!] \leqslant c\big(\{\!\{\!\{\tilde g_0\}\!\}\!\}_\beta+[\![\bar g]\!]_\beta|\!|\!|\tilde g|\!|\!|+|\!|\!|\tilde g|\!|\!|^2\big).
$$

Similar argument gives

$$
[\![\![T[\tilde g]-T[\tilde h]]\!]\!] \leqslant c\big([\![\bar g]\!]_\beta|\!|\!|\tilde g-\tilde h|\!|\!|+|\!|\!|\tilde g+\tilde h|\!|\!||\!|\!|\tilde g-\tilde h|\!|\!|\big),
$$

for some constant c.

Finally, the smallness assumptions on $\{\!\{\!\{\tilde g_0\}\!\}\!\}_\beta$ and $[\![\bar g]\!]_\beta$ coming from the smallness assumption on the boundary data b_0 guarantee that the nonlinear map T is contractive in a small neighborhood of the original in the Banach space with the norm $[\![\![\cdot]\!]\!]$. Therefore, there is a unique fixed point which implies that (2.149) has a unique global in time solution converging to 0 with the algebraic rate $(1+t)^{-\frac32}$ as $t\to\infty$. This completes the proof of Theorem 2.28.

3. Exterior problem—flow past an obstacle

3.1. *Formulation of problem*

The flow past an obstacle is one of the classical problems in gas dynamics and fluid mechanics. In the context of the Boltzmann equation, this problem is formulated as follows. Denote the domain occupied by the obstacle by $\mathcal{O}\subset\mathbb{R}^n$, its exterior domain $\mathbb{R}^n\setminus\mathcal{O}$ by Ω, and the boundary by $\partial\Omega=\partial\mathcal{O}$. We consider a gas flow in the domain Ω which is in an equilibrium state at infinity with a prescribed nonzero bulk velocity $c\in\mathbb{R}^n$. Moreover, we assume that there is no external force nor source. Then, the problem is,

$$
\begin{aligned}
&\partial_t f+\xi\cdot\nabla_x f=Q(f,f), && t>0,\ x\in\Omega,\ \xi\in\mathbb{R}^n,\\
&f\to\mathbf{M}_c(\xi)\ (|x|\to\infty), && t>0,\ \xi\in\mathbb{R}^n,\\
&f|_{t=0}=f_0, && x\in\Omega,\ \xi\in\mathbb{R}^n,\\
&\gamma^-f=\mathbb{B}\gamma^+f, && t>0,\ x\in\partial\Omega,\ \xi\in\mathbb{R}^n,\ n(x)\cdot\xi<0.
\end{aligned}
\tag{3.1}
$$

Here, $\mathbf{M}_c$ is the Maxwellian for the far field. By suitable scalings of f and velocity variables, we can take

$$\mathbf{M}_c(\xi) = \mathbf{M}_{[1,c,1]}(\xi), \tag{3.2}$$

without loss of generality. The third equation in the above is the initial condition and the last equation is the boundary condition on the boundary $\partial\Omega$. $\mathbb{B}$ is the boundary operator and $\gamma^{\pm}$ are trace operators, both introduced in Section 1.3 with $n(x)$ being the outward normal to the boundary $\partial\Omega$ at point $x \in \partial\Omega$ (inward with respect to $\mathcal{O}$).

The aim of this section is to present the existence theorem of the stationary solutions to (3.1) for sufficiently small c. The presentation follows [64], with a part of proof renewed. As for the stability of the stationary solutions, we refer the reader to [65]. The case where the bulk velocity c at infinity is large, especially, close to the Mach number 1, is a physically interesting problem in connection with the transonic flow with a shock waves emanating from the surface of the obstacle. This is a big open problem for the Boltzmann equation, however.

Thus, the stationary problem we shall solve is

$$\begin{aligned} &\xi \cdot \nabla_x f = Q(f, f), && x \in \Omega,\ \xi \in \mathbb{R}^n, \\ &f \to \mathbf{M}_c(\xi) \quad \big(|x| \to \infty\big), && \xi \in \mathbb{R}^n, \\ &\gamma^- f = \mathbb{B}\gamma^+ f, && x \in \partial\Omega,\ \xi \in \mathbb{R}^n,\ n(x)\cdot\xi < 0, \end{aligned} \tag{3.3}$$

where $f = f(x, \xi)$. In the below, we assume that the Maxwellian $\mathbf{M}_c$ is not a stationary solution except $c = 0$. Since $\mathbf{M}_c$ satisfies the first and second equations of (3.3) for any c, this is equivalent to assuming that the boundary operator $\mathbb{B}$ preserves the Maxwellian $\mathbf{M}_c$ for $c = 0$ but not for $c \neq 0$. All the examples of boundary conditions presented in Section 1.3 satisfy this assumption. Then, we can expect that the non-Maxwellian stationary solution exists at least for nonzero but small bulk velocity c.

The strategy of proof is as follows. Write $\mathbf{M}_0 = \mathbf{M}_{c=0}$. We look for a solution in the form,

$$f = \mathbf{M}_c + \mathbf{M}_0^{1/2} u, \tag{3.4}$$

which reduces (3.3) to

$$\begin{aligned} &\xi \cdot \nabla_x u = \mathbf{L}_c u + \Gamma(u, u), && x \in \Omega,\ \xi \in \mathbb{R}^n, \\ &u \to 0\ (x \to \infty), && \xi \in \mathbb{R}^n, \\ &\gamma^- u = \mathbb{B}_0 \gamma^+ u + h_c, && x \in \partial\Omega,\ \xi \in \mathbb{R}^n,\ n(x)\cdot\xi < 0, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\mathbf{L}_c u = 2\mathbf{M}_0^{-1/2} Q\big(\mathbf{M}_c, \mathbf{M}_0^{1/2} u\big), \quad \Gamma(u, v) = 2\mathbf{M}_0^{-1/2} Q\big(\mathbf{M}_0^{1/2} u, \mathbf{M}_0^{1/2} v\big), \\ &\mathbb{B}_0 = \mathbf{M}_0^{-1/2} \mathbb{B} \mathbf{M}_0^{1/2}, \end{aligned} \tag{3.6}$$

$$h_c = \mathbf{M}_0^{-1/2}(\mathbb{B}\gamma^+\mathbf{M}_c - \gamma^-\mathbf{M}_c).$$

Here, $\mathbf{L}_c$ is not the same operator as the linearized collision operator $\mathbf{L}$ in (1.23) but Γ is the same as in (1.23).

One might expect that the setting $f = \mathbf{M}_c + \mathbf{M}_c^{1/2}u$ is another choice that seems more convenient in analysis of the problem because, then, $\mathbf{L}_c$ becomes self-adjoint. However, in this setting, the boundary operator $\mathbb{B}_0$ turns out to be unbounded and hence the problem (3.5) becomes ill-posed.

Let B_c be the linearized operator,

$$B_c = -\xi \cdot \nabla_x + \mathbf{L}_c, \tag{3.7}$$

coupled with the boundary condition $\gamma^- u = \mathbb{B}_0\gamma^+ u$ and suppose that it has an inverse B_c^{-1}. Then, (3.5) can be reduced to

$$u + B_c^{-1}\Gamma(u,u) = \phi_c, \tag{3.8}$$

where ϕ_c is a solution to the linear stationary problem

$$\begin{aligned} &\xi \cdot \nabla_x \phi_c = \mathbf{L}_c\phi_c, && x \in \Omega,\ \xi \in \mathbb{R}^n, \\ &\phi_c \to 0\ (x \to \infty), && \xi \in \mathbb{R}^n, \\ &\gamma^-\phi_c = \mathbb{B}_0\gamma^+\phi_c + h_c, && x \in \partial\Omega,\ \xi \in \mathbb{R}^n,\ n(x)\cdot\xi < 0. \end{aligned} \tag{3.9}$$

Once the existence of B_c^{-1} and ϕ_c is established, (3.8) can be solved by the implicit function theorem.

A delicate problem is the construction of the inverse operator B_c^{-1}. Indeed, the point 0 is in the continuous spectrum of B_c. However, B_c can have a bounded inverse by a suitable choice of function spaces for the domain of definition and range of it. This argument is in the same spirit as in the principle of limiting absorption which is familiar in the scattering theory. Actually, we will do this on the operator B_c^∞ introduced as the extension of B_c to the whole space by ignoring the boundary condition. The limiting absorption principle is carried out using a semi-explicit formula of the inverse operator of B_c^∞ which is obtained based on the spectral analysis of the operator B_0^∞ $(c = 0)$ developed in [58,59,61]. Then, the inverse of B_c will be constructed as a compact perturbation of B_c^∞. The compactness property is based on the "velocity averaging" which was used in [58] and has been generalized later extensively, see, e.g., [30,51].

This section is a close reproduction of the results from the paper [64]. There, a detailed argument was developed on the continuity property of various operators and resolvents with respect to the spectral parameter λ as well as the flow parameter c. However, they will be skipped here because, as far as the stationary problem is concerned, the estimates of the resolvent are needed only for $\lambda = 0$ while the continuity properties in c can be checked with somewhat straightforward modification of the computation in estimating norms of various operators for each fixed c.

On the other hand, in this section, the key compactness argument is revised by the aid of the "velocity averaging" property. This compactness property is essential in the construction of the inverse B_c^{-1}. In [64,65], this was established by using a compact integral operator whose integral kernel can be visualized only by a rather complicated change of variables specific to the particular reflection law of particles on the wall, so that rather restrictive conditions were required on the reflection boundary conditions. These conditions will be somewhat relaxed in this section.

As is noted in [9,50], the velocity averaging argument for the interior problem has a difficulty arising from the fact that even in the force-free field, particles can be reflected by the wall infinitely many times for a finite time. On the other hand, in the exterior problem, there is no such difficulty if the obstacle is a convex domain: The particle in the force-free field is reflected by the wall at most once, that is, a reflected particle does never return to the obstacle.

The plan of this section is as follows. After a brief description on the property of the linearized collision operator $\mathbf{L}_c$ and the trace operators $\gamma^{\pm}$ in the next subsections, we will study, in Section 3.3, the linearized Boltzmann operator in the whole space. Denote this operator by B_c^{∞}. First, based on the spectral analysis, a semi-explicit formula of its resolvent is derived. This formula shows that 0 is a spectral point of the operator B_c^{∞} but an unbounded inverse operator $(B_c^{\infty})^{-1}$ exists, say, in the L^2 space. Fortunately, this formula provides a far-reaching information on the spectral singularity at 0, ensuring that the inverse operator turns out to be a bounded operator if the domain of definition and range space are appropriately chosen. This is exactly in the same spirit as the limiting absorption principle for the resolvent operator in the scattering theory.

Starting from the bounded operator B_c^{∞} thus established, in Section 3.4, the inverse B_c^{-1} will be constructed by the perturbation argument. Since our obstacle is "compact," the perturbation can be expected "compact". In fact, we will derive a semi-explicit expression of B_c^{-1} in terms of B_c^{∞} and a related compact operator. It is at this stage that the "velocity averaging" argument is used. Finally in Section 3.5, the nonlinear problem (3.8) is solved by the aid of the contraction mapping principle.

3.2. *Preliminaries*

3.2.1. *Properties of* $\mathbf{L}_c$. Although the operator $\mathbf{L}_c$ in (3.6) and $\mathbf{L}$ in (1.23) are not the same operators, since $\mathbf{L}_c$ is defined with the normalized Maxwellian $\mathbf{M}_{[1,c,1]}$, and owing to the translation invariance property (1.31) of Q, the relation

$$\mathbf{L}_c = \theta_c^{-1}\mathbf{L}\theta_c, \quad \theta_c u(\xi) = e^{-(\xi+c)\cdot c/2}u(\xi + c) \tag{3.10}$$

holds for all $c \in \mathbb{R}^n$. This and Proposition 1.3, then, imply that $\mathbf{L}_c$ has the decomposition of the form

$$\mathbf{L}_c u = -\nu_c(\xi)u + \int_{\mathbb{R}^n} K_c(\xi, \xi_*)u(\xi_*)\, d\xi_*,$$

$$\nu_c(\xi) = \nu(\xi - c), \quad K_c(\xi, \xi_*) = K(\xi, \xi_*)\exp\left\{-\frac{1}{2}c \cdot (\xi - \xi_*)\right\}. \tag{3.11}$$

With an evident modification, the function $\nu_c(\xi)$ enjoys the estimate (1.26) with the same constants ν_0, ν_1, while the kernel $K_c(\xi, \xi_*)$ enjoys the estimates (1.34) and (1.35), because the estimates (1.36) can absorb the quantity $\exp\{\frac{1}{2}c \cdot (\xi - \xi_*)\}$ with the factors $1/4, 1/8$ replaced by smaller ones in the exponents in (1.36).

This indicates that Lemmas 1.4, 1.5, and 1.8 are valid for K_c except for the self-adjointness. Moreover, it is seen that K_c is continuous in c in relevant operator norms. Also, observe that since as before, the quantity $\exp\{\alpha(|\xi| - |\xi_*|)\}$ is absorbed by K, the proof of Lemma 1.4 works to prove that

$$K_c : L^p_\alpha \to L^q_\alpha \tag{3.12}$$

is bounded for any $\alpha \in \mathbb{R}$ and $1 \leqslant p \leqslant q \leqslant \infty$, where

$$L^p_\alpha = \{u \mid e^{\alpha|\xi|}u \in L^p(\mathbb{R}^n_\xi)\}.$$

Although the operator $\mathbf{L}_c$ is no longer self-adjoint, it still enjoys Proposition 1.6 except (2).

PROPOSITION 3.1. *Define* $\mathbf{L}_c$ *with the domain of definition*

$$D(\mathbf{L}_c) = \{u \in L^2 \mid \nu_c(\xi)u \in L^2(\mathbb{R}^n_\xi)\}.$$

Then,

(i) *The spectrum* $\sigma(L_c)$ *outside* $(-\infty, -\nu_*)$ *consists of discrete eigenvalues which are invariant in* c.

(ii) *The null space* $\mathcal{N}_c$ *of* $\mathbf{L}_c$ *is invariant in* c *and hence the corresponding eigenprojection* $\mathbf{P}_c$ *is just* $\mathbf{P}$ *of Proposition* 1.6.

PROOF. Since K_c is a compact operator (Lemma 1.5), the spectrum outside $(-\infty, -\nu_*)$ consists of only discrete eigenvalues. Let $u \in L^2$ be an eigenfunction of $\mathbf{L}_c$ corresponding to such an eigenvalue λ: $\lambda u = \mathbf{L}_c u$. Rewrite this as

$$u = Hu, \quad H = (\lambda + \nu_c)^{-1}K_c.$$

Owing to (1.26) and (3.12), H is bounded on L^2_α and so is $H_1 = \chi(|\xi| > a)H$ for any $a > 0$. Moreover, the norm of H_1 can be made smaller than 1 if a is chosen sufficiently large, so that the inverse $(I - H_1)^{-1}$ exists and is bounded on L^2_α. On the other hand, $H_2 = \chi(|\xi| < a)H : L^2 \to L^2_\alpha$ is bounded. Noting that u satisfies $u = (I - H_1)^{-1}H_2 u$, we conclude $u \in L^2_\alpha$ for $\alpha > 0$. Therefore, $v = \theta_c u \in L^2$ and in view of (3.10), v satisfies $\lambda v = \mathbf{L}v$. The converse is also true, so the proof of (i) is complete. The proof of (ii) follows if we note that

$$\mathcal{N}_c = \theta_c^{-1}\mathcal{N} = \mathcal{N}. \qquad \square$$

3.2.2. *Trace theorem.* The boundary values, or traces, can be defined only for functions having some regularity. Here, we shall establish the trace theorem for functions having the regularity associated with the differential operator B_c. Throughout this section, we assume that

$$\begin{aligned}&\mathcal{O} \text{ is a bounded convex domain of } \mathbb{R}^n \\ &\text{and the boundary } \partial\mathcal{O} \text{ is piecewise } C^1.\end{aligned} \tag{3.13}$$

Recall $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, $\partial\Omega = \partial\mathcal{O}$ and set

$$D = \Omega \times \mathbb{R}^n, \tag{3.14}$$

and introduce the linear operators

$$\Lambda^{\pm} u = \xi \cdot \nabla_x u \pm h(\xi) u, \quad (x, \xi) \in D \quad \text{(same sign throughout)}, \tag{3.15}$$

for functions $h = h(\xi)$ satisfying

$$h \in L^{\infty}_{\mathrm{loc}}\big(\mathbb{R}^n\big), \qquad h_0 = \inf_{\xi} \operatorname{Re} h(\xi) > 0. \tag{3.16}$$

Our trace spaces are

$$\begin{aligned} W^{p,\pm} &= \big\{ u \in L^p(D) \;\big|\; \Lambda^{\pm} u \in L^p(D) \big\}, \\ Y^{p,\pm} &= L^p\big(S^{\pm}, \rho(x,\xi)\, d\sigma_x\, d\xi\big), \end{aligned} \tag{3.17}$$

for $p \in [1, \infty]$, where $S^{\pm}$ are defined by (1.49), ρ is a weight function defined by

$$\rho(x, \xi) = \big| n(x) \cdot \xi \big|,$$

and σ_x is the Lebesgue measure on $\partial\Omega$. These spaces are all Banach spaces endowed with obvious norms. The main result of this subsection reads as follows.

THEOREM 3.2. *Assume* (3.13) *and* (3.16). *Then, for any* $p \in [1, \infty]$, *there exist operators* $\gamma^{\pm}$ *satisfying*

(i) $\gamma^{\pm} : W^{p,\pm} \to Y^{p,\pm}$ *are linear bounded operators,*

(ii) $\gamma^{\pm} f = f|_{S^{\pm}} \quad \big(\forall f \in C^1_0(\bar{D})\big)$.

They are unique extensions of the operators defined by (ii), *for the case* $p < \infty$.

PROOF. We will present the proof for the case '+' only, because the proof for '−' is the same. Let $(x, \xi) \in S^+$. Then, the point $x - t\xi$ is in Ω for any $t \geqslant 0$, and it holds that for any function $u \in C_0^1(\bar{D})$,

$$\frac{\partial}{\partial t}\bigl(e^{-h(\xi)t}u(x - t\xi)\bigr) = -e^{-h(\xi)t}(\Lambda^+ u)(x - t\xi).$$

Integrate this over $t \in [0, \infty)$ to deduce

$$u(x, \xi) = \int_0^\infty e^{-h(\xi)t}(\Lambda^+ u)(x - t\xi)\,dt. \tag{3.18}$$

By the Hölder inequality, we get for $p \in [0, \infty)$, $1/p + 1/q = 1$,

$$\bigl|u(x, \xi)\bigr|^p \leqslant (qh_0)^{p/q}\int_0^\infty \bigl|(\Lambda^+ u)(x - t\xi)\bigr|^p\,dt.$$

Set for each $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
&\partial\Omega^\pm(\xi) = \bigl\{x \in \partial\Omega \bigm| n(x)\cdot\xi \gtrless 0\bigr\},\\
&\Omega^\pm(\xi) = \bigl\{x \mp t\xi \bigm| x \in \partial\Omega^\pm(\xi),\ t \geqslant 0\bigr\},
\end{aligned}$$

and compute

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_{\partial\Omega^+(\xi)}\bigl|u(x, \xi)\bigr|^p\rho(x, \xi)\,d\sigma_x\right)d\xi\\
&\quad\leqslant (qh_0)^{p/q}\int_{\mathbb{R}^n}\left(\int_{\partial\Omega^+(\xi)}\int_0^\infty\bigl|(\Lambda^+ u)(x - t\xi)\bigr|^p\,dt\,\rho(x, \xi)\,d\sigma_x\right)d\xi\\
&\quad= (qh_0)^{p/q}\int_{\mathbb{R}^n}\left(\int_{\Omega^+(\xi)}\bigl|(\Lambda^+ u)(x)\bigr|^p\,dx\right)d\xi,
\end{aligned}$$

which yields

$$\|\gamma^+ u\|_{Y^{p,+}} \leqslant (qh_0)^{-1/q}\|\Lambda^+ u\|_{L^p(D)}. \tag{3.19}$$

Here, we have used Fubini's theorem,

$$\int_{\Omega^\pm(\xi)} w(x)\,dx = \int_{\partial\Omega^\pm(\xi)}\int_0^\infty w(x \mp t\xi)\rho(x, \xi)\,dt\,d\sigma_x\,d\xi. \tag{3.20}$$

So far, $u \in C_0^1(\bar{D})$ has been assumed, but it is seen from density argument that (3.19) still holds for $u \in W^{p,+}$ if $p \in [1, \infty)$, whence follows the theorem for γ^+, and also for γ^- in exactly the same way.

Actually, the above argument implies more: $u \in W^{p,+}$ is absolute continuous on the line $\{x - t\xi, t > 0\}$ for almost all $(x,\xi) \in S^+$, on which holds

$$(\gamma^+ u)(x,\xi) = -\lim_{t\to 0+} \int_t^\infty e^{-h(\xi)s} (\Lambda^+ u)(x - s\xi)\, ds, \quad \text{a.a. } (x,\xi) \in S^+. \tag{3.21}$$

See (3.18).

Now, consider the case $u \in W^{\infty,+}$. Its cutoff χu with a smooth cutoff function χ of support in $\{|x| < R_1, |\xi| < R_2\}$ is in $W^{p,+}$ for any $p \in [1,\infty)$ if R_1 is fixed so large that the ball $\{|x| < R_1\}$ can contain $\mathcal{O}$. Then, (3.21) applies to χu, which gives, after letting $R_2 \to \infty$,

$$\|\gamma^+ u\|_{Y^{\infty,+}} \leqslant h_0^{-1} \|\Lambda^+ u\|_{L^\infty(D)} + c_0 \|u\|_{L^\infty(D)}, \tag{3.22}$$

where $c_0 > 0$ is a constant depending only on $\nabla_x \chi$ and R_1. This proves the theorem for the case $p = \infty$.

Observe that $u \in W^{p,+}$ does not imply $u \in W^{p,-}$ unless $hu \in L^p(D)$, so that it may not have the trace $\gamma^- u \in Y^{p,-}$. However, it has a trace $\gamma^- u \in Y^{p,-}_{\text{loc}}$ as seen by the truncation with respect to ξ.

The situation differs for the functions defined on the whole space $D^\infty = \mathbb{R}^n \times \mathbb{R}^n$. □

PROPOSITION 3.3. *Let $p \in [1,\infty]$ and $u \in L^p(D^\infty)$. If u satisfies either $\Lambda^+ u \in L^p(D^\infty)$ or $\Lambda^- u \in L^p(D^\infty)$, then, u has both the traces $\gamma^+ u \in Y^{p,+}$ and $\gamma^- u \in Y^{p,-}$.*

PROOF. Notice that for $u \in C_0^1(D^\infty)$, (3.18) holds for $(x,\xi) \in S^-$ as well as for $(x,\xi) \in S^+$. This is not possible for the exterior domain case because $(x,\xi) \in S^-$ results in $x - t\xi \notin \Omega$ for $t > 0$ small. Instead of (3.19), we now have

$$\|\gamma^\pm u\|_{Y^{p,\pm}} \leqslant \left(\frac{p-1}{ph_0}\right)^{-(p-1)/p} \|\Lambda^+ u\|_{L^p(D^\infty)}.$$

The rest of the proof is similar to that of Theorem 3.2. □

Another difference is in the

PROPOSITION 3.4. *Assume* $\operatorname{Im} h(\xi) \in L^\infty(\mathbb{R}^n)$. *Then*

$$W^{2,+}(D^\infty) = W^{2,-}(D^\infty) = \{u \in L^2(D^\infty) \mid \xi \cdot \nabla_x u,\ h(\xi)u \in L^2(D^\infty)\}.$$

PROOF. The function $u(x,\xi) \in L^2(D^\infty)$ has a Fourier transform $\hat{u}(k,\xi) \in L^2(D^\infty)$ ($k \in \mathbb{R}^n$) with respect x, and the Parseval relation gives

$$\|\Lambda^\pm u\|_{L^2(D^\infty)} = \|(ik \cdot \xi \pm h(\xi))\hat{u}\|_{L^2(D^\infty)}.$$

Now, the equality $|ik\cdot\xi \pm h(\xi)|^2 = |\mathrm{Re}\, h(\xi)|^2 + |(k\cdot\xi \pm \mathrm{Im}\, h(\xi)|^2$ proves the proposition. □

The following Green's formula will be used to prove the dissipativity of the operator B_c.

LEMMA 3.5. *Let $u \in W^{2,+}$ be such that $hu \in L^2(D)$ and $\gamma^- u \in Y^{2,-}$. Then,*

$$\mathrm{Re}(\Lambda^+ u, u)_{L^2(D)} = \frac{1}{2}\big(\|\gamma^+ u\|^2_{Y^{2,+}} - \|\gamma^- u\|^2_{Y^{2,-}}\big) + \mathrm{Re}(hu, u)_{L^2(D)}.$$

PROOF. This is immediate for $u \in C_0^1(\bar{D})$ by integration by parts and the general case follows by density argument. □

3.3. *Operator B_c^∞*

As stated in the introduction, we will construct the inverse of the operator B_c as a perturbation of the whole space case. If this operator is defined, say, in the space $L^2(D^\infty)$, its spectrum contains the point 0, which means that either the inverse does not exist, or if exists, it is unbounded. Actually, the latter case occurs. This will be shown, in the next subsection, by constructing a semi-explicit expression based on the spectral analysis developed in [58,59,61].

Fortunately, this expression tells us much more. In Section 3.3.2, it will be shown, with the help of some additional boundedness properties obtained by a bootstrap argument based on the smoothing effect of a related transport operator, that the inverse can be viewed as a bounded operator if an appropriate choice of functions spaces is made for the domain of definition and for the range space. This procedure is called the limiting absorption principle in the scattering theory.

3.3.1. *Spectral analysis of B_c^∞.* The operator we consider is

$$B_c^\infty u = -\xi\cdot\nabla_x u + \mathbf{L}_c u, \quad (x,\xi) \in D^\infty. \tag{3.23}$$

Proposition 3.4 says that it is well defined in the space $L^2(D^\infty)$ endowed with the domain of definition

$$D\big(B_c^\infty\big) = \big\{u \in L^2(D^\infty) \,\big|\, \xi\cdot\nabla_x u,\ \nu_c(\xi)u \in L^2(D^\infty)\big\}. \tag{3.24}$$

As is easily seen, this is a densely defined closed operator, and its spectral property can be best studied by the Fourier transformation with respect to x since the coefficients in this operator are all constant in x. Let

$$\hat{u}(k,\xi) = \mathcal{F}(u) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ik\cdot x} u(x,\xi)\,dx, \quad k \in \mathbb{R}^n,\ i = \sqrt{-1}, \tag{3.25}$$

be the Fourier transform of the function $u = u(x, \xi)$ with respect to x. Formally, we have,

$$\mathcal{F}\big(B_c^\infty u\big) = (-i\xi \cdot k + \mathbf{L}_c)\hat{u}. \tag{3.26}$$

This suggests the study of the operator

$$\hat{B}_c^\infty(k)w = (-i\xi \cdot k + \mathbf{L}_c)w \tag{3.27}$$

acting on functions $w = w(\xi)$ of the variables ξ only, with $k \in \mathbb{R}^n$ fixed.

In the sequel, we consider this operator in the space $L^2 = L^2(\mathbb{R}^n_\xi)$ with the domain of definition,

$$D\big(\hat{B}_c^\infty(k)\big) = \big\{w \in L^2 \,\big|\, (k \cdot \xi)w,\ \nu(\xi)w \in L^2\big\}, \tag{3.28}$$

and construct its resolvent

$$\Phi(\lambda, k, c) = \big(\lambda - \hat{B}_c^\infty(k)\big)^{-1}. \tag{3.29}$$

Actually, we can have its semi-implicit expression based on the spectral analysis which has been carried out for the case $c = 0$ in [58,59,61].

THEOREM 3.6. *Assume* (1.25). *Then, there exist positive numbers* $c_0, \kappa_0, \sigma_0, \sigma_1$ *such that* $\sigma_0 \leqslant \sigma_1$*, and the following holds for each* $c \in \mathbb{R}^n$, $|c| \leqslant c_0$.

(1) *The spectrum of the operator* $\hat{B}_c^\infty(k)$ *is empty in the half-plane* $\{\operatorname{Re}\lambda > -\sigma_0\}$ *if* $|k| \geqslant \kappa_0$ *while it consists only of* $n+2$ *discrete semi-simple eigenvalues* $\lambda_j(k, c)$, $j = 0, \dots, n+1$, *in the half-plane* $\{\operatorname{Re}\lambda > -\sigma_1\}$ *if* $|k| \leqslant \kappa_0$.

(2) *The asymptotic expansion*

$$\lambda_j(k, c) = ik \cdot c + i\lambda_{j,1}|k| - \lambda_{j,2}|k|^2 + O\big(|k|^3\big), \tag{3.30}$$

holds for $|k| \leqslant \kappa_0$ *uniformly in* $|c| \leqslant c_0$, *with some constants*

$$\lambda_{j,1} \in \mathbb{R}, \qquad \lambda_{j,2} > 0.$$

The corresponding eigenprojection, denoted by $P_j(k, c)$, *has an asymptotic expansion,*

$$P_j(k, c) = P_{j,0}(\tilde{k}) + |k|P_{j,1}(k, c), \quad \tilde{k} = k/|k| \in S^{n-1}, \tag{3.31}$$

with orthogonal projections $P_{j,0}(\tilde{k})$ *on* L^2 *which are orthogonal to each other and satisfy*

$$\sum_{j=0}^{4} P_{j,0}(\tilde{k}) = \mathbf{P} \quad \big(\tilde{k} \in S^{n-1}\big), \tag{3.32}$$

where $\mathbf{P}$ *is the orthogonal projection* (1.45) *onto the null space* $\mathcal{N}$ *of* $\mathbf{L}$. *Moreover, the operator*

$$P_{j,1}(k,c): L^2 \to L^\infty_\beta \tag{3.33}$$

is a bounded operator for any $\beta \geqslant 0$ *and a continuous function of* (k,c) *in the region* $|k| \leqslant \kappa_0, |c| \leqslant c_0$. *Here,* $L^\infty_\beta = L^\infty(\mathbb{R}^n_\xi; (1+|\xi|)^\beta d\xi)$.

(3) *The formula*

$$\Phi(\lambda,k,c) = \sum_{j=0}^{n+2} \Phi_j(\lambda,k,c), \quad \operatorname{Re}\lambda > -\nu_*,\ k \in \mathbb{R}^n,\ |c| \leqslant c_0, \tag{3.34}$$

holds where

(a) *for* $j = 0, \ldots, n+1$,

$$\Phi_j(t,k,c) = \frac{1}{\lambda - \lambda_j(k,c)} P_j(k,c)\chi(k), \tag{3.35}$$

$\chi(k)$ *being the characteristic function for* $|k| \leqslant \kappa_0$, *and*

(b) $\Phi_{n+2}(\lambda,k,c)$ *is a linear bounded operator on* L^2 *whose operator norm enjoys the estimate*

$$\left\| \Phi_{n+2}(\lambda,k,c) \right\| \leqslant b_0, \quad \operatorname{Re}\lambda \geqslant -\sigma_0,\ k \in \mathbb{R}^3,\ |c| \leqslant c_0, \tag{3.36}$$

for some constants b_0 *independent of* λ, k, *and* c.

REMARK 3.7. The coefficients $\lambda_{j,m}$ in the asymptotic expansion (3.30) have well-known physical meanings. Under a suitable numbering of j, they are characterized as follows.

(a) $\lambda_{0,1} = -\lambda_{0,n+1} = c_s \neq 0$ (sound speed), $\lambda_{1,1} = \lambda_{1,n+1} = \kappa$ (heat diffusivity).

(b) $\lambda_{0,j} = 0$, $\lambda_{1,j} = \nu$ (viscosity coefficient), $j = 1, \ldots, n$.

These physical constants are for the gas in the equilibrium state specified by $\mathbf{M}_{[1,0,1]}$. See, e.g., [11,67].

PROOF OF THEOREM 3.6. This will be given in four steps.

Step 1. We shall show that the spectrum of $\hat{B}^\infty_c(k)$ in the right half plane $\{\operatorname{Re}\lambda > -\nu_*\}$ consists only of discrete eigenvalues with possible accumulation points on the line $\{\operatorname{Re}\lambda = -\nu_*\}$ or at infinity. To this end, introduce an auxiliary operator

$$\hat{A}^\infty_c(k)w = \left(-i\xi\cdot k - \nu_c(\xi)\right)w, \qquad D\left(\hat{A}^\infty_c(k)\right) = D\left(\hat{B}^\infty_c(k)\right). \tag{3.37}$$

Since this is a multiplication operator in L^2, its spectrum consists of an essential spectrum given by

$$\sigma\left(\hat{A}^\infty_c(k)\right) = \left\{-i\xi\cdot k - \nu_c(\xi) \,\middle|\, \xi \in \mathbb{R}^n\right\} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant -\nu_*\} \tag{3.38}$$

and the resolvent has an explicit expression

$$\big(\lambda - \hat{A}_c^\infty(k)\big)^{-1} w = \frac{1}{\lambda + i\xi \cdot k + \nu_c(\xi)} w, \quad \lambda \notin \sigma\big(\hat{A}_c^\infty(k)\big), \tag{3.39}$$

which implies that its L^2 operator norm satisfies

$$\big\| \big(\lambda - \hat{A}_c^\infty(k)\big)^{-1} \big\| \leqslant \frac{1}{\operatorname{Re}\lambda + \nu_*} \tag{3.40}$$

for $\operatorname{Re}\lambda > -\nu_*$.

Since $\hat{B}_c^\infty(k) = \hat{A}_c^\infty(k) + K_c$ and K_c is a compact operator, owing to Weyl's lemma on the compact perturbation of the spectrum, [40], $\hat{B}_c^\infty(k)$ has the same essential spectrum and its spectrum in the solvent set of $\hat{A}_c^\infty(k)$, in particular in the region $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -\nu_*\}$, consists only of discrete eigenvalues with possible accumulation points on the boundary of the resolvent set of $\hat{A}_c^\infty(k)$. This proves the claim.

Step 2. Let (λ, ϕ) be a pair of an eigenvalue and eigenfunction of $\hat{B}_c^\infty(k)$ for $\operatorname{Re}\lambda > -\nu_*$. By virtue of (3.10) it holds that

$$(\lambda + ic \cdot k)\psi = \hat{B}_0^\infty(k)\psi, \quad \psi = \theta_c^{-1}\phi.$$

Repeat the proof of Proposition 3.1 with $(\lambda + ic \cdot k + \nu_c)^{-1}$ in place of $(\lambda + \nu_c)^{-1}$, to conclude $\psi \in D(\hat{B}_0^\infty(k))$. Thus, $\lambda + ic \cdot k$ is an eigenvalue of $\hat{B}_0^\infty(k)$.

Step 3. On the other hand, the spectrum of $\hat{B}_0^\infty(k)$ in $\{\operatorname{Re}\lambda > -\nu_*\}$ has been studied extensively in [27,58,59,61], which leads to the following lemma, whose proof will be given at the end of this section.

LEMMA 3.8. *There are positive numbers $c_0, \kappa_0, \sigma_0, \sigma_1, \sigma_*$ such that $\sigma_0 \leqslant \sigma_1$, and the following holds if $|c| \leqslant c_0$.*

(1) *If $|k| \geqslant \kappa_0$, then, $\hat{B}_c^\infty(k)$ has no eigenvalues in the half-plane $\{\operatorname{Re}\lambda \geqslant -\sigma_0\}$ and*

$$\big\| \Phi(\lambda, k, c) w \big\|_{L^2} \leqslant b_1 \|w\|_{L^2}, \quad \operatorname{Re}\lambda \geqslant -\sigma_0,\ w \in L^2, \tag{3.41}$$

holds for a constant $b_1 > 0$ independent of c, k, λ, w.

(2) *If $|k| \leqslant \kappa_0$, then, $\lambda_j(k, c)$, $j = 0, \ldots, n+1$, given in Theorem 3.6(1) are the only eigenvalues of $\hat{B}_c^\infty(k)$ lying in the half-plane $\{\operatorname{Re}\lambda \geqslant -\sigma_1\}$ and satisfying*

$$\operatorname{Re}\lambda_j(k, c) \geqslant -\sigma_1/2, \quad j = 0, \ldots, n+1,\ |k| \leqslant \kappa_0,\ |c| \leqslant c_0.$$

Further, put

$$\tau_0 = \max\big\{ \big|\operatorname{Im}\lambda_j(k, c)\big| \bigm| j = 0, \ldots, n+1,\ |c| \leqslant c_0,\ |k| \leqslant \kappa_0 \big\},$$

which is a finite number by virtue of Theorem 3.6(2). Then,

$$\big\| \Phi(\lambda, k, c) w \big\|_{L^2} \leqslant b_1 \|w\|_{L^2}, \quad \lambda \in \Sigma_*,\ w \in L^2, \tag{3.42}$$

holds for a constant $b_1 > 0$ independent of c, k, λ, w and

$$\Sigma_* = \{\mathrm{Re}\,\lambda > \sigma_*\} \cup \left\{\mathrm{Re}\,\lambda > -\sigma_1,\ |\mathrm{Im}\,\lambda| \geqslant 2\tau_0\right\}.$$

Admit this for a while to conclude the proof of Theorem 3.6. The statement (1) in it is now evident from the above lemma, while the asymptotic expansions of $\lambda_j(k,0)$ and $P_j(k,0)$ for the case $c = 0$ have been discussed in [27], whence (2) of Theorem 3.6 follows.

Step 4. It remains to prove the statement (3). Recall that we are considering λ with $\mathrm{Re}\,\lambda > -\sigma_0$. The proof is carried out separately for $|k| > \kappa_0$ and $|k| \leqslant \kappa_0$. For the former, Lemma 3.8 (1) indicates that $\Phi_{n+2} = \Phi$ and therefore, gives (3.36). For the latter, owing to the second part of (2) in Lemma 3.8, this is the same also when $\lambda \in \Sigma_*$.

Now, it remains to check the case where $|k| \leqslant \kappa_0$ and $|\mathrm{Im}\,\lambda| \leqslant 2\tau_0$. (3.34) is then the Laurant expansion of Φ: The principal term is given by (3.35) because $\lambda_j(k,c)$ are seen to be semi-simple, and the remainder is given by the contour integral

$$\Phi_{n+2}(\lambda, k, c) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Phi(\zeta, k, c)}{\zeta - \lambda}\, d\zeta,$$

where $\mathcal{C}$ is a simple closed rectifiable curve enclosing the points λ and λ_j, $j = 0, \ldots, n+1$, but no other points in the spectrum of $\hat{B}_c^{\infty}(k)$ are inside nor on $\mathcal{C}$. In view of Lemma 3.8(2), $\mathcal{C}$ can be chosen in the domain, say, $\{\zeta \in \mathbb{C} \mid -\sigma_1 \leqslant \mathrm{Re}\,\zeta \leqslant \sigma_*, |\mathrm{Im}\,\zeta| \leqslant 3\tau_0\}$ so that all of the points λ and $\lambda_j(k,c)$ can stay away from $\mathcal{C}$ uniformly for k, c. Then, (3.36) follows if we note that $\Phi(\zeta, k, c)$ is analytic in ζ on $\mathcal{C}$ and hence its operator norm is uniformly bounded in ζ, k, c. This completes the proof of the theorem. □

The rest of this section is devoted to the proof of Lemma 3.8. The following proposition, implying the "velocity averaging," was originally proved in [58] for the case $c = 0$. In the sequel, we write $\lambda = \sigma + i\tau$.

PROPOSITION 3.9. *There are positive numbers C, c_0 such that the following estimates hold whenever $|c| \leqslant c_0$, $\delta > 0$ and $\sigma > -\nu_* + \delta$.*

(1) *For any $k \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, it holds that*

$$\left\| \left(\lambda - \hat{A}_c^{\infty}(k)\right)^{-1} K_c \right\| \leqslant C\delta^{-1+2/(3n+4)}|k|^{-2/(3n+4)}.$$

(2) *For any $\kappa_0 > 0$, there is a constant τ_0 such that*

$$\left\| \left(\lambda - \hat{A}_c^{\infty}(k)\right)^{-1} K_c \right\| \leqslant C\delta^{-1+2/(n+2)}|\tau|^{-2/(n+2)},$$

holds for any $|k| \leqslant \kappa_0$, $|\tau| \geqslant \tau_0$.

Here, $\|\ \|$ stands for the operator norm in L^2_ξ.

PROOF. Fix c and put $G(\lambda,k)=(\lambda-\hat{A}_c^{\infty}(k))^{-1}K_c$. Let $\chi(D)$ be the characteristic function of the domain $D\subset\mathbb{R}^n$. By virtue of Lemma 1.4(a) with $p=2,\ r=\infty$, we get

$$\left\|\chi\big(|\xi|<R\big)G(\lambda,k)\right\|\leqslant C\left(\int_{|\xi|<R}\left|\lambda+ik\cdot\xi+\nu_c(\xi)\right|^{-2}d\xi\right)^{1/2}.$$

Denote the last integral by J. Set $\lambda=\sigma+i\tau$ and

$$\begin{aligned}\Sigma_1&=\big\{\xi\in\mathbb{R}^n\ \big|\ |\xi|<R,\ |\tau+k\cdot\xi|\leqslant\epsilon|k|\big\},\\ \Sigma_2&=\big\{\xi\in\mathbb{R}^n\ \big|\ |\xi|<R\big\}\setminus\Sigma_1,\end{aligned}\tag{3.43}$$

for any $\epsilon>0$. It is easy to see that there is a constant $C>0$ independent of k,R,ϵ,τ such that

$$\operatorname{mes}\Sigma_1\leqslant C\epsilon R^{n-1},\qquad \operatorname{mes}\Sigma_2\leqslant CR^n,$$

hold, mes being the Lebesgue measure in $\mathbb{R}^n$. Let $\sigma\geqslant-\nu_0+\delta$. Then, we get,

$$J=\int_{\Sigma_1}+\int_{\Sigma_2}\leqslant C\big(\delta^{-2}\epsilon R^{n-1}+\big(\epsilon|k|\big)^{-2}R^n\big).$$

Choose $\epsilon=R^{1/3}(\delta/|k|)^{2/3}$ to deduce

$$\left\|\chi\big(|\xi|<R\big)G(\lambda,k)\right\|\leqslant C\delta^{-2/3}R^{(3n-2)/6}|k|^{-1/3}.$$

On the other hand, by virtue of (1.34),

$$\left\|\chi\big(|\xi|>R\big)G(\lambda,k)\right\|\leqslant C\delta^{-1}R^{-1}.$$

Choosing $R=(|k|/\delta)^{2/(3n+4)}$ leads to (1) of the proposition.

To prove (2), let $|\tau|>2\kappa_0R$ for which

$$|\tau+k\cdot\xi|\geqslant|\tau|-|k||\xi|\geqslant|\tau|-\kappa_0R\geqslant|\tau|/2,$$

whenever $|k|\leqslant\kappa_0$ and $|\xi|\leqslant R$, whence

$$J\leqslant C\big(\delta^2+|\tau|^2\big)^{-1}R^n.$$

Choosing $R=(|\tau|/\delta)^{2/(n+2)}$ leads to (2) with the choice $\tau_0=(2\kappa_0)^{1+2/n}\delta^{-2/n}$. This completes the proof of the proposition. □

Notice that although the estimates in (1) and (2) make no sense if $k=0$ and $\tau=0$, G is still uniformly bounded owing to (3.40).

REMARK 3.10. Introduce the transport operator

$$A_c^\infty = -\xi \cdot \nabla_x + \mathbf{L}_c. \tag{3.44}$$

The estimate (1) in Proposition 3.9 states that for a given $f \in L^2(D^\infty)$, the function $u = (\lambda - A_c^\infty)^{-1} K_c f$ gains a regularity in x in such a way that $u \in L^2(\mathbb{R}^n_\xi, H^\ell_x)$, $\ell = 2/(3n+4) > 0$, where H^ℓ_x is the usual Sobolev space. Notice that u is a unique solution of the *transport equation*

$$\big(\lambda - A_c^\infty\big) u = K_c f, \quad f \in L^2_{x,\xi}. \tag{3.45}$$

This property has been extended extensively and is now called the "velocity averaging", see, e.g., [30,51]. A significant consequence of this is that the operator

$$\big(\lambda - A_c^\infty\big)^{-1} K_c : L^2\big(D^\infty\big) \to L^2\big(D' \times \mathbb{R}^n\big)$$

is a compact operator for any compact domain $D' \subset \mathbb{R}^n$. Roughly, this is seen from Rellich's theorem [40] that a bounded set of H^ℓ_x is locally compact in L^2 and from Lemma 1.5 that K_c is a compact operator on L^2_ξ. See Lemma 3.22 in Section 3.4.

Now, we are ready to give the

PROOF OF LEMMA 3.8. First, we consider the eigenvalues of $\hat{B}_c^\infty(k)$. Let $|c| \leqslant c_0$ with c_0 of Proposition 3.9. Let (λ, φ) be an eigenpair in the plane $\{\operatorname{Re}\lambda > -\nu_* + \delta\}$, $\delta > 0$. Then, the eigenvalue problem

$$\big(\lambda - \hat{B}_c^\infty(k)\big)\varphi = 0, \quad \varphi \in D\big(\hat{B}_c^\infty(k)\big), \quad \varphi \neq 0, \tag{3.46}$$

can be rewritten as

$$\varphi = G(\lambda, k)\varphi$$

which, in turn, gives rise to

$$\big\| G(\lambda, k) \big\| \geqslant 1.$$

However, this is violated, that is, $\|G(\lambda, k)\| < 1$, for the following three cases.

(a) $\operatorname{Re}\lambda > \nu_{**}$ and $k \in \mathbb{R}^n$ with some constant $\nu_{**} > 0$ (by (3.40)),

(b) $\operatorname{Re}\lambda > -\nu_* + \delta$ and $|k| \geqslant \kappa_1$ with some constant $\kappa_1 > 0$ (by Proposition 3.9(1)),

(c) $\operatorname{Re}\lambda > -\nu_* + \delta$, $|k| \leqslant \kappa_1$, and $|\tau| > \tau_1$ with κ_1 of (b) and some constant $\tau_1 > 0$ (by Proposition 3.9(2)).

In other words, such eigenvalues can exist only when $|k| \leqslant \kappa_1$ and lie exclusively in the region

$$\Sigma^* = \big\{\lambda = \sigma + i\tau \in \mathbb{C} \;\big|\; -\nu_* + \delta \leqslant \sigma \leqslant \nu_{**},\ |\tau| \leqslant \tau_1\big\}.$$

The number of such eigenvalues is uniformly bounded for $|k| \leqslant \kappa_1$.

We can show $\nu_{**} = 0$. For this, it suffices to consider the case $c = 0$ as seen from the proof of Theorem 3.6: The eigenvalue for the case $c \neq 0$ is given by the shift by $ic \cdot k$ of an eigenvalue for the case $c = 0$. Recalling that $\mathbf{L}$ is nonpositive self-adjoint, we compute the inner product of $(3.46)_{c=0}$ and φ,

$$0 = \mathrm{Re}\big(\big(\lambda - \hat{B}_0^\infty(k)\big)\varphi, \varphi\big)_{L^2} = \sigma\|\varphi\|^2 - (\mathbf{L}\varphi, \varphi) \geqslant \sigma\|\varphi\|^2, \quad \lambda = \sigma + i\tau,$$

which is a contradiction if $\sigma > 0$, proving $\nu_{**} = 0$.

Furthermore, this computation shows that

(i) no eigenvalues exist on the imaginary axis when $k \neq 0$ and

(ii) the point 0 is the only eigenvalue on the imaginary axis when $k = 0$.

In fact, if we set $\sigma = 0$ in the above computation, we get $(\mathbf{L}\varphi, \varphi)_{L^2} = 0$, and hence by virtue of Proposition 1.6, $\varphi \in \mathcal{N}$. Then, the eigenvalue equation (3.46) is reduced to

$$(\tau + k \cdot \xi)\varphi = 0,$$

which is impossible, however, for $\varphi \neq 0$ unless $\tau = 0$ and $k = 0$.

It can be also shown that

(iii) as $k \to 0$, the eigenvalue in Σ^* either goes out of the region Σ^* or approaches one of the eigenvalues $\mu \in [-\nu_* + \delta, 0]$ of $\hat{B}_c^\infty(0) = \mathbf{L}_0 = \mathbf{L}$, and $\lambda_j(k, c)$ in Theorem 3.6(2) are the only eigenvalues which tend to 0.

The first half is easy to prove, see Proposition 1.6(3)(b)(c), while the second half is due to [27]. Let $\mu_* < 0$ be the largest eigenvalue of $\mathbf{L}$ other than 0, and set $\sigma_1 = |\mu_*|/2$. Evidently, then, there is a number $\kappa_0 > 0$ such that for all $|k| \leqslant \kappa_0$, the only eigenvalues lying in $\Sigma^* \cap \{\mathrm{Re}\,\lambda \geqslant -\sigma_1\}$ are $\lambda_j(k, c)$, $j = 0, \ldots, n+1$, given in Theorem 3.6(2), with $\mathrm{Re}\,\lambda_j(k, c) \geqslant -\sigma_1/2$.

Now, with this κ_0 chosen smaller if necessary so that $\kappa_0 \in (0, \kappa_1]$, let Λ^* denote the totality of the eigenvalues in Σ^* for $|k| \in [\kappa_0, \kappa_1]$, and put

$$\sigma_0 = \frac{1}{2}\inf\{-\mathrm{Re}\,\lambda \mid \lambda \in \Lambda^*\}.$$

A consequence of the above (i) is that $\sigma_0 > 0$. This proves the first halves of both (1) and (2) of Lemma 3.8.

It remains to evaluate $\|\Phi(\lambda.k, c)\|$. First, the second resolvent equation can be written in the form

$$\Phi(\lambda, k, c) = \big(I - G(\lambda, k)\big)^{-1}\big(\lambda - \hat{A}_c^\infty(k)\big)^{-1},$$

which implies that if both inverses on the right hand side exist, then, the resolvent $\Phi(\lambda, k, c)$ exists and is given by the above formula.

In view of (3.40), $(\lambda - \hat{A}_c^\infty(k))^{-1}$ exists and uniformly bounded for all λ with Re $\lambda \geqslant -\sigma_1$ since $\sigma_1 < \nu_*$. In the regions (a), (b), (c) in the above, $\|G\| < 1$, so the inverse $(I - G)^{-1}$ exists as seen by the aid of the Neumann series. Further, in these regions, we can have, say, $\|G\| \leqslant 1/2$ and hence $\|(I-G)^{-1}\| \leqslant 2$, uniformly in λ, k, c if $\nu_{**}, \kappa_1, \tau_1$ are chosen sufficiently large, which comes from (3.40) and thanks to Proposition 3.9. Thus, we proved (3.41) for $|k| \geqslant \kappa_1$ and (3.42) for $|k| \leqslant \kappa_1$ and $\lambda \in \Sigma_*$ with $\sigma_*, 2\tau_0$ replaced by ν_{**}, τ_1 respectively. The rest of relevant values of λ, k forms a bounded set on which there are no eigenvalues belonging to Σ^*, so $\Phi(\lambda, k, c)$ is uniformly bounded there. This completes the estimates (3.41) and (3.42), and hence the proof of Lemma 3.8. □

3.3.2. *Limiting absorption principle.* The inverse Fourier transform of $\Phi(\lambda, k, c)$ is the resolvent $(\lambda - B_c^\infty)^{-1}$. Thus, the inverse of the operator B_c^∞, which we denote by $U^\infty(c)$ in the sequel, is given by

$$U^\infty(c) \equiv \left(B_c^\infty\right)^{-1} = \sum_{j=0}^{n+2} U_j^\infty(c),$$
$$U_j^\infty(c) = -\mathcal{F}^{-1}\left\{\Phi_j(0,k,c)\right\}\mathcal{F}. \tag{3.47}$$

Actually, $U_j^\infty(c)$ is not a bounded operators for $j = 0, \dots, n+1$: $\Phi_j(0,k,c)$ behaves like $1/\lambda_j(k,c)$ near $k = 0$ and $\lambda_j(0,c) = 0$, as seen from (3.30). That is, $\Phi_j(0,k,c)$ has a singularity at $k = 0$ and therefore, is not a Fourier multiplier in $L^2(D^\infty)$.

The singularity behaves differently according to the classification of $\lambda_j(k,c)$ given in Remark 3.7, but it is integrable near $k = 0$. Consequently, $U^\infty(c)$ can be extended as a bounded operator by modifying the spaces of its domain and range appropriately. This is in the spirit of the limiting absorption principle.

To be precise, introduce the function space,

$$L_\beta^{p,r} = \left\{u = u(x,\xi) \;\middle|\; \langle\xi\rangle^\beta u \in L^r\left(\mathbb{R}_\xi^n; L^p\left(\mathbb{R}_x^n\right)\right)\right\}, \qquad L^{p,r} = L_0^{p,r}. \tag{3.48}$$

PROPOSITION 3.11. *There is a positive number c_0 such that for each $|c| \leqslant c_0$, $j = 0, \dots, n+1$, and $m = 0, 1$, the operators*

$$|c|^{\gamma\theta} U_j^\infty(c)(1-\mathbf{P})^m : L_0^{q,2} \to L_\beta^{p,\infty}$$

are all bounded operators and their operator norms are uniformly bounded in c, whenever

$$n \geqslant 3, \quad \beta \geqslant 0, \quad m = 0, 1, \quad 1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty,$$
$$\gamma \equiv 1/q - 1/p > (l-m)/(n+\theta),$$

where

$l = 1,\ \theta = 0$ *for the case* (a) *of Remark* 3.7,

$l = 2,\ \theta \in [0,1)$ *for the case* (b) *of Remark* 3.7.

Here $\mathbf{P} = \mathbf{P}_c$ is the projection in Proposition 3.1.

PROOF. The proof relies on the following property of the Fourier transformation:

$$\mathcal{F}, \mathcal{F}^{-1} : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \tag{3.49}$$

is a bounded operator whenever $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$ and $1/q + 1/p = 1$.

First, we prove the case $m = 0$. Let $p \geqslant 2$, $s \geqslant 1$ and $1/p + 1/p' = 1$. We have

$$\begin{aligned}
\left\| U_j^\infty(c) u(\cdot, \xi) \right\|_{L^p(\mathbb{R}^n_x)} &\leqslant \left\| \Psi_j(0, k, c) \hat{u}(\cdot, \xi) \right\|_{L^{p'}(\mathbb{R}^n_k)} \quad \text{(by (3.49))} \\
&= \left(\int_{|k| \leqslant \kappa_0} \left| (\lambda_j(k, c) \right|^{-p'} \left| P_j(k, c) \hat{u}(k, \xi) \right|^{p'} dk \right)^{1/p'} \\
&\quad \text{(by Theorem 3.6 (3a))} \\
&\leqslant J^\gamma \left(\int_{|k| \leqslant \kappa_0} \left| P_j(k, c) \hat{u}(k, \xi) \right|^{p's} dk \right)^{1/p's} \\
&\quad \text{(by Hölder)} \\
&\leqslant C J^\gamma \left(\int_{|k| \leqslant \kappa_0} \langle \xi \rangle^{-p's\beta} \left\| \hat{u}(k, \cdot) \right\|_{L^2_\xi}^{p's} dk \right)^{1/(p's)} \\
&\quad \text{(by Theorem 3.6(2))} \\
&\leqslant C \langle \xi \rangle^{-\beta} J^\gamma \|u\|_{L^{q,2}}, \quad 1/q = 1 - 1/(p's) \quad \text{(by (3.49))}
\end{aligned}$$

where J is the integral

$$J = \int_{|k| \leqslant \kappa_0} \left| \lambda_j(k, c) \right|^{-1/\gamma} dk, \quad \gamma = 1/(p'r) = 1/q - 1/p \ (1/s + 1/r = 1).$$

Consider the case (b) of Remark 3.7, that is, the case where $\lambda_{j,0} = 0$ in the asymptotic expansion (3.30) of Theorem 3.6. Its higher order term $O(|k|^3)$ may be ignored by choosing κ_0 smaller if necessary, and we compute the integral

$$J_0 = \int_{|k| \leqslant \kappa_0} \left| ic \cdot k + a|k|^2 \right|^{-1/\gamma} dk$$

for, say, $a = \lambda_{j,2}/2 > 0$. Use the spherical coordinates of $k \in \mathbb{R}^n$ to deduce

$$J_0 = C \int_0^{\kappa_0} J_1\big(\kappa, |c|\big) \kappa^{n-1} \, d\kappa,$$

where

$$J_1\big(\kappa, |c|\big) = \int_0^1 \frac{(1 - \mu^2)^{(n-3)/2}}{(|c|^2 \kappa^2 \mu^2 + a^2 \kappa^4)^{1/(2\gamma)}} \, d\mu.$$

The change of variable $\mu = (\kappa/c)y$ and the Hölder inequality yield

$$\begin{aligned} J_1 &\leqslant |c|^{-1}\kappa^{1-2/\gamma} \int_0^{|c|/\kappa} \bigl(y^2+a^2\bigr)^{-1/(2\gamma)}\,dy \\ &\leqslant |c|^{-1}\kappa^{1-2/\gamma} \biggl(\int_0^{|c|/\kappa} dy\biggr)^{1/r'} \biggl(\int_0^{|c|/\kappa} \bigl(y^2+a^2\bigr)^{-s'/(2\gamma)}\,dy\biggr)^{1/s'}, \\ &\quad 1/r'+1/s'=1 \\ &\leqslant C|c|^{-1+1/r'}\kappa^{1-2/\gamma-1/r'}, \quad \text{if } s'/\gamma > 1. \end{aligned}$$

Put $\theta = 1 - 1/r'$. Then, $s'/\gamma > 1$ means $\gamma\theta < 1$ and

$$J_0 \leqslant C|c|^{-\theta} \int_0^{\kappa_0} \kappa^{n-2/\gamma-1/r'}\,d\kappa \leqslant C|c|^{-\theta}.$$

The last inequality follows if $n - 2/\gamma - 1/r' > -1$ or $\gamma > 2/(n+\theta)$, which proves the proposition for the case $l = 2$ and $m = 0$.

In the computation for the case $m = 1$, $\hat{u}$ is replaced by $(I - \mathbf{P})\hat{u}$. Then, since Theorem 3.6(2) gives

$$P_j(k,c)(I-\mathbf{P}) = |k|P_{j,1}(k,c)(I-\mathbf{P}),$$

it suffices to compute the integral

$$\begin{aligned} J_0 &= \int_{|k|\leqslant\kappa_0} |k|^{1/\gamma}\bigl|\lambda_j(k,c)\bigr|^{-1/\gamma}\,dk, \\ &\gamma = 1/(p'r) = 1/q - 1/p \ (1/s+1/r=1). \end{aligned}$$

Now, still in the case (b) of Remark 3.7, the preceding computation leads to

$$J_0 \leqslant C|c|^{-\theta} \int_0^{\kappa_0} \kappa^{n-1/\gamma-1/r'}\,d\kappa \leqslant C|c|^{-\theta}.$$

The last inequality holds if $n - /\gamma - 1/r' > -1$ or $\gamma > 1/(n+\theta)$, which proves the proposition for the case $l = 2$ and $m = 1$.

Consider the case (a) of Remark 3.7. Then, since $\lambda_{j,0} \neq 0$ and if $c_0 > 0$ is chosen smaller if necessary, $\lambda_j(k,c)$ behaves like $ib|k| + a|k|^2$ with $a > 0$, $b \neq 0$ for $|k| \leqslant \kappa_0$ and $|c| \leqslant c_0$. Then, the integral J_0 takes the form, in the spherical coordinates,

$$J_0 = C\int_0^{\kappa_0} \kappa^{-(1-m)/\gamma+n-1}\bigl(b^2+\kappa^2\bigr)^{-1/(2\gamma)}\,d\kappa$$

which is bounded if $\gamma > (1-m)/n$, proving the proposition for the case $l = 1$ and $m = 0, 1$. This completes the proof of the lemma. □

REMARK 3.12. The operator $U_j^\infty(c)$ has a singularity $|c|^{-\gamma\theta}$ if $\theta > 0$. The freedom of choice of p, q is the most restrictive in the case $l = 2, m = 0$.

The remainder term $U_{n+2}^\infty(c)$ is easy to evaluate: Theorem 3.6(3b) and Parseval's relation lead to

LEMMA 3.13. *$U_{n+2}^\infty(c)$ is a bounded operator on $L^2 = L^{2,2}$ for each $|c| \leqslant c_0$ with uniformly bounded operator norm.*

We can now deduce the main result of this subsection.

THEOREM 3.14. *There are positive numbers C, c_0 such that for each $|c| \leqslant c_0$, it holds that*

$$\begin{aligned}&\left\| U^\infty(c)(1-\mathbf{P})^m \nu_c^\alpha u\right\|_{L_\beta^{p,r}}\\ &\quad\leqslant C\left(\|u\|_{L_\beta^{p,r}} + \left\|\nu_c^\alpha u\right\|_{L^2} + |c|^{-\gamma\theta}\left\|\nu_c^\alpha u\right\|_{L^{q,2}}\right)\end{aligned} \tag{3.50}$$

whenever

$$\begin{aligned}&n \geqslant 3, \quad \beta \geqslant 0, \quad m = 0, 1, \quad 1 \leqslant q \leqslant 2 \leqslant p, \quad r \leqslant \infty,\\ &\alpha \in [0, 1], \quad \theta \in [0, 1),\\ &\gamma \equiv 1/q - 1/p > (2-m)/(n+\theta).\end{aligned}$$

PROOF. To simplify the notation, put

$$R = U^\infty(c) = \left(B_c^\infty\right)^{-1}, \qquad R' = \left(A_c^\infty\right)^{-1}, \qquad G = R'K_c,$$

where A_c^∞ is as in (3.44). An argument similar to that in Section 3.2.2 yields an explicit expression of R':

$$R'u = \int_0^\infty e^{-\nu_c(\xi)t} u(x - t\xi, \xi)\, dt, \quad (x, \xi) \in D^\infty. \tag{3.51}$$

We need the following lemma.

LEMMA 3.15. *Let $c_0 > 0$, $p, r \in [1, \infty]$, and $\beta \geqslant 0$. Then, the following operators are all bounded with uniform bounds of operator norms for $|c| \leqslant c_0$.*

(1) $R'\nu_c^\alpha : L_\beta^{p,r} \to L_\beta^{p,r}$ $\quad\big(\forall \alpha \in [0, 1]\big)$,
(2) $G^\ell : L_\beta^{p,r} \to L_\beta^{p,r}$ $\quad(\forall \ell \in \mathbb{N})$,
(3) $G^\ell : L^q \to L_\beta^{p,r}$ $\quad\big(\forall q \in [1, p),\ \exists \ell \in \mathbb{N}\big)$.

Admit this for a while to complete the proof of the theorem. Write the second resolvent equation, [40], as $R = R' + GR$. Iterating this gives

$$R = \sum_{h=0}^{\ell} G^h R' + G^\ell R$$

and hence

$$R(I - \mathbf{P})^m \nu_c^\alpha = \sum_{h=0}^{\ell} G^h R'(I - \mathbf{P})^m \nu_c^\alpha + G^\ell R(I - \mathbf{P})^m \nu_c^\alpha \equiv H_1 + H_2.$$

Apply (1), (2) of the above lemma to H_1, which gives the first term on the right hand side of (3.50). Its second term is a contribution from $G^\ell U_{n+2}^\infty(c)$ to H_2 evaluated by using Lemma 3.13 and (3) of the above lemma. Finally, The contribution from $G^\ell U_j^\infty(c)$ for other j is just the third term in view of Proposition 3.11 and (2) of the above lemma. Here, we are forced to use the worst case $\ell = 2$. This completes the proof of the theorem. □

We shall also discuss the boundary operators $\gamma^\pm U^\infty(c)$ which are well-defined in $Y^{p,\pm}$ as seen in Proposition 3.3. We need them in different boundary spaces. Define

$$Y_\beta^{p,r,\pm} \ni u \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} \left\{ \left(1 + |\xi|\right)^\beta \left(\int_{\Omega^\pm(\xi)} \left|u(x, \xi)\right|^p \rho(x, \xi)\, d\sigma_x \right)^{1/p} \right\}^r d\xi < \infty. \tag{3.52}$$

Notice $Y^{p,\pm} = Y_0^{p,p,\pm}$. See (3.17).

THEOREM 3.16. *There are positive numbers C, c_0 such that for each $|c| \leqslant c_0$, it holds that*

$$\begin{aligned} &\left\|\gamma^\pm U^\infty(c)(1 - \mathbf{P})^m \nu_c^\alpha u\right\|_{Y_\beta^{p,r,\pm}} \\ &\quad \leqslant C\left(\left\|\nu_c^{\alpha/p} u\right\|_{L_\beta^{p,r}} + \left\|\nu_c^\alpha u\right\|_{L^2} + |c|^{-\gamma\theta}\left\|\nu_c^\alpha u\right\|_{L^{q,2}}\right) \end{aligned} \tag{3.53}$$

whenever

$$\begin{aligned} &n \geqslant 3, \quad \beta \geqslant 0, \quad m = 0, 1, \quad 1 \leqslant q \leqslant 2 \leqslant p, \quad r \leqslant \infty, \\ &\alpha \in [0, 1], \quad \theta \in [0, 1), \\ &\gamma \equiv 1/q - 1/p > (2 - m)/(n + \theta). \end{aligned}$$

PROOF. We use again the second resolvent equation, but now in the form

$$\gamma^{\pm}R(I-\mathbf{P})^m\nu_c^{\alpha} = \gamma^{\pm}R'\sum_{h=0}^{\ell}(G')^h(I-\mathbf{P})^m\nu_c^{\alpha} + \gamma^{\pm}G^{\ell}R(I-\mathbf{P})^m\nu_c^{\alpha}$$
$$\equiv H_1^{\pm} + H_2^{\pm},$$

where $G' = K_c R'$. The first term $H_1^{\pm}$ gives the first term on the right hand side of (3.53) if we note that Lemma 3.15(2) is valid also for G' and if we admit

$$\left\|\gamma^{\pm}R'\nu^{\alpha}u\right\|_{Y_{\beta}^{p,r,\pm}} \leqslant C\left\|\nu_c^{\alpha/p}u\right\|_{L_{\beta}^{p,r}}. \tag{3.54}$$

This will be proved later, together with Lemma 3.15.

The second term on the right hand side of (3.53) is a contribution from the part $v = G^{\ell}U_{n+2}^{\infty}(c)(I-\mathbf{P})^m\nu_c^{\alpha}$ contained in $H_2^{\pm}$. In fact, it comes from Lemma 3.13 combined with Lemma 3.15(3) for ℓ large enough to realize $p = r = \infty$ and the elementary inequality

$$\|\gamma^{\pm}v\|_{Y_{\beta}^{p,r,\pm}} \leqslant C\|v\|_{L_{\beta'}^{\infty}}, \quad \beta' \geqslant \beta + 1/p + 1/r \text{ (since } \rho(x,\xi) \leqslant |\xi|).$$

The third term is obtained similarly by Proposition (3.11) for $U_j^{\infty}(c)$ for $j = 0,\dots,n+1$. This completes the proof of the theorem. □

The rest of this section is devoted to the proof of Lemma 3.15 and the claim (3.54).

PROOF OF LEMMA 3.15. Taking the L_x^p norm of the formula (3.51) yields

$$\left\|R'\nu_c^{\alpha}u(\cdot,\xi)\right\|_{L_x^p} \leqslant \int_0^{\infty} e^{-\nu_c(\xi)t}\nu_c(\xi)^{\alpha}\left\|u(\cdot,\xi)\right\|_{L_x^p}dt \leqslant \nu_*^{-1+\alpha}\left\|u(\cdot,\xi)\right\|_{L_x^p},$$

whence Lemma 3.15(1) follows, and also (2) exactly in the same way, combined with Lemma 1.4.

To prove (3), we again start from the formula (3.51) but take its L_{ξ}^q norm. Let $p, q \in [1,\infty]$ be such that

$$q \leqslant p, \qquad \gamma \equiv \frac{1}{q} - \frac{1}{p} < \frac{1}{n}.$$

Then,

$$\left\|R'u(x,\xi)\right\|_{L_{\xi}^q} \leqslant \int_0^{\infty} e^{-\nu_* t}\left\|u(x-t\xi,\xi)\right\|_{L_{\xi}^q}dt.$$

By the change of variables $\xi \to y = x - t\xi$, we get

$$\left\|u(x-t\xi,\xi)\right\|_{L_{\xi}^q} = t^{-n/q}\left\|u\big(y,(y-x)/t\big)\right\|_{L_y^q}.$$

Write the last norm as $w(t,x)$. Put $r = p/q > 1$. Then, again by the change of variables $x \to \xi = (x-y)/t$,

$$\|w(t,x)\|_{L_x^p}^q = \|w(t,x)^q\|_{L_x^r} \leqslant \int_{\mathbb{R}^n} \| |u(y,(y-x)/t)|^q \|_{L_x^r} \, dy = t^{n/r} \|u\|_{L_x^q(L_\xi^p)}.$$

Combining these estimates yields

$$\|R'u\|_{L_x^p(L_\xi^q)} \leqslant \int_0^\infty e^{-\nu_* t} t^{-n\gamma} \|u\|_{L_x^q(L_\xi^p)} \, dt = C\|u\|_{L_x^q(L_\xi^p)}.$$

Now, the bootstrap argument based on this and Lemma 1.4(1) proves Lemma 3.15(3). □

PROOF OF (3.54). Put $w = R'\nu_c^\alpha u$. Then, by Hölder inequality and (3.51),

$$|w(x,\xi)|^p \leqslant \int_0^\infty \left| \nu_c(\xi)^{\alpha+1/p-1} u(x-t\xi,\xi) \right|^p dt.$$

Multiply this by $\rho(x,\xi)$ and integrate over $S^\pm$. Then, we get the desired estimate (3.54). □

3.4. *Operator B_c*

Since the obstacle $\mathcal{O}$ is "compact," the operator B_c may be expected to be a compact perturbation of B_c^∞. Actually, we will derive a semi-explicit formula of the inverse operator B_c^{-1} in terms of the inverse $U^\infty(c) = (B_c^\infty)^{-1}$ and related operators, in which the compactness argument based on the "velocity averaging" plays a key role.

Our working hypotheses are (1.25) for the collision operator Q, (3.13) for the domain Ω, and the one for the boundary operator $\mathbb{B}_0$ is stated as

$$[\mathrm{B}] \quad \begin{array}{lll} (1) & \mathbb{B}_0\gamma^+\mathbf{M}_0^{1/2} = \gamma^-\mathbf{M}_0^{1/2}, & \forall (x,\xi) \in S^-, \\ (2) & \|\mathbb{B}_0 u\|_{Y^{2,-}} \leqslant \|u\|_{Y^{2,+}}, & \forall u \in Y^{2,+}, \\ (3) & \mathbb{B}_0 : Y_\beta^{p,+} \to Y_\beta^{p,-} \text{ bounded} & \forall p \in [2,\infty],\ \beta \geqslant 0. \end{array} \tag{3.55}$$

[B](1) requires that $\mathbb{B}$ preserves the standard Maxwellian ($c = 0$) and (2) that the boundary condition is conservative or dissipative in L^2 sense. It is easy to check that any convex linear combination of the examples (2), (3), and (4) with $T_w = 1$ in (1.57) introduced in Section 1.3 fulfill [B].

For simplicity of notation, throughout this section, the inverse B_c^{-1} will be denoted by $U(c)$. Thus, $U(c)$ is a solution operator of the boundary value problem

$$\begin{cases} -\xi \cdot \nabla_x u + \mathbf{L}_c u = f, & x \in \Omega,\ \xi \in \mathbb{R}^n, \\ \mathbb{M}u = 0, & (x,\xi) \in S^-, \\ u \to 0\ (|x| \to \infty), & \xi \in \mathbb{R}^n, \end{cases} \tag{3.56}$$

where $\mathbb{M}$ is the boundary operator

$$\mathbb{M}u = \gamma^- u - \mathbb{B}_0 \gamma^+. \tag{3.57}$$

Let r denote the restriction operator from D^∞ to D, and e the extension operator from D to D^∞ by 0. $U(c)$ will be constructed in the following steps. Suppose, first, that $U(c)$ exists. Then, $u_1 = U(c)f - rU^\infty(c)ef$ must solve the inhomogeneous boundary value problem

$$\begin{cases} -\xi \cdot \nabla_x u_1 + \mathbf{L}_c u_1 = 0, & x \in \Omega,\ \xi \in \mathbb{R}^n, \\ \mathbb{M}u = H, & (x, \xi) \in S^-, \\ u_1 \to 0\ (|x| \to \infty), & \xi \in \mathbb{R}^n, \end{cases} \tag{3.58}$$

where

$$H = \mathbb{M}rU^\infty(\lambda, c)ef. \tag{3.59}$$

Introduce an auxiliary problem

$$\begin{cases} -\xi \cdot \nabla_x u_2 - \nu_c(\xi) u_2 = 0, & x \in \Omega,\ \xi \in \mathbb{R}^n, \\ \gamma^- u_2 = H, & (x, \xi) \in S^-, \\ u_2 \to 0\ (x \to \infty), & \xi \in \mathbb{R}^n. \end{cases} \tag{3.60}$$

This can be solved explicitly as $u_2 = R(c)H$ with the solution operator $R(c)$ given by

$$R(c)h = \begin{cases} e^{-\nu_c(\xi) t^-(x,\xi)} h(x - t^-(x, \xi)\xi), & t^-(x, \xi) < \infty, \\ 0, & t^-(x, \xi) = \infty, \end{cases} \tag{3.61}$$

where t^- is the "backward exit time"

$$t^-(x, \xi) = \inf\{t \geqslant 0 \mid x - t\xi \in \mathcal{O}\}, \quad (x, \xi) \in D.$$

The operator $R(c)$ will be studied later in Lemma 3.21, which shows, among others, that $\gamma^+ R(c) = 0$. Admitting this for a while, we now see that $u_3 = u_1 - u_2$ solves the inhomogeneous problem

$$\begin{cases} -\xi \cdot \nabla_x u_3 + \mathbf{L}_c u_3 = K_c u_2, & x \in \Omega,\ \xi \in \mathbb{R}^n, \\ \mathbb{M}u_3 = 0, & (x, \xi) \in S^-, \\ u_3 \to 0\ (|x| \to \infty), & \xi \in \mathbb{R}^n. \end{cases} \tag{3.62}$$

Thus, $u_3 = U(c) K_c u_2$ if $U(c)$ exists.

Now, gathering all the above quantities, we arrive at the operator identity

$$U(c) = S_0 + R(c)S_1 + U(c)S_2S_1 \tag{3.63}$$

with

$$S_0 = rU^\infty(c)e, \qquad S_1 = \mathbb{M}S_0, \qquad S_2 = K_c R(c).$$

Define the operator

$$T = T(c) = S_1 S_2 = \mathbb{M}rU^\infty(c)eK_c R(c). \tag{3.64}$$

Then, if the operator $I - T(c)$ has an inverse, (3.63) gives a semi-explicit expression of $U(c)$:

$$\begin{aligned} U(c) &= S_0 + R(c)S_1 + \big(S_0 S_2 + R(c)T\big)(1-T)^{-1}S_1 \\ &= S_0 + \big(S_0 S_2 + R(c)\big)(1-T)^{-1}S_1 \\ &= S_0 + \big(\gamma^- rU^\infty(c)e\big)^\dagger(1-T)^{-1}S_1, \end{aligned} \tag{3.65}$$

where † means the adjoint operator. Notice that if we put $T(c) = S_2 S_1$, we have another expression. However, the present one has an advantage that Theorem 3.14 can be used with $m = 1$ which gives the best possible estimate when combined with the nonlinear term $\Gamma(u,u)$.

Although the derivation given above is formal, it is clear that if all the operators appearing on its right hand side are well-defined, the expression (3.65) is substantiated and provides the desired inverse operator $U(c)$.

Thus, our task is now to establish estimates of those operators. The operator $U^\infty(c)$ has been already studied extensively and $R(c)$ is given explicitly. A crucial point is the invertibility of the operator $1 - T$. Write $Y_\beta^{p,-} = Y_\beta^{p,p,-}$ in (3.52):

$$Y_\beta^{p,-} = \big\{u \bigm| \langle\xi\rangle^\beta u \in Y^{p,-}\big\}.$$

The following theorem is a key theorem to our theory, whose proof will be given at the end of this section.

THEOREM 3.17. *For $n \geqslant 3$, there exist positive numbers $p_0 \geqslant 2$ and c_0 such that for any*

$$p \in [p_0, \infty], \quad \beta > n(1/2 - 1/p), \quad |c| \leqslant c_0, \tag{3.66}$$

the following holds for the operator

$$T(c) : Y_\beta^{p,-} \to Y_\beta^{p,-}.$$

(1) *$T(c)$ is a bounded operator whose operator norm is uniformly bounded in c.*
(2) *$T(c)$ is a compact operator for $p \in [p_0, \infty)$ and so is $T(c)^3$ for $p = \infty$.*
(3) *For $p \in [p_0, \infty]$, the inverse $(I - T(c))^{-1}$ exists and its operator norm is uniformly bounded in c.*

We can have $p_0 = 2$ when $n \geqslant 5$.

REMARK 3.18. For $p = \infty$, it is not yet known whether or not $T(c)$ itself is compact.

Now, Theorems 3.14 and 3.17, together with Lemma 3.21 given below, can justify the expression (3.65). To show this, put

$$L_\beta^{p,r} = L_\beta^{p,r}(D) = \{u = u(x,\xi) \mid \langle\xi\rangle^\beta u \in L^r(\mathbb{R}_\xi^n; L^p(\Omega_x))\},$$
$$X_\beta^p = L_\beta^{p,\infty} \cap L_{\beta-1/p}^\infty, \qquad Z_q = L_0^{2,2} \cap L_0^{q,2}. \tag{3.67}$$

THEOREM 3.19. *Let*

$$n \geqslant 3, \quad 1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty,$$
$$\alpha \in [0,1], \quad \theta \in [0,1), \quad m = 0,1, \quad \beta > n/2,$$
$$1/q - 1/p > (2-m)/(n+\theta), \quad 1/p < 1 - 2/(n+2). \tag{3.68}$$

Further, set $\gamma = 1 + 1/p - 1/q$. Then, there is a constant c_0 such that

$$\left\| U(c)(1-\mathbf{P})^m \nu_c^\alpha u \right\|_{L_{\beta-1/p}^{p,\infty}} \leqslant C|c|^{-\theta\gamma} \left(\|u\|_{X_\beta^p} + \|\nu_c^\alpha u\|_{Z_q} \right) \tag{3.69}$$

holds for $|u| \leqslant c_0$.

REMARK 3.20. Compared with Theorem 3.14, $U(c)$ behaves worse than $U^\infty(c)$ near $c = 0$.

In order to prove this theorem, first, we shall study the operator $R(c)$ defined in (3.61). The statement (2) of the following lemma has been used in the derivation of equation (3.60).

LEMMA 3.21. *For any $c \in \mathbb{R}^n$, the following holds.*

(1) *Let $p, q, r, s \in [1,\infty]$ with $r \leqslant p$, $s \leqslant q$ and $\beta \geqslant 0$. Put*

$$\gamma_0 = \frac{1}{r} - \frac{1}{p} + n\left(\frac{1}{s} - \frac{1}{q}\right).$$

The operators

$$R(c) : Y_{\beta+\gamma}^{p,q,-} \to L_\beta^{r,s} \quad (\gamma > \gamma_0, \text{ or } \gamma \geqslant \gamma_0 \text{ if } q = s),$$
$$K_c R(c) : Y_{\beta+\gamma}^{p,q,-} \to L_\beta^{r,s} \quad (\gamma > \gamma_0 - 1)$$

are bounded operators with operator norms locally uniformly bounded in c.

(2) $\gamma^+ R(c) = 0$.

PROOF. Put $u = R(c)h$ and set $w = |u|^r$ in (3.20) to reduce

$$\|u(\cdot,\xi)\|^r_{L^r_x} = \int_{\Omega^-(\xi)} |u(x,\xi)|^r \, dx \leqslant \frac{1}{r\nu_*} \int_{\partial\Omega^-(\xi)} |h(x,\xi)|^r \rho(x,\xi)\, d\sigma_x$$

for each ξ. Then, (1) for the first operator follows for the case $p = r$, $q = s$, $\gamma = 0$. The other cases come from this by a simple observation that $\partial\Omega$ is bounded, $\rho(x,\xi) \leqslant |\xi|$, and the injection

$$Y^{p,q,-}_{\beta+\gamma} \subset Y^{r,s,-}_{\beta} \quad \text{is continuous if} \quad \gamma > \gamma_0 \ (q \neq s) \text{ or } \gamma \geqslant \gamma_0 \ (q = s). \tag{3.70}$$

Use Lemma 1.4 for the second operator. To prove (2), let $h \in Y^{2,-}$. Then, $u = R(c)h \in W^{2,+}$ since $u \in L^2$ as shown above and $\Lambda^+ u = (\xi\cdot\nabla + \nu_c)u = 0 \in L^2$ by (3.60). Thus, (3.19) implies $\gamma^+ u = 0$. This completes the proof of the lemma. □

Now we are in the position for the

PROOF OF THEOREM 3.19. We shall evaluate each term on the right hand side of (3.65). Its first term S_0 was already evaluated in Theorem 3.14, from which the first condition of (3.68) comes.

To evaluate the second term, combine Theorem 3.14 for $r = \infty$, $q = \infty$, $m_0 = \alpha = 0$ with Lemma 3.21 for $q = 1$ to evaluate $S_0 S_2$. Then, if the second condition of (3.68) is satisfied,

$$\begin{aligned}\big\|\big(S_0S_2 + R(c)\big)h\big\|_{L^{p,\infty}_{\beta-1/p}} &\leqslant C\big(\|h\|_{Y^{p,\infty,-}_{\beta-1/p}} + \|S_2 h\|_{L^2} + |c|^{-\theta\gamma'}\|S_2 h\|_{L^{1,2}}\big)\\ &\leqslant C\big(1 + \|c\|^{-\theta\gamma'}\big)\|h\|_{Y^{p,\infty,-}_{\beta-1/p}}\end{aligned}$$

holds with $\gamma' = 1 - 1/p$. The estimate of $R(c)$ is due to Lemma 3.21. Now, put $h = (I - T(c))^{-1}h'$. Notice that the injection $Y^{p,\infty,-}_{\beta-1/p} \subset Y^{\infty,-}_{\beta}$ is continuous, and use Theorem 3.17 for $p = \infty$, to get

$$\|h\|_{Y^{p,\infty,-}_{\beta-1/p}} \leqslant C\|h\|_{Y^{\infty,-}_{\beta}} \leqslant C\|h'\|_{Y^{\infty,-}_{\beta}}. \tag{3.71}$$

Finally, put $h' = \mathbb{M}rU^\infty(c)e(I-P)^m\nu^\alpha_c u$, and use Theorem 3.16, for $p = r = \infty$ to obtain estimates of h' in $Y^{\infty,-}_\beta$ and $Y^{2,-}$. The restriction for q in Theorem 3.16 is absorbed in the first condition of (3.68). This completes the proof of Theorem 3.19. □

The rest of this section is devoted to the proof of Theorem 3.17.

PROOF OF THEOREM 3.17(1). Choose $p_0 \geqslant 2$ so that

$$p_0 > n/(n-2).$$

Put $u = eK_cR(c)h$ in Theorem 3.16 with $p = r \geqslant p_0$, $q = 1$, $\alpha = \theta = m = 0$. Then, the condition for p, q in the theorem is satisfied for $n \geqslant 3$ since $1/q - 1/p \geqslant 1 - 1/p_0 >$

$2/n$. The three terms on the right hand side of (3.53) can be evaluated by means of Lemma 3.21(1), which gives the desired estimate of $T(c)$. proving Theorem 3.17(1). Clearly, we can choose $p_0 = 2$ for $n \geqslant 5$. □

The proof of Theorem 3.17(2) relies on the "velocity averaging" stated in Remark 3.10, which is now stated as follows.

LEMMA 3.22. *Let* $Re\,\lambda > -\nu_*$, $\beta \geqslant 0$ *and* $c \in \mathbb{R}^n$. *The operator*

$$\gamma^{\pm}\big(\lambda - A_c^{\infty}\big)^{-1} K_c : L_\beta^{p,p} \to Y_\beta^{p,-}$$

is bounded for $p \in [2, \infty]$ *and compact for* $p \in [2, \infty)$.

PROOF. Proposition 3.3 assures that the relevant operator is bounded for the case $p \in [1, \infty]$ and $\beta = 0$, but the same proof given there works also for the case $\beta > 0$.

For the compactness, therefore, it suffices to prove the case $p = 2$: Other cases come by means of the interpolation [13]. Put $W = (\lambda - A_c^{\infty})^{-1} K_c$. Proposition 3.3 gives the trace estimates

$$\|\gamma^{\pm} W u\|_{Y^{2,\pm}} \leqslant C \|K_c u\|_{L^2(D^{\infty})}.$$

On the other hand, the Fourier transform of W is the operator $G = G(\lambda, k) = (\lambda - \hat{A}_c^{\infty}(k))^{-1} K_c$ studied in Proposition 3.9 on the decay property in k. We decompose G into three parts

$$\begin{aligned} G &= \chi\big(|\xi| > R\big) G + \chi(\Sigma_1)\chi\big(|\xi| < R\big) G + \big(1 - \chi(\Sigma_1)\big)\chi\big(|\xi| < R\big) G \\ &= G_1 + G_2 + G_3, \end{aligned}$$

where $\Sigma_1 = \Sigma_1(\epsilon, R)$ is given by (3.43) and $R, \epsilon > 0$. Correspondingly, write $W = W_1 + W_2 + W_3$.

First, note that the computation in the proof of Proposition (3.9) can be rewritten as

$$\begin{aligned} &\|G_1\| \leqslant C\delta^{-1} R^{-1}, \qquad \|G_2\| \leqslant C\delta^{-1}\epsilon^{1/2} R^{(n-1)/2}, \\ &\|G_3\| \leqslant C\big(\epsilon|k|\big)^{-1} R^{n/2}, \end{aligned} \tag{3.72}$$

which hold for any $c, k \in \mathbb{R}^n$ and $\mathrm{Re}\,\lambda \geqslant -\nu_* + \delta$, $\delta > 0$, where $\|\ \|$ is the operator norm on L_ξ^2 and $C > 0$ is independent of c, k, δ, λ.

Now, it is important to observe that the same computation, combined with Proposition 3.3, yields

$$\begin{aligned} &\|\gamma^{\pm} W_1 u\|_{Y^{2,\pm}} \leqslant C\|\Theta_1 K_c u\|_{L^2(D^{\infty})} \leqslant C R^{-1} \|u\|_{L^2(D^{\infty})}, \\ &\|\gamma^{\pm} W_2 u\|_{Y^{2,\pm}} \leqslant C\|\Theta_2 K_c u\|_{L^2(D^{\infty})} \leqslant C\delta^{-1}\epsilon^{1/2} R^{(n-1)/2} \|u\|_{L^2(D^{\infty})}, \end{aligned}$$

where $\Theta_1 = \chi(|\xi| > R)$ and $\Theta_2 = \mathcal{F}^{-1}\chi(\Sigma_1)\chi(|\xi| < R)\mathcal{F}$. Note that $\Theta_j,\ j = 1, 2$, commutes with $(\lambda - A_c^\infty)^{-1}$. Take $\epsilon = R^{-n-1}$. Then,

$$\|\gamma^\pm W - \gamma^\pm W_3\| \leqslant C\delta^{-1}R^{-1} \to 0 \quad (R \to \infty),$$

in the relevant operator norm. Since the operator norm limit of compact operators is a compact operator [40], it now suffices to prove the compactness of $\gamma^\pm W_3$.

For this, it is essential to observe that the estimate of G_3 stated above implies that

$$W_3 : L^2(D^\infty) \to L^2\big(\mathbb{R}^n_\xi; H^1\big(\mathbb{R}^n_x\big)\big)$$

is a bounded operator. According to the Sobolev–Rellich theorem [1], any bounded set of $H^\ell(\mathbb{R}^n)$, $\ell > 1/2$, is compactly embedded in $L^2(S)$ for any $(n-1)$-dimensional compact manifold S of $\mathbb{R}^n$. Thus, we can conclude that

$$\gamma^\pm W_3 : L^2(D^\infty) \to Y^{2,\pm}$$

has a compactness property with respect to x.

On the other hand, the compactness with respect to ξ is expected from the compactness of K_c stated in Lemma 1.5. To see this, however, we need to do a little more. First, decompose W_3 as

$$W_3 = \chi\big(|k| < \kappa_0\big)W_3 + \chi\big(|k| > \kappa_0\big)W_3 = W_{31} + W_{32}$$

for $\kappa_0 > 0$. The estimate of G_3 gives

$$\|W_{32}u\|_{L^2(\mathbb{R}^n_\xi; H^{1-\eta}(\mathbb{R}^n_x))} \leqslant C\kappa_0^{-\eta}\|u\|_{L^2(D^\infty)}$$

for any $\eta \in [0, 1]$. This and the Sobolev–Rellich theorem indicate that the operator norm of

$$\gamma^\pm W_{32} : L^2(D^\infty) \to Y^{2,\pm}$$

tends to 0 as $\kappa_0 \to \infty$.

Thus, we shall consider W_{31}. Its Fourier transform is given by

$$G_{31} = \Psi(k, \xi)K_c,$$
$$\Psi(k, \xi) = \big(\lambda + ik \cdot \xi + \nu_c(\xi)\big)^{-1}\big(1 - \chi(\Sigma_1)\big)\chi\big(|k| < \kappa_0\big)\chi\big(|\xi| < R\big).$$

We claim that

$$\sup_{k \in \mathbb{R}^n} \int_{\mathbb{R}^n} \big|\Psi(k, \xi + h) - \Psi(k, \xi)\big|^2 d\xi = o\big(|h|\big) \quad \big(h \in \mathbb{R}^n,\ |h| \to 0\big). \tag{3.73}$$

If this is valid, then, putting $w = W_{31}u$ and $\tau_h w = w(x, \xi + h)$, we have

$$\left\|(\tau_h - I)w\right\|_{L^2(\mathbb{R}^n_\xi; H^\ell(\mathbb{R}^n_x))} \leqslant C\left\|(\tau_h - I)K_c u\right\|_{L^2(D^\infty)} + o\big(|h|\big)\|u\|_{L^2(D^\infty)}.$$

Here, $\ell > 0$ can be taken arbitrarily large because of the presence of the cutoff function $\chi(|k| < \kappa_0)$. As is well known [26], a bounded set of L^2 is compact if and only if it is uniformly bounded and equi-continuous. Apply this to $K_c u$ and see that the first term on the right hand side in the above tends to 0 as $h \to 0$ uniformly for u in any bounded set of $L^2(D^\infty)$. Now, again by [26], we conclude the compactness of $\gamma^\pm W_{31}$ in ξ.

Finally, we see that the integral in (3.73) is bounded by

$$C|h|^2 + C\int_{|\xi|<R}\left|(I - \tau_h)\nu_c(\xi)\right|^2 d\xi + C\int_{|\xi|<R}\left|(I - \tau_h)\chi(\Sigma_1)(k, \xi)\right|^2 d\xi.$$

The second term on the right hand side is evaluated by $o(|h|)$ since $\nu_c(\xi) \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ by (1.27) while a simple geometric consideration gives

$$\begin{aligned}&\operatorname{mes}\big\{\xi \bigm| |\xi| < R,\ (I - \tau_h)\chi(\Sigma_1)(k, \xi) \neq 0\big\}\\ &\quad\leqslant 2\operatorname{mes}\big\{\xi \bigm| |\xi| < R, \left|k\cdot(\xi + h)\right| > \epsilon|k| > |k\cdot\xi|\big\} \leqslant C|h|R^{n-1},\end{aligned}$$

which indicates that the third term is of $O(|h|)$. This proves (3.73) and hence completes the proof of the lemma. □

Now, we can finish the proof of Theorem 3.17.

PROOF OF THEOREM 3.17(2). We need five steps.

Step 1. We claim that $T(c)$ is a compact operator for $p < \infty$ sufficiently large. For the proof, write $V^\infty(c) = (A_c^\infty)^{-1}$ and recall the second resolvent equation $U^\infty(c) = V^\infty(c) - V^\infty(c)K_c U^\infty(c)$, [40], which yields

$$\begin{aligned}T(c) &= \mathbb{M}r V^\infty(c)eK_c R(c) - \mathbb{M}r V^\infty(c)K_c U^\infty(c)eK_c R(c)\\ &= T_1 + T_2.\end{aligned} \tag{3.74}$$

By virtue of Lemma 3.22 combined with Lemma 3.21, T_1 has the desired compactness property for any $p < \infty$. The compactness property for T_2 comes again from Lemma 3.22 if the factor $U^\infty(c)eKR(c)$ is a bounded operator from $Y_\beta^{p,-} \to L_\beta^{p,p}$ for some $p < \infty$ and β. This can be proved exactly in the same way as Theorem 3.17(1) by using the estimates in Theorem 3.14 instead of those in Theorem 3.16. There, we should choose $p_0 = p$ so large that $1 - 1/p > 2/n$. Clearly, $p_0 = 2$ is possible for $n \geqslant 5$.

Step 2. We have to show that $T(c)^3$ is compact for $p=\infty$. For this, consider various cross products of T_1 and T_2. Lemmas 1.4 and 3.15 imply the smoothing effect that the operator

$$T_1^2 : Y_\beta^p \to Y_\beta^\infty$$

is bounded. Combining this with Lemma 3.22 and the first inequality of (3.71) implies that T_1^3 has the desired property. The proof is similar for other products. The detail is omitted.

Step 3. Thus, if $p \in [p_0, \infty)$, the spectrum of $T(c)$ consists only of nonzero discrete eigenvalues and the origin $\lambda = 0$, but this is also true for $p = \infty$ since $T(c)^3$ is a compact operator though it is not clear whether $T(c)$ itself is compact or not, see [26, p. 579].

Now, to establish the existence of the inverse $(I - T(c))^{-1}$, we first prove that 1 is not an eigenvalue of $T(0)$. For this, suppose the contrary so that there exists an eigenfunction,

$$\phi \in Y_\beta^{p,-}, \quad \phi \neq 0, \quad \phi = T(0)\phi.$$

Define w and u by

$$w = U^\infty(0) K e R(0)\phi, \quad u = w - eR(0)\phi, \tag{3.75}$$

where K is for K_0 $(c=0)$. Notice from Step 1 and by the aid of Lemma 3.21 that u is in L^p and solves the boundary value problem

$$B_0 u = -\xi \cdot \nabla_x u + \mathbf{L}u = 0 \quad \text{in } \Omega, \qquad \mathbb{M}u = 0 \quad \text{on } \partial\Omega.$$

First, consider the case $n \geqslant 5$. Then, we can take $p_0 = 2$ so that w, and hence u, are in $L^2(D^\infty)$. Thus, Green's formula (3.5) applies to this problem in Ω, to deduce, by virtue of the assumption [B](1) and the nonpositivity of $\mathbf{L}$ stated in Proposition 1.6,

$$0 = (B_0 u, u) = \frac{1}{2}\big(\|\mathbb{B}_0 \gamma^+ u\|_{Y^{2,-}} - \|\gamma^+ u\|_{Y^{2,+}}\big) + (\mathbf{L}u, u) \leqslant 0,$$

where $(\,,\,)$ is the inner product of $L^2(D)$. Now, we have obtained $(\mathbf{L}u, u) = 0$, and thereby $(I - \mathbf{P})u = 0$. Then, the equation $B_0 u = 0$ reduces to $\xi \cdot \nabla_x(\mathbf{P}u) = 0$ but this is possible only when $\mathbf{P}u = 0$. Thus, $u = 0$ or $w = R(0)\phi$ in D. And this implies $\gamma^+ w = 0$, owing to Lemma 3.21(2).

On the other hand, w in (3.75) is defined also in $\mathcal{O} \times \mathbb{R}^n_\xi$ and still satisfies the equation $B_0^\infty w = 0$ because $eR(0)\phi = 0$ there by definition of the extension operator e, and $\gamma^+ w = 0$ because w is absolutely continuous on the characteristic line $x + t\xi$, see Section 3.2.2. Since $w \in L^2(\mathcal{O} \times \mathbb{R}^n)$, we can apply Green's formula in $\mathcal{O}$, to deduce

$$\begin{aligned} 0 = \big(B_0^\infty w, w\big) &= \frac{1}{2}\big(\|\gamma^+ w\|^2_{Y^{2,+}} - \|\gamma^- w\|^2_{Y^{2,-}}\big) + (\mathbf{L}w, w) \\ &= -\|\gamma^- w\|^2_{Y^{2,-}} + (\mathbf{L}w, w) \leqslant 0, \end{aligned}$$

where $(,)$ is the inner product of $L^2(\mathcal{O} \times \mathbb{R}^n)$, whence follows $\gamma^- w = 0$. Note that the direction of the outward unit normal to $\partial\mathcal{O}$ is inward to Ω. Summarizing, we reached the contradiction $\phi = T(0)\phi = \mathbb{M}w = \gamma^- w - \mathbb{B}_0\gamma^+ w = 0$.

Now, we shall consider the case $n = 3, 4$, for which w in (3.75) is not in L^2: We know only $w \in L^p_\beta$ for $p \geqslant p_0 > 2$. As a consequence, Green's formula cannot be used directly. Fortunately, however, w has a special structure,

$$
\begin{aligned}
& w = w_1 + w_2, && (3.76)\\
& \text{(a)} \quad J_R \equiv \int_{S_R \times \mathbb{R}^n} \left(\xi \cdot \frac{x}{|x|} \right) \bigl| w_1(x, \xi) \bigr|^2 \, d\sigma_x \, d\xi = 0, \quad \mathbf{L} w_1 = 0, \\
& \text{(b)} \quad \bigl| w_1(x, \xi) \bigr| \leqslant C \bigl(1 + |x| \bigr)^{n-2}, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \\
& \text{(c)} \quad \bigl(1 + |\xi| \bigr) \nabla_x w_1, \ \bigl(1 + |\xi| \bigr) w_2 \in L^2(D^\infty).
\end{aligned}
$$

Here, S_R is the sphere of radius R with center at origin which is assumed to be inside $\mathcal{O}$ without loss of generality.

We will prove this at the end of this section and here proceed as follows. Let $\chi_R(x)$ be a smooth cutoff function such that it is 1 for $|x| \leqslant R - 1$ and 0 for $|x| \geqslant R$, and put

$$
u' = u_1 + \chi_R u_2, \quad u_1 = w_1, \ u_2 = w_2 - eR(0)\phi.
$$

This is in $L^2(D_R)$ for any $R > 0$ where $D_R = (\Omega \cap B_R) \times \mathbb{R}^n$, B_R being the ball of radius R with center at origin. Note that

$$
\begin{aligned}
& 0 = \chi_R(B_0 u) = B_0 u' + J, \quad J = (1 - \chi_R)\xi \cdot \nabla_x u_1 + (\xi \cdot \nabla_x \chi_R) u_2, \\
& \mathbb{M}u' = \mathbb{M}u = 0.
\end{aligned}
$$

By Green's formula (3.5) in D_R, the assumption [B](1) in (3.55), and Proposition 1.6, we get

$$
\begin{aligned}
0 &= (B_0 u' + J, u') \\
&= \frac{1}{2} \bigl(\|\mathbb{B}_0 \gamma^+ u\|_{Y^{2,-}} \bigr) - \|\gamma^+ u\|_{Y^{2,+}} + (\mathbf{L}u', u') + (J, u') + J_R \\
&\leqslant \bigl| (J, u') \bigr| + J_R.
\end{aligned}
$$

Here, $(,)$ is the inner product of $L^2(D_R)$.

By virtue of (3.76)(a), $J_R = 0$, while, with the norm $\| \ \|_R$ of $L^2(\{R - 1 < |x| < R\} \times \mathbb{R}^n)$, we have by (b) that

$$
\|w_1\|_R \leqslant C R^{-(n-2)} (\text{mes}\, S_R)^{1/2} \leqslant C_0.
$$

Also by (c), it holds that $J, w_2 \in L^2$, which yields, by the Schwarz inequality,

$$
\bigl| (J, u') \bigr| \leqslant \|J\|_R \bigl(C_0 + \|w_2\|_R \bigr) \to 0 \quad (R \to \infty),
$$

for any $n \geqslant 3$. Now, we have obtained $\lim(\mathbf{L}u', u') = 0$, and thereby $(I - \mathbf{P})u = 0$. This leads to the contradiction $\phi = 0$ by proceeding exactly in the same way as for the case $n \geqslant 5$.

Step 4. By the Fredholm alternative, therefore, $(I - T(0))^{-1}$ exists as a bounded operator. This is valid also for $(I - T(c))^{-1}$ for $|c| \leqslant c_0$ with some small $c_0 > 0$ because $T(c)$ is a continuous function of c in the relevant operator norm. Although the proof of this continuity can be carried out by repeating almost the same computation as presented so far for establishing boundedness of various operator norms, it is too lengthy to reproduce here. The interested reader is referred to [64,65].

Step 5. It remains to prove (3.76). We recall the formula (3.47) and put

$$W_j(x, \xi) = U_j(0)eKR(0)\phi, \quad j = 0, \dots, n+2.$$

Notice that since $eKR(0)\phi \in L^1 \cap L^2$ by Lemma 3.21, and if we follow the numbering j in Remark 3.7, W_1, W_{n+1} and W_{n+2} are all in L^2 by virtue of Proposition 3.11 for $q = 1$, $p = 2$, $l = 1$, $m = 0$, $\theta = 0$, and by Lemma 3.13. Moreover, by the bootstrap as in Lemma 3.15(3), we see $(1 + |\xi|)W_j \in L^2$. Therefore, they shall go to the component w_2.

We shall extract L^2 parts of other W_j, which also necessarily go to w_2. First, compute, by the aid of (3.20),

$$\Theta(k, \xi) \equiv \mathcal{F}\big(eR(c)\phi\big) = \int_{\partial\Omega^-(\xi)} \frac{1}{\nu_c(\xi) + i\xi \cdot k}\phi(x, \xi)e^{ik\cdot x}\rho(x, \xi)\, d\sigma_x.$$

Clearly, this is a smooth function of k so that for small $|k|$, it has an expansion,

$$\Theta(k, \xi) = \Theta(0, \xi) + |k|\Theta_1(k, \xi).$$

Since we are now in the class (b) of Remark 3.7 and hence $\lambda_{j,1} = 0$, recalling the asymptotic expansions of eigenvalues and eigenprojections in Theorem 3.6(2) and combining the above expression, we make a new decomposition

$$\begin{aligned} W_j &= \mathcal{F}^{-1}\left(\frac{1}{\lambda_{j,2}|k|^2} P_{j,0}(\tilde{k})(K\Theta)(0, \xi)\chi\big(|k| \leqslant \kappa_0\big)\right) \\ &\quad + \mathcal{F}^{-1}\left(\frac{1}{|k|}\Psi(k, c)\chi\big(|k| \leqslant \kappa_0\big)\right) \\ &\equiv A_1 + A_2, \end{aligned}$$

where all the terms having the factor $|k|$ were gathered in the term A_2, so that $\|\Psi(k, c)\|_{L^2_\xi}$ is uniformly bounded for $|k| \leqslant \kappa_0$ and $|c| \leqslant c_0$. Observe that the singularity of A_2 near $k = 0$ is, therefore, of order $|k|^{-2}|k| = |k|^{-1}$, which is square integrable over $|k| \leqslant \kappa_0$, and hence, by Parseval's relation, $A_2 \in L^2(D^\infty)$, for $n \geqslant 3$. This, combined with the property (3.33), shows that A_2 goes to w_2.

On the other hand, ∇_x gives rise to the factor $|k|$ in A_1 so that the singularity is of order -1, which implies, by the aid of Proposition 1.6, $(1+|\xi|)\nabla_x A_1 \in L^2(D^\infty)$ even for $n = 3, 4$, proving (3.76)(c). Now, for $n = 3, 4$, w_1 is the sum of A_1 for $j = 1, \dots, n$. Denote them by $a_j = a_j(x, \xi)$. Since $P_{j,0}(\tilde{k})$ depends only on $\tilde{k}$, it has the form

$$a_j(x, \xi) = \int_{|k| \leqslant \kappa_0} |k|^{-2} e^{ik\cdot x} \vartheta_j(\tilde{k}, \xi)\, dk$$

with some function ϑ_j of $\tilde{k}$, ξ only. This implies that a_j is rotation invariant with respect to x, that is, it is a radial function $a_j = a_j(|x|, \xi)$, and hence, so is $w_1 = w_1(|x|, \xi)$, which gives

$$\int_{S_R} \left(\xi \cdot \frac{x}{|x|}\right) \bigl|w_1(R, \xi)\bigr|^2 \, d\sigma_x = 0,$$

whence $J_R = 0$ and (3.76)(a) follows. Further, going to the spherical coordinates of k, we see

$$\int_{-1}^{1} \bigl(1-\mu^2\bigr)^{(n-3)/2} a(\mu) \left\{ \int_0^{\kappa_0} r^{n-3} e^{ir|x|\mu} \, dr \right\} d\mu$$
$$= O\bigl(|x|^{-(n-2)}\bigr) \quad \bigl(|x| \to \infty\bigr),$$

if $a(\mu)$ is a smooth function on $[-1, 1]$, which proves (3.76)(b). This completes the proof of (3.76). □

3.5. *Stationary solution*

It is now easy to see that the solution of the inhomogeneous linear boundary value problem (3.9) is obtained in the form

$$\phi_c = R(c)h_c - U(c)K_c R(c)h_c. \tag{3.77}$$

Although $U(c)$ may have a singularity as $c \to 0$, it is compensated by the fact that

$$\|h_c\| = O\bigl(|c|\bigr) \quad \text{in } Y_\beta^{\infty,-},\ \beta > n, \tag{3.78}$$

which comes from the assumption [B](3) in (3.55). Theorem 3.19 and this prove the

THEOREM 3.23. *Let*

$$n \geqslant 3, \quad p \in [2, \infty], \quad \theta \in [0, 1), \quad 1/p < 1 - 2/(n+\theta), \quad \beta > n. \tag{3.79}$$

Then, ϕ_c *defined by* (3.77) *solves* (3.9) *with*

$$\|\phi_c\| = O\bigl(|c|^{1-\theta\gamma}\bigr) \quad (c \to 0) \quad \text{in } L_\beta^{p,\infty},\ \gamma = 2 - 1/p. \tag{3.80}$$

In order to solve the nonlinear problem (3.8), we use Theorems 1.1 and 3.19, to deduce

PROPOSITION 3.24. *Let*

$$n \geqslant 3, \quad \theta \in [0,1), \quad \beta > n/2+1,$$
$$p \in [2,4] \cap \big[(n+2)/(n+\theta-2), n+\theta\big), \tag{3.81}$$

and put $\gamma = 1 + 2/p$. *Then, there are constants* $C_0 > 0$ *and* $c_0 > 0$ *such that for* $|c| \leqslant c_0$,

$$\big\| U(c)\Gamma(u,v)\big\| \leqslant C_0 |c|^{-\theta\gamma} \|u\|\|v\| \quad \textit{in } X_\beta^p. \tag{3.82}$$

Notice that although the choice $\theta = 0$ in the above is not possible in the physically important case $n = 3$, the singularity in (3.82) as $c \to 0$ can be compensated by the nice behavior of ϕ_c in Theorem 3.23. To see this, note that for $\theta \in [0, 2/7)$ and $p \geqslant 2$, we can find α such that

$$\alpha_1 = \theta(1+1/p) < \alpha < 1 - \theta(2-1/p) = \alpha_2.$$

Put $v = |c|^\alpha u$ and rewrite (3.5) as

$$v = \Phi(v,c) \equiv -|c|^\alpha U(c)\Gamma(v,v) + |c|^{-\alpha}\phi_c.$$

By virtue of Theorems 3.23 and 3.24, we can have

$$\big\|\Phi(v,c)\big\| \leqslant C_1|c|^\sigma \|v\|^2 + C_2|c|^\tau,$$
$$\big\|\Phi(v,c) - \Phi(w,c)\big\| \leqslant C_1|c|^\sigma \big(\|v\| + \|w\|\big)\|v-w\|,$$

both in X_β^p, where $\sigma = \alpha - \alpha_1$, $\tau = \alpha_2 - \alpha$, and C_1, C_2 are positive constants independent of c, v, w. This shows that if c is small, $\Phi(\cdot, c)$ is contractive. To see this, choose $c \in \mathbb{R}^n$ so small that

$$D \equiv 1 - 4C_1C_2|c|^{\sigma+\tau} > 0,$$

and with $a_* = (1-\sqrt{D})/(2C_1|c|^\sigma)$, set

$$W_* = \big\{v \in X_\beta^p \bigm| \|v\| \leqslant a_*\big\}.$$

Notice that a_* is the smaller positive root of the quadratic equation $C_1|c|^\sigma a^2 - a + C_2|c|^\tau = 0$, and that W_* is a complete metric space with the distance function induced by the relevant norm. Suppose now that $v, w \in W_*$. Then,

$$\big\|\Phi(v,c)\big\| \leqslant C_1|c|^\sigma a_*^2 + C_2|c|^\tau = a^*,$$
$$\big\|\Phi(v,c) - \Phi(w,c)\big\| \leqslant \mu\|v-w\|, \quad \mu \equiv 2C_1 a_*|c|^\sigma = 1 - \sqrt{D} < 1,$$

which shows that Φ is a contraction map. Thus, we proved the following theorem.

THEOREM 3.25. *Let*

$$n \geqslant 3, \quad \theta \in [0, 2/7), \quad \beta > n/2 + 1,$$

and assume also (3.79) *and* (3.81). *Then, there is a constant* $c_0 > 0$ *such that for any* $|c| \leqslant c_0$, (3.5) *has a unique solution* $u = u_c$ *in* X^p_β. *It satisfies*

$$\|u_c\| = O\big(|c|^{\alpha+\tau}\big), \quad \alpha + \tau = \alpha_1 = 1 - \theta(2 - 1/p). \tag{3.83}$$

Further, we can prove that u_c is continuous in c in the space X^p_β and that u_c is in the trace space $W^{p,+}$ with $\gamma^{\pm} u_c \in Y^{p,\pm}$ so that u_c solves (3.5) in L^p sense. See [64].

In [65], we have shown the asymptotic stability of u_c for small perturbation. The proof relies on the decay property in t of the semi-group e^{tB_c}. First, $e^{tB_c^\infty}$ is constructed as the inverse Laplace transform of the corresponding resolvent, i.e. (3.47), to which Theorem 3.6 applies to derive decay estimates of $e^{tB_c^\infty}$. On the other hand, (3.65) gives, with due modification, a semi-explicit expression of the resolvent of B_c in terms of that of B_c^∞, and hence, its inverse Laplace transform gives an expression of e^{tB_c} in terms of $e^{tB_c^\infty}$. The resulting expression then enables us to derive the decay rate of e^{tB_c}. The proof, however, is too lengthy to reproduce here. In the next section, we will discuss the decay estimate of $e^{tB_c^\infty}$ for the case $c = 0$ in a similar context.

4. Time-periodic solution

4.1. *Problem and basic strategy*

The purpose of this section is to study the Boltzmann equation (1.1) for the case where the inhomogeneous term S is a time-periodic function. More precisely, we assume $F = 0$ and look for a time-periodic solution near a uniform Maxwellian $\mathbf{M}$. Thus, we put

$$f = \mathbf{M} + \mathbf{M}^{1/2} u,$$

and consider the inhomogeneous equation

$$\frac{\partial u}{\partial t} = -\xi \cdot \nabla_x u + \mathbf{L}u + \Gamma(u, u) + S, \quad (t, x, \xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{4.1}$$

where $\mathbf{L}$ and Γ are those in (1.23). Here, we denote the modified inhomogeneous term by the same symbol S as in (1.1), but no confusion will arise.

The inhomogeneous term S stands for the distributional density of the external source of gas particles and in the case where S is time-periodic, (4.1) is the most basic model problem in the study of the generation and propagation of sound waves in a gas with an oscillating source. The existence and stability of the time-periodic solutions have been studied for various fluid dynamical equations including the Euler and Navier–Stokes equations, see, e.g., [28,46,70] and references therein, but only a little is known for the Boltzmann equation.

The below is our recent result [63] for the case $F = 0$. A physically more interesting case $F \neq 0$, that is, the case where the gas is shaken by the time-periodic force, has been solved for the space dimension $n \geqslant 5$ but is still open for $n = 3, 4$, see [25].

The time-periodic solution is viewed as a stationary solution under the periodic boundary condition in time. A variety of methods have been developed for solving the time-periodic boundary value problem for nonlinear PDE's. The strategy we adopt here is a combination of the time-decay estimate of the relevant linearized problem and the contraction mapping principle. The method developed here is applicable to a wide class of semi-linear evolution equations.

In order to explain the strategy, consider the Cauchy problem for (4.1) in the form

$$\begin{cases} \frac{du}{dt} = Bu + \Gamma(u,u) + S(t) \quad (t > t_0), \\ u(t_0) = u_0, \end{cases} \tag{4.2}$$

where B is the linearized Boltzmann operator,

$$B = -\xi \cdot \nabla_x + \mathbf{L}. \tag{4.3}$$

Notice that we do not fix the initial time t_0 but must consider arbitrary $t_0 \in \mathbb{R}$. Let e^{tB} denote the semi-group generated by B and consider the Duhamel form of (4.2),

$$u(t) = e^{(t-t_0)B} u_0 + \int_{t_0}^{t} e^{(t-\tau)B} \bigl\{ \Gamma\bigl(u(\tau), u(\tau)\bigr) + S(\tau) \bigr\}\, d\tau, \quad t \geqslant t_0. \tag{4.4}$$

As usual, the solution of the integral equation (4.4) is called a mild solution of (4.2).

Let T_0 denote the period of the inhomogeneous term $S = S(t, x, \xi)$. Suppose, first, that (4.4) has a t-periodic solution u^{per} with period T_0. Thus, it solves (4.2), and hence (4.4), with the particular initial data $u_0 = u^{\mathrm{per}}(t_0)$ for each $t_0 \in \mathbb{R}$. Choose $t_0 = -kT_0$ for $k \in \mathbb{N}$. Since $u_0 = u^{\mathrm{per}}(-kT_0) = u^{\mathrm{per}}(0)$, (4.4) is written as

$$u^{\mathrm{per}}(t) = e^{(t+kT_0)B} u^{\mathrm{per}}(0) + \int_{-kT_0}^{t} e^{(t-\tau)B} \bigl\{ \Gamma\bigl(u^{\mathrm{per}}(\tau), u^{\mathrm{per}}(\tau)\bigr) + S(\tau) \bigr\}\, d\tau.$$

We shall now suppose that e^{tB} has a nice decay property as $t \to \infty$. Then, going to the limit as $k \to \infty$ yields

$$u^{\mathrm{per}}(t) = \int_{-\infty}^{t} e^{(t-\tau)B} \bigl\{ \Gamma\bigl(u^{\mathrm{per}}(\tau), u^{\mathrm{per}}(\tau)\bigr) + S(\tau) \bigr\}\, d\tau, \tag{4.5}$$

provided that the last integral converges. Admit this for the time being and define the nonlinear map,

$$\Phi[u](t) = \int_{-\infty}^{t} e^{(t-\tau)B} \bigl\{ \Gamma\bigl(u(\tau), u(\tau)\bigr) + S(\tau) \bigr\}\, d\tau. \tag{4.6}$$

Then, (4.5) indicates that u^{per} is a fixed point of $\varPhi$.

Conversely, suppose that $\varPhi$ has a fixed point. It may not be time-periodic. But, let us suppose that the fixed point of $\varPhi$ is unique. Then, we can claim that if S is time-periodic with period T_0, so is this fixed point. For the proof, denote this unique fixed point by $\bar{u} = \bar{u}(t)$ and put $v(t) = \bar{u}(t + T_0)$. We have,

$$\begin{aligned} v(t) = \varPhi[\bar{u}](t + T_0) &= \int_{-\infty}^{t+T_0} e^{(t+T_0-\tau)B}\bigl\{\varGamma\bigl(\bar{u}(\tau), \bar{u}(\tau)\bigr) + S(\tau)\bigr\}\,d\tau \\ &= \int_{-\infty}^{t} e^{(t-\tau)B}\bigl\{\varGamma\bigl(\bar{u}(\tau + T_0), \bar{u}(\tau + T_0)\bigr) + S(\tau + T_0)\bigr\}\,d\tau \\ &= \int_{-\infty}^{t} e^{(t-\tau)B}\bigl\{\varGamma\bigl(v(\tau), v(\tau)\bigr) + S(\tau)\bigr\}\,d\tau \quad \text{(by periodicity of } S) \\ &= \varPhi[v](t). \end{aligned}$$

Thus, v is another fixed point but then, the uniqueness assumption says $v(t) = \bar{u}(t)$ for all $t \in \mathbb{R}$, proving the periodicity of $\bar{u}$ with the period T_0. It is evident that if this unique fixed point is differentiable with respect to t, it is a desired periodic solution to (4.2). Notice also that the above method covers the special case where S is time-independent and provides the existence of the stationary solution.

This section is organized as follows. The decay property of e^{tB} is established in Section 4.2. In Section 4.3, it is shown that $\varPhi$ is well defined and becomes a contraction map if S is small. The stability of u^{per} will be also discussed.

4.2. *Semi-group e^{tB}*

The convergence of the integral in (4.6) solely depends on the integrability of $e^{(t-s)B}$ over $s \in (-\infty, t]$ or that of e^{tB} over $t \in [0, \infty)$ because we can not assume the integrability of $\varGamma(u, u)$ nor S: They are t-periodic functions in our setting. e^{tB} will be integrable if it has a strong decay property. In order to establish such a decay property, we shall appeal to the theory of semi-groups that the inverse Laplace transform of the resolvent gives the corresponding semi-group, [37,40,72]. In our case, Theorem 3.6 provides informations of the resolvent which are enough to derive the desired decay property in various function spaces. Our function space, on the other hand, should be a Banach algebra which enables us to manipulate the nonlinearity of $\varGamma$. Among possible function spaces, our choice here is

$$X_\beta = L^2 \cap L^\infty_\beta, \tag{4.7}$$

where

$$L^2 = L^2\bigl(\mathbb{R}^n_x \times \mathbb{R}^n_x\bigr), \qquad L^\infty_\beta = \bigl\{u \bigm| \bigl(1 + |\xi|\bigr)^\beta u \in L^\infty\bigl(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\bigr)\bigr\}.$$

The norms are properly defined. This is a rather large space in the sense that no regularity of functions are required. As for another possible choice, see [63,67].

In [66], we proved that B generates a C_0 semi-group on L^2 and its semi-group e^{tB} can be well defined on the space X_β with the following decay property. Introduce an auxiliary space for initial data,

$$Z_q = L^2\big(\mathbb{R}^n_\xi; L^q(\mathbb{R}^n_x)\big), \quad q \in [1,2]. \tag{4.8}$$

Notice that $Z_2 = L^2$. The decay rate is characterized by the quantity

$$\sigma_{q,k} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{k}{2}. \tag{4.9}$$

THEOREM 4.1. (See [66, Theorem 2.20].) *Let $q \in [1,2]$ and $\beta \geqslant 0$. Then, there is a positive constant c_0 such that for any $u \in X_\beta \cap Z_q$, it holds that*

$$\big\|e^{tB}u\big\|_{X_\beta} \leqslant c_0(1+t)^{-\sigma_{q,0}}\big\{\|u\|_{X_\beta} + \|u\|_{Z_q}\big\}, \tag{4.10}$$

$$\big\|e^{tB}(I-\mathbf{P})u\big\|_{X_\beta} \leqslant c_0(1+t)^{-\sigma_{q,1}}\big\{\|u\|_{X_\beta} + \|u\|_{Z_q}\big\}. \tag{4.11}$$

Moreover, if u satisfies the condition

$$u \in X_\beta, \quad \big(1+|x|\big)u \in Z_1, \quad \int_{\mathbb{R}^3} \mathbf{P}u\,dx = 0, \quad a.e.\ \xi \in \mathbb{R}^n, \tag{4.12}$$

it holds that

$$\big\|e^{tB}u\big\|_{X_\beta} \leqslant c_0(1+t)^{-\sigma_{1,1}}\big\{\|u\|_{X_\beta} + \big\|\big(1+|x|\big)u\big\|_{Z_1}\big\}, \tag{4.13}$$

while if u has the form $u = \partial_x\tilde{u}$ for some function $\tilde{u} \in Z_q$, then,

$$\big\|e^{tB}u\big\|_{X_\beta} \leqslant c_0(1+t)^{-\sigma_{q,1}}\big\{\|u\|_{X_\beta} + \|\tilde{u}\|_{Z_q}\big\}. \tag{4.14}$$

The decay estimates (4.10) and (4.11) were first established in [58,59] in the space $L^2(\mathbb{R}^n_\xi; H^\ell(\mathbb{R}^n_x))$ starting from the expression (3.34) for $c = 0$. More precisely, since the operator B of (4.3) is nothing but the operator B_0^∞ $(c = 0)$ of (3.23), the semi-group e^{tB} is given by the inverse Laplace transform of $\Phi(\lambda, k, 0)$. Thus, consider the inverse Laplace transform of Φ_j in (3.34) for $c = 0$,

$$\Psi_j(t,k) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{\lambda t}\Phi_j(\lambda, k, 0)\,d\lambda,$$

where $\sigma_0 > 0$ is so chosen that the Bromwich integration path $(\sigma_0 - i\infty, \sigma_0 + i\infty)$ lies in the resolvent set of B. Now, we have the expression

$$e^{tB} = \sum_{j=0}^{n+2} V_j(t), \qquad V_j(t) = \mathcal{F}^{-1}\bigl\{\Psi_j(t,k)\bigr\}\mathcal{F}. \tag{4.15}$$

For $j = 0, \dots, n+1$, since $\Phi_j(\lambda, k, 0)$ has a simple pole only, it holds that

$$\Psi_j(t,k) = e^{\lambda_j(k)t} P_j(k)\chi\bigl(|k| \leqslant \kappa_0\bigr). \tag{4.16}$$

Similar to the proof of Proposition 3.11, we have, for example,

$$\bigl\|V_j(t)u\bigr\|_{Z_2} \leqslant CJ^{1/2}\|u\|_{Z_1}, \qquad J = \int_{|k|\leqslant\kappa_0} e^{-(\lambda_{j,2}/2)|k|^2 t}\,dk$$

and J is easily seen to have the polynomial decay $O(t^{-2\sigma_{1,0}})$. A much more delicate computation gives the exponential decay of V_{n+2}. Thus, (4.10) for $q = 1$ holds in the space $L^2_{x,\xi}$. In [66], the same decay property was established in the space X_β by means of a bootstrap argument similar to that for Lemma 3.15(3). The other estimates in Theorem 4.1 can be derived similarly. as in [66].

According to (4.10), e^{tB} is integrable if $\sigma_{q,0} > 1$ or $n > 4q/(2-q)$. The best choice is $q = 1$ and hence $n \geqslant 5$. For the nonlinear term Γ, we can use the estimate (4.11) due to Proposition (1.6), which has the extra decay rate $1/2$ so that $\sigma_{1,1} > 1$ for $n \geqslant 3$. For the case $n = 3, 4$, the source term S should be assumed to have a property required by one of (4.11), (4.12), or (4.14).

The above estimates are not enough, however, to control the unbounded factor ν of the nonlinear operator Γ, see Theorem 1.1. Fortunately, similar to the inverse operator B_c^{-1}, the semi-group e^{tB} also has a smoothing effect if coupled with the t-integration. Thus, consider the integral

$$\Theta[h](t) = \int_{-\infty}^{t} e^{(t-s)B}\nu h(s)\,ds. \tag{4.17}$$

Write

$$X^\beta = L^\infty(\mathbb{R}; X_\beta), \qquad Y^\beta = L^\infty\bigl(\mathbb{R}; L^\infty_\beta\bigr), \qquad Z^q = L^\infty(\mathbb{R}; Z_q), \tag{4.18}$$

and $|||\cdot|||_\beta = |||\cdot|||_{X^\beta}$.

LEMMA 4.2. *Let $n \geqslant 3$, $\beta > n/2$, and $\delta \in [0,1)$, and assume $h \in Y^\beta \cap Z^1$ and $\nu^{(1-\delta)}h \in Z^2$. Further, assume one of the following four conditions.*

(a) $n \geqslant 5$.
(b) $\mathbf{P}(\nu h) = 0$ *a.a.* $(t,x,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.
(c) $(1+|x|)h \in Z^1$, $\int_{\mathbb{R}^n} \mathbf{P}(\nu h)\,dx = 0$ *a.a.* $(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n$.

(d) $h = \partial_x \tilde{h}$, $\tilde{h} \in Z^1$.

Then, the integral (4.17) *converges in the norm of* X^β *and for a constant* $C_0 > 0$, *the following estimates hold depending on the cases in the above*:

(a) *and* (b) $|||\Theta[h]|||_\beta \leqslant C_0(|||h|||_\beta + \|h\|_{Z^1} + \|\nu^{(1-\delta)}h\|_{Z^2})$,

(c) $|||\Theta[h]|||_\beta \leqslant C_0(|||h|||_\beta + \|(1+|x|)h\|_{Z^1} + \|\nu^{(1-\delta)}h\|_{Z^2})$,

(d) $|||\Theta[h]|||_\beta \leqslant C_0(|||h|||_\beta + \|\tilde{h}\|_{Z^1} + \|\nu^{(1-\delta)}h\|_{Z^2})$.

This lemma was proved in [66], by using the decay estimates in Theorem 4.1 and a little more precise structure of e^{tB} including ν, together with a bootstrap argument. The detail is omitted.

4.3. *Existence and stability*

We now construct a fixed point of the map Φ. First, notice that the map (4.6) can be written as

$$\Phi[u] = \Theta\big[\nu^{-1}\Gamma(u,u)\big] + \Theta\big[\nu^{-1}S\big]. \tag{4.19}$$

Also, we need the estimate suggested by Theorem 1.1 and stated explicitly in Lemma 2.6 of [66] that

$$\begin{aligned} &\big\|\nu^{-1}\Gamma(u,v)\big\|_{X_\beta} + \big\|\nu^{(1-\delta)}\Gamma(u,v)\big\|_{Z_2} + \big\|\nu^{-1}\Gamma(u,v)\big\|_{Z_1} \\ &\quad \leqslant C\|u\|_{X_\beta}\|v\|_{X_\beta}, \end{aligned} \tag{4.20}$$

for any $\delta \in [0,1]$ and $\beta > n/2 + \gamma(1-\delta)$ where $\gamma \in [0,1]$ is the index of the cutoff potential in (1.25). Proposition 1.6(3) implies that $\nu h = \Gamma(u,v)$ satisfies the condition (b) in Lemma 4.2, so that we can have by the aid of (4.20),

$$\big|\big|\big|\Theta\big[\nu^{-1}\Gamma(u,v)\big]\big|\big|\big|_\beta \leqslant C_1|||u|||_\beta|||v|||_\beta$$

for each $\beta > n/2$. The constant $C_1 > 0$ is independent of u, v but depends on β in such a way that $C_1 \to \infty$ as $\beta \to n/2 + 0$. For, we must have $\delta < 1$ in Lemma 4.2.

On the other hand, we shall suppose that the inhomogeneous term satisfies one of the conditions (a)–(d) of Lemma 4.2 applied to $h = \nu^{-1}S$. That is, we assume

$$\big|\big|\big|\Theta\big[\nu^{-1}S\big]\big|\big|\big|_\beta \leqslant C_0S_0,$$

where $C_0 > 0$ is a constant independent of S and

$$S_0 = \begin{cases} |||S|||_\beta + \|S\|_{Z^1}, & \text{for the cases (a), (b),} \\ |||S|||_\beta + \|(1+|x|)S\|_{Z^1}, & \text{for the cases (c),} \\ |||S|||_\beta + \|\tilde{S}\|_{Z^1}, & \text{for the cases (d) where } S = \partial_x\tilde{S}. \end{cases} \tag{4.21}$$

Now, combining these two estimates yields

$$\big|\!\big|\!\big|\Phi[u]\big|\!\big|\!\big|_\beta \leqslant C_0 S_0 + C_1 |\!|\!| u |\!|\!|_\beta^2,$$
$$\big|\!\big|\!\big|\Phi[u] - \Phi[v]\big|\!\big|\!\big|_\beta \leqslant C_1 |\!|\!| u + v |\!|\!|_\beta |\!|\!| u - v |\!|\!|_\beta. \tag{4.22}$$

The latter comes from the bilinear symmetry of Γ.

As before, these two estimates are enough to show that the map Φ becomes contractive if S_0 is chosen small: We choose S_0 so small that

$$D \equiv 1 - 4C_0 C_1 S_0 > 0$$

and define the complete metric space

$$W_* = \big\{ u \in X^\beta \bigm| |\!|\!| u |\!|\!|_\beta \leqslant a_* \big\}, \quad a_* = \frac{1}{2C_1}(1 - \sqrt{D}),$$

a_* being the smaller positive root of the quadratic equation $C_1 a^2 - a + C_0 S_0 = 0$. Now, for any $u, v \in W_*$, it follows from (4.22) that

$$\big|\!\big|\!\big|\Phi[u]\big|\!\big|\!\big|_\beta \leqslant C_0 S_0 + C_1 a_*^2 = a_*, \quad \big|\!\big|\!\big|\Phi[u] - \Phi[v]\big|\!\big|\!\big|_\beta \leqslant \mu |\!|\!| u - v |\!|\!|_\beta,$$

where $\mu = 2C_1 a_* = 1 - \sqrt{D} < 1$, showing that Φ is a contraction on W_*. Thus, we proved the

THEOREM 4.3 (Existence of time-periodic solution). *Let $n \geqslant 3$, $\beta > n/2$ and assume the cutoff potential* (1.25) *for $\gamma \in [0, 1]$. Then, there are two positive constants a_0 and a_1 such that if S is a time-periodic function with period T_0 and satisfies one of the conditions in Lemma* 4.2 *for $h = \nu^{-1} S$ with*

$$S_0 \leqslant a_0,$$

where S_0 is defined by (4.21), *then,* (4.1) *has a time-periodic mild solution u^{per} with the same period T_0 which is unique in the function class*

$$u^{\mathrm{per}} \in L^\infty(\mathbb{R}; X_\beta), \quad \sup_{t \in \mathbb{R}} \big\| u^{\mathrm{per}}(t) \big\|_{X_\beta} \leqslant a_1 S_0. \tag{4.23}$$

Notice that $a_1 \leqslant S_0 / a_*$ and can be chosen independently of S_0.

As for the regularity of u^{per}, we quote the result of [66] here without proof.

THEOREM 4.4. *Under the same assumption as in Theorem* 4.3,

$$u^{\mathrm{per}} \in C^0\big(\mathbb{R}; L^2\big), \tag{4.24}$$

and if $\partial_x^\alpha S \in L^\infty(\mathbb{R}, X_\beta)$, $|\alpha| \leqslant \ell$ for some $\ell \geqslant 2$, then, u^{per} is a classical solution.

Finally, in order to discuss the asymptotic stability of u^{per}, we should consider the Cauchy problem (4.2) assuming that the initial data u_0 is a small perturbation of u^{per}. Thus, put

$$v = u(t) - u^{\mathrm{per}}(t), \qquad v_0 = u_0 - u^{\mathrm{per}}(t_0) \tag{4.25}$$

and reduce (4.2) to

$$\begin{cases} \frac{dv}{dt} = Bv + L^{\mathrm{per}}(t)v + \Gamma(v, v) \quad (t > t_0), \\ v(t_0) = v_0. \end{cases} \tag{4.26}$$

Here the inhomogeneous term S is suppressed at the cost of the extra linear term

$$L^{\mathrm{per}}(t)v = 2\Gamma\bigl(u^{\mathrm{per}}(t), v\bigr).$$

Without loss of generality, we may put $t_0 = 0$ and work on the Duhamel formula

$$v(t) = e^{tB}v_0 + \int_0^t e^{(t-\tau)B}\bigl\{L^{\mathrm{per}}(\tau)v(\tau) + \Gamma\bigl(v(\tau), v(\tau)\bigr)\bigr\}\,d\tau. \tag{4.27}$$

We shall define the map Φ by this right hand side and establish the existence of its fixed point. Again, this can be done by a combination of the decay estimates in Section 4.2 and the contraction mapping principle, though in a different context from the proof of Theorem 4.3.

Similar to the map Θ in (4.17), define

$$\Psi(h)(t) = \int_0^t e^{(t-\tau)B}\nu h(\tau)\,d\tau, \tag{4.28}$$

for $h = h(t, x, \xi)$. Then, (4.27) has the form

$$v(t) = e^{tB}v_0 + \Psi\bigl(\nu^{-1}L^{\mathrm{per}}v\bigr)(t) + \Psi\bigl(\nu^{-1}\Gamma(v, v)\bigr)(t). \tag{4.29}$$

The function space and norm in our setting are

$$W_{\beta,\sigma} = \bigl\{u \in L^\infty(0, \infty; X_\beta) \bigm| |||u|||_{\beta,\sigma} < \infty\bigr\},$$
$$|||u|||_{\beta,\sigma} = \sup_{t\geqslant 0}(1+t)^\sigma \|u(t)\|_{X_\beta},$$

for $\beta, \sigma \geqslant 0$.

The boundedness of the integral in Ψ as $t \to \infty$ is guaranteed by the decay property of e^{tB} in Theorem 4.1 and the unbounded factor ν can be controlled as in Lemma 4.2, by the smoothing effect of e^{tB} under the t-integration. The following lemma is a simple variant of Lemma 3.1 of [66].

LEMMA 4.5. *Let $n \geqslant 3$, $\delta \in [0,1)$, $\beta > n/2 + \gamma(1-\delta)$, and $\sigma > 0$. Then, there exists a constant $C_0 > 0$ such that for any $h \in X^\beta \cap Z^1$ satisfying $\mathbf{P}\nu h = 0$, it holds that*

$$\left\|\left|\Psi(h)\right|\right\|_{\beta,\sigma_*} \leqslant C_0 [\![h]\!]_{\beta,\sigma},$$

where $\sigma_* = \min(\sigma_{1,1}, \sigma)$ and

$$[\![h]\!]_{\beta,\sigma} = \|h\|_{\beta,\sigma} + \sup_{t \geqslant 0}(1+t)^\sigma \left\|\nu^{(1-\delta)} h\right\|_{Z^2} + \sup_{t \geqslant 0}(1+t)^\sigma \|h\|_{Z^1}.$$

Take this for granted and use (4.20) to deduce

$$\begin{aligned}
&\left\|\left|\Psi\left(\nu^{-1} L^{\mathrm{per}} v\right)\right|\right\|_{\beta,\sigma_*} \leqslant 2C_1 \left\|\left|u^{\mathrm{per}}\right|\right\|_{\beta,0} |\!|\!| v |\!|\!|_{\beta,\sigma} \leqslant C_2 S_0 |\!|\!| v |\!|\!|_{\beta,\sigma},\\
&\quad C_2 = 2a_1 C_1,\ \sigma_* = \min(\sigma_{1,1}, \sigma),\\
&\left\|\left|\Psi\left(\nu^{-1} \Gamma(v,v)\right)\right|\right\|_{\beta,\sigma_{**}} \leqslant C_1 |\!|\!| u |\!|\!|_{\beta,\sigma} |\!|\!| v |\!|\!|_{\beta,\sigma}, \quad \sigma_{**} = \min(\sigma_{1,1}, 2\sigma),
\end{aligned}$$

where we used (4.23).

Choose $\sigma = \sigma_{q,0}$ in the above. It follows from (4.9) that $\sigma_* = \sigma_{**} = \sigma_{q,0}$ for $q \in [1,2]$. As a consequence, we get

$$\begin{aligned}
&\left\|\left|\Phi[v]\right|\right\|_{\beta,\sigma} \leqslant C_0 V_0 + C_2 S_0 |\!|\!| v |\!|\!|_{\beta,\sigma} + C_1 |\!|\!| v |\!|\!|_{\beta,\sigma}^2,\\
&\left\|\left|\Phi[u] - \Phi[v]\right|\right\|_{\ell,\beta,\sigma} \leqslant C_2 S_0 |\!|\!| u - v |\!|\!|_{\beta,\sigma} + C_1 |\!|\!| u + v |\!|\!|_{\beta,\sigma} |\!|\!| u - v |\!|\!|_{\beta,\sigma},
\end{aligned}$$

where

$$V_0 = \|v_0\|_{X_\beta} + \|v_0\|_{Z_q}.$$

The remaining argument is almost the same as before: If S_0 and then V_0 are chosen sufficiently small, it turns out that Φ is a contraction in a ball of the space X_β. We summarize the result as follows.

THEOREM 4.6 (Stability). *Under the same situation as in Theorem 4.3, there are positive constants a_0, b_0, b_1 such that if $S_0 \leqslant a_0$ and if the initial data u_0 satisfies*

$$u_0 \in X_\beta \cap Z_q, \quad U_0 \equiv \left\|u_0 - u^{\mathrm{per}}(t_0)\right\|_{X_\beta} + \left\|u_0 - u^{\mathrm{per}}(t_0)\right\|_{Z_q} \leqslant b_0, \tag{4.30}$$

for some $q \in [1,2]$, the Cauchy problem (4.2) has a global solution satisfying

$$u \in L^\infty(t_0, \infty; X_\beta), \tag{4.31}$$

$$\left\|u(t) - u^{\mathrm{per}}(t)\right\|_{X_\beta} \leqslant b_1 U_0 (1 + t - t_0)^{-\sigma_{q,0}}, \quad a.a.\ t \geqslant t_0, \tag{4.32}$$

where $\sigma_{q,0}$ is as in (4.9).

REMARK 4.7. The above two theorems cover the case where S is t-independent, implying the existence and stability of the stationary solutions.

REMARK 4.8. The existence and stability theorems can be also established on the torus $\mathbb{T}^n$. In this case, we have the exponential decay in (4.11) with the same function spaces X_β and Z_q defined, of course, with $\mathbb{T}^n_x$ replaced by $\mathbb{R}^n_x$. This is also established based on the spectral analysis in Theorem 3.6, in which the Fourier variables k are to be in $\mathbb{N}^n$. Thus, for $V_j(t)$ in (4.16), $j = 0, \dots, n+1$,

$$V_j(t,0)(I-\mathbf{P}) = P_{j,0}(\tilde{k})(I-\mathbf{P}) = 0, \qquad V_j(t,k) = O(1)e^{-\sigma_0 t} \quad \big(|k| \geqslant 1\big),$$

with $\sigma_0 = \lambda_{j,2}/2 > 0$, see Theorem 3.6(2). Also, $V_{n+2}(t,k)$ enjoys the exponential decay. In conclusion, if S satisfies the condition

$$S \in X^\beta, \qquad \int_{\mathbb{T}^n} (\mathbf{P}S)(t,x,\cdot)\,dx = 0, \quad \text{a.a. } t \in \mathbb{R},$$

and if S is mall, then, the existence and stability of the unique periodic solution u^{per} can be concluded also on $\mathbb{T}^n$, see [66].

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CHAPTER 6

Existence and Stability of Spikes for the Gierer–Meinhardt System

Juncheng Wei

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail: *wei@math.cuhk.edu.hk*

Contents

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1. Introduction

It is a common belief that diffusion is a smoothing and trivializing process. Indeed, this is the case for a single diffusion equation. Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{1.1}$$

Assume that $u_0(x)$ is continuous. It is known that $u(x,t)$ is smooth for $t > 0$ (*smoothing*), and $u(x,t) \to \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx$ as $t \to +\infty$ (*trivializing*). A similar result holds when a source/sink term (or a reaction term) is present. Namely, for the problem

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{1.2}$$

it is known that when Ω is convex, the only stable solutions are constants [5,46]. Thus there are only trivial patterns (constant solutions) for single reaction–diffusion equations (on convex domains).

On the other hand, it is important to be able to use diffusion (and reaction) to model pattern formations in various branches of science (e.g., biology and chemistry). One important question is: can we get *nontrivial* patterns (stable nontrivial solutions) for systems of reaction–diffusion equations?

Let us consider the following system of reaction–diffusion equations:

$$\begin{cases} u_t = D_u \Delta u + f(u,v) & \text{in } \Omega \times (0, +\infty), \\ v_t = D_v \Delta v + g(u,v) & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{1.3}$$

In 1957, Turing [68] proposed a mathematical model for morphogenesis, which describes the development of complex organisms from a single shell. He speculated that localized peaks in concentration of a chemical substance, known as an inducer or morphogen, could be responsible for a group of cells developing differently from the surrounding cells. He then demonstrated, with linear analysis, how a nonlinear reaction diffusion system like (1.3) could possibly generate such isolated peaks. Later in 1972, Gierer and Meinhardt [21] demonstrated the existence of such solution numerically for the following (so-called Gierer–Meinhardt system)

$$\text{(GM)} \quad \begin{cases} \frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{h^q}, & x \in \Omega, t > 0, \\ \tau \frac{\partial h}{\partial t} = D\Delta h - h + \frac{a^r}{h^s}, & x \in \Omega,\ t > 0, \\ \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Here, the unknowns $a = a(x,t)$ and $h = h(x,t)$ represent the respective concentrations at point $x \in \Omega \subset \mathbb{R}^N$ and at time t of the biochemical called an activator and an inhibitor; $\epsilon > 0$, $D > 0$, $\tau > 0$ are all positive constants; $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; Ω is a smooth bounded domain in $\mathbb{R}^N$; $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. The exponents (p,q,r,s) are assumed to satisfy the condition

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geqslant 0, \quad \text{and} \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1.$$

Gierer–Meinhardt system was used in [21] to model head formation in the hydra. *Hydra*, an animal of a few millimeters in length, is made up of approximately 100,000 cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments on *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if and only if the transplanted area is sufficiently far from the (old) head. These observations have led to the assumption of the existence of two chemical substances—a *slowly* diffusing (i.e., $\epsilon \ll 1$) activator a and a *fast* diffusing (i.e., $D \gg \epsilon$) inhibitor h.

To understand the dynamics of (GM), it is helpful to consider first its corresponding "kinetic system"

$$\begin{cases} a_t = -a + a^p/h^q, \\ \tau h_t = -h + a^r/h^s. \end{cases} \tag{1.4}$$

This system has a unique constant steady state $a \equiv 1$, $h \equiv 1$. For $0 < \tau < \frac{qr}{(p-1)(s+1)}$ it is easy to see that the constant solution $a \equiv 1, h \equiv 1$ is stable as a steady state of (ODE).

However, if $\frac{\epsilon}{\sqrt{D}}$ is small, it is not hard to see that the constant steady state $a \equiv 1, h \equiv 1$ of (GM) becomes unstable and bifurcation may occur. This phenomenon is generally referred to as *Turing's diffusion-driven instability*. (A general criteria for this can be found in Murray's book [47].)

There are many other reaction–diffusion systems which exhibit Turing's diffusion-driven instability: they include Gray–Scott model from chemical reactor theory, Schnakenberg model, Sel'kov model, Lengyl–Epstein model, Thomas model, Keener–Tyson model, Brusselator, Oregonator, etc. For introduction and discussion on these general Turing models, we refer to the book [47]. A survey of mathematical modeling of biological and chemical phenomena using reaction–diffusion systems is given in [38]. Mathematical modeling of patterns in biological morphogenesis using extensions of GM model are discussed in [36] and [48].

Several common characteristics of Turing type reaction–diffusion systems include: first, they are *nonvariational*, i.e., they do not have Lyapunov or energy functional so standard variational (or energy) method cannot be applied; second, they are *noncooperative*, i.e., they do not have Maximum Principles so sub-super-solution method cannot be applied; third, they support finite-amplitude spatial-temporal patterns of remarkable diversity and complexity, such as stable spikes, layers, stripes, spot-splitting, traveling waves, etc. (See [63].) The study of these RD systems not only increases our knowledge on Turing patterns, but also induces new tools and techniques to deal with other problems which may share similar characteristics.

The most interesting phenomena associated with (GM) is the existence of stable spikes and stripes. The numerical studies of [21] and more recent those of [31] have revealed that in the limit $\epsilon \to 0$, the (GM) system seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\bar{\Omega}$. Moreover, as $\epsilon \to 0$, the pattern exhibits a "*spike layer phenomenon*" by which we mean that the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain number of points as $\epsilon \to 0$, whereas the maximum value of the activator concentration diverges to $+\infty$.

Such kind of point-condensation phenomena has generated a lot of interests both mathematically and biologically in recent years. The purpose of this chapter is to report on the current trend and status of such studies (up to June, 2006). We shall not give most of proofs. For more details, please see the references and therein.

In the study of spiky patterns (or concentration phenomena), two fundamental methods emerge. The first one is the so-called "Localized Energy Method", or *LEM* in short. LEM is a combination of traditional Lyapunov–Schmidt reduction method with variational techniques. This is a very useful tool to construct solutions with various concentration behavior, such as spikes, layers, or vortices. The second method is the so-called "Nonlocal Eigenvalue Problem Method", or *NLEP* in short. This deals with eigenvalue problems which are nonselfadjoint. It plays fundamental role in the study of stability of spike patterns. In this survey, I shall illustrate these two methods in details in the hope that they may find applications in other problems.

Throughout this chapter, unless otherwise stated, we always assume that

$$\epsilon \ll 1, \quad D \text{ is finite}, \quad \tau \geqslant 0. \tag{1.5}$$

2. Steady states in shadow system case

2.1. *Reduction to single equation*

In general, the full (GM) system is very difficult to study. A very useful idea, which goes back to Keener and Nishiura, is to consider the so-called *shadow system*. Namely, we let $D \to +\infty$ first. Suppose that the quantity $-h + a^p/h^q$ remains bounded, then we obtain

$$\Delta h \to 0, \quad \frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Thus $h(x,t) \to \xi(t)$, a constant. To derive the equation for $\xi(t)$, we integrate both sides of the equation for h over Ω and then we obtain the following so-called shadow system

$$\begin{cases} a_t = \epsilon^2 \Delta a - a + a^p/\xi^q & \text{in } \Omega, \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega a^r \, dx/\xi^s, \\ a > 0 \quad \text{in } \Omega \quad \text{and} & \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

The advantage of shadow system is that by a simple scaling,

$$a = \xi^{\frac{q}{p-1}} u, \quad \xi = \left(\frac{1}{|\Omega|} \int_\Omega u^r \right)^{\frac{p-1}{(p-1)(s+1)-qr}}, \tag{2.3}$$

the stationary shadow system can be reduced to a single equation

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

whose energy functional is given by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u_+^{p+1} \right) dx,$$
$$\text{where } u_+ = \max(u, 0), \tag{2.5}$$

for $u \in H^1(\Omega)$.

First we give some definitions on solutions to (2.4). A family of solutions $\{u_\epsilon\}$ to (2.4) are called *concentrated solutions* if there exists a subset $\Gamma \subset \bar{\Omega}$ such that $u_\epsilon \to 0$ in $C^0_{loc}(\bar{\Omega} \backslash \Gamma)$ and $\max_{x \in \Gamma} u_\epsilon(x) \geqslant c_0 > 0$. If Γ consists of only points in $\bar{\Omega}$, these kind solutions are called *point condensations*. Among point condensations, there are two kinds: *spikes* and *bubbles*. *Spikes* are those concentrated solutions such that $\max_{x \in \bar{\Omega}} u_\epsilon \leqslant C$, while *bubbles* are those with $\max_{x \in \bar{\Omega}} u_\epsilon \to +\infty$. If the dimension of Γ is positive, concentrated solutions are also called *layers*. (Similar definitions can also be given for solutions of the full Gierer–Meinhardt system by considering the activator a only.)

In the following, we discuss the existence of all kinds of *concentrated solutions* to (2.4).

2.2. *Subcritical case: spikes to (2.4)*

Let us assume first that $1 < p < (\frac{N+2}{N-2})_+$ ($= \frac{N+2}{N-2}$ if $N \geqslant 3$; $= +\infty$ when $N = 1, 2$). In this case, problem (2.4) can be studied by traditional variational methods, for example, Mountain-Pass method, or Nehari's solution manifold method. For Mountain-Pass method, by taking a function $e(x) \equiv k$ for some constant k in Ω, and choosing k large enough, we have $J_\epsilon(e) < 0$, for all $\epsilon \in (0, 1)$. Then for each $\epsilon \in (0, 1)$, we can define the so-called mountain-pass value

$$c_\epsilon = \inf_{h \in \Gamma} \max_{0 \leqslant t \leqslant 1} J_\epsilon\big[h(t)\big] \tag{2.6}$$

where $\Gamma = \{h\colon [0, 1] \to H^1(\Omega) \mid |h(t) \text{ is continuous}, \ h(0) = 0, \ h(1) = e\}$.

It is easy to see that (Lemma 2.1 of [57]), c_ϵ can be characterized by

$$c_\epsilon = \inf_{u \not\equiv 0, \ u \in H^1(\Omega)} \sup_{t > 0} J_\epsilon[tu], \tag{2.7}$$

which can be shown to be the least among all nonzero critical values of J_ϵ. (This formulation (2.7) is sometimes referred to as the Nehari manifold technique.) Moreover, c_ϵ is attained by some function u_ϵ which is then called a *least-energy solution*.

In a series of papers [57] and [58], Ni and Takagi studied the so-called *least energy* solutions and proved the following theorem

THEOREM 2.1. (See [57,58].) *For ϵ sufficiently small, there exists a mountain-pass solution u_ϵ which is also least-energy solution such that u_ϵ has only one local maximum point $P_\epsilon \in \partial\Omega$ and $u_\epsilon \to 0$ in $C^2_{\mathrm{loc}}(\bar{\Omega}\backslash\{P_\epsilon\})$. Moreover, as $\epsilon \to 0$,*

$$H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P),$$

where $H(P)$ is the mean curvature function for $P \in \partial\Omega$, and $u_\epsilon(P_\epsilon + \epsilon y) \to w(y)$ uniformly in $\Omega_{\epsilon,P_\epsilon} = \{y \mid P_\epsilon + \epsilon y \in \Omega\}$, where $w(y)$ is the unique solution of the following

$$\begin{cases} \Delta w - w + w^p = 0, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y\in\mathbb{R}^N} w(y), & w \to 0 \text{ at } \infty. \end{cases} \tag{2.8}$$

REMARK 2.2.1. The existence of ground state to (2.8) is well known. The radial symmetry of w follows from the famous Gidas–Ni–Nirenberg theorem [22]. The uniqueness of w is proved in [39].

REMARK 2.2.2. The proof of Theorem 2.1 is by expansion of energy:

$$c_\epsilon = \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + o(\epsilon)\right] \tag{2.9}$$

where

$$I[w] = \int_{\mathbb{R}^N}\left(\frac{1}{2}\big(|\nabla w|^2 + w^2\big) - \frac{1}{p+1}w^{p+1}\right)$$

is the energy of the ground state. A further expansion of c_ϵ up to the ϵ^2 order is given by [89]

$$c_\epsilon = \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + \epsilon^2\big[c_2\big(H(P_\epsilon)\big)^2 + c_3R(P_\epsilon)\big] + o\big(\epsilon^2\big)\right] \tag{2.10}$$

where c_1, c_2, c_3 are generic constants and $R(P_\epsilon)$ is the scalar curvature at P_ϵ. In particular $c_1, c_3 > 0$. (When $N = 2$, a further expansion to the order of ϵ^3 is also given in [90].) Some applications of the formula (2.10) are given in [89].

Since then there has been a lot of studies on problem (2.4). A general principle is that boundary spike solutions are related to the boundary mean-curvature $H(P)$, $P \in \partial\Omega$, while interior spike solutions are related to the distance function $d(P, \partial\Omega)$. Note also that for boundary spike the order is usually $O(\epsilon)$ while for interior spikes the order is $O(e^{-\frac{d}{\epsilon}})$ for some $d > 0$.

Let me mention some results on multiple boundary and interior peaked solutions.

For single and multiple boundary spikes, Gui [26] first constructed multiple boundary spike solutions at multiple local maximum points of $H(P)$, using variational method. Wei [72], Wei and Winter [81,82] (independently by Bates, Dancer and Shi [4]) constructed single and multiple boundary spike solutions at multiple nondegenerate critical points of $H(P)$, using Lyapunov–Schmidt reduction method. Y.Y. Li [41], del Pino, Felmer and Wei [16] constructed single and multiple boundary spikes in the degeneracy case. Using Localized Energy method (LEM), a clustered solution is also constructed by Gui, Wei and Winter [29] (independently by Dancer and Yan [9]).

THEOREM 2.2. (See [9,29].) *Let Γ be a subset of $\partial\Omega$, where it holds*

$$\min_{\partial\Gamma} H(P) > \min_{\Gamma} H(P). \tag{2.11}$$

Then for any fixed positive integer k, there exists ϵ_k such that for $\epsilon < \epsilon_k$, problem (2.4) *has a solution u_ϵ with k boundary local maximum points $P_{j,\epsilon} \in \Gamma$. Furthermore, $H(P_{j,\epsilon}) \to \min_\Gamma H(P)$.*

The energy expansion for K-boundary spikes is

$$J_\epsilon[u_\epsilon] = \epsilon^N \Bigg[\frac{K}{2} I[w] - c_1 \epsilon \sum_{j=1}^{K} H(P_{j,\epsilon}) - \sum_{i \neq j} \big(\gamma_0 + o(1)\big) w\left(\frac{|P_{i,\epsilon} - P_{j,\epsilon}|}{\epsilon}\right)\Bigg]. \tag{2.12}$$

For single and multiple interior peaked solutions, the situation is quite different, as the errors are *exponentially small*. Wei [78,73] first constructed single interior peak solution at a strictly local maximum point of $d(P, \partial\Omega)$. Gui and Wei [27] proved the following

THEOREM 2.3. (See [27].) *For any fixed positive integer k, there exists ϵ_k such that for $\epsilon < \epsilon_k$, problem* (2.4) *has a solution u_ϵ with k interior local maximum points $P_{j,\epsilon} \in \Omega$. Moreover, $(P_{1,\epsilon}, \ldots, P_{k,\epsilon})$ approaches a limiting sphere-packing position, i.e.,*

$$\varphi_k(P_{1,\epsilon}, \ldots, P_{k,\epsilon}) \to \max_{(P_1,\ldots,P_k)\in\Omega^k} \varphi_k(P_1, \ldots, P_k) \tag{2.13}$$

where

$$\varphi_k(P_1, \ldots, P_k) = \min_{i,j,l,i\neq j} \big(|P_i - P_j|, 2d(P_l, \partial\Omega)\big). \tag{2.14}$$

The energy expansion for K-interior spikes is

$$J_\epsilon[u_\epsilon] = \epsilon^N\Bigg[K I[w] - \gamma_0 \sum_{j=1}^{K} e^{-\frac{2d(P_{j,\epsilon},\partial\Omega)}{\epsilon}} - \gamma_1 \sum_{i\neq j} w\left(\frac{|P_{i,\epsilon} - P_{j,\epsilon}|}{\epsilon}\right)\Bigg]. \tag{2.15}$$

Grossi, Pistoia and Wei [30] further showed that there is an one-to-one correspondence between the (sub-differential) critical points of φ_k and k-interior peaked solutions.

Concerning the existence of mixed-boundary-interior-spikes, the following theorem gives a complete answer.

THEOREM 2.4. (See [28].) *For any two fixed positive integers k, l, there exists $\epsilon_{k,l}$ such that for $\epsilon < \epsilon_{k,l}$, problem* (2.4) *has a solution u_ϵ with k interior local maximum points and l boundary maximum points*.

Theorems 2.2, 2.3 and 2.4 imply that the number of solutions to (2.4) goes to infinity as $\epsilon \to 0$. Recently, the following lower bound on number of solutions is obtained:

THEOREM 2.5. (See [44].) *There exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and for each integer K bounded by*

$$1 \leqslant K \leqslant \frac{\alpha_{N,\Omega,f}}{\epsilon^N(|\ln\epsilon|)^N}$$

where $\alpha_{N,\Omega,p}$ is a constant depending on N, Ω and p only, there exists a solution with K interior peaks. (*An explicit formula for $\alpha_{N,\Omega,p}$ is also given*.) *As a consequence, we obtain that for ϵ sufficiently small, there exists at least $[\frac{\alpha_{N,\Omega,p}}{\epsilon^N(|\ln\epsilon|)^N}]$ number of solutions*. *Moreover, for each $\beta \in (0, N)$ there exists solution with energy in the order of $\epsilon^{N-\beta}$*.

Theorems 2.2, 2.3, 2.4 and 2.5 can all be proved by the powerful method—*Localized Energy Method*—which was first introduced in [27]. We shall discuss it next.

2.3. *Localized energy method (LEM)*

We illustrate a general method in finding solutions with concentrating behavior—the so-called *Localized Energy Method*, or *LEM* in short. The advantage of such method is that it can be applied to subcritical, critical or supercritical problems, as long as the limiting solution is well analyzed. This method was introduced in Gui and Wei [27] in dealing with spikes.

In the following, we show how to prove Theorem 2.5 by LEM. We need to introduce some notation first.

Theorem 2.5 actually holds for a slightly more general equation than (2.4), namely,

$$\begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

We will always assume that $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}$ for some $0 < \sigma \leqslant 1$ and satisfies the following conditions (f1)–(f2):

(f1) $f(u) \equiv 0$ for $u \leqslant 0$, $f(0) = f'(0) = 0$.

(f2) The following equation

$$\begin{cases} \Delta w - w + f(w) = 0, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), & w \to 0 \text{ at } \infty, \end{cases} \tag{2.17}$$

has a unique solution $w(y)$ and w is nondegenerate, i.e.,

$$\mathrm{Kernel}\big(\Delta - 1 + f'(w)\big) = \mathrm{span}\left\{ \frac{\partial w}{\partial y_1}, \ldots, \frac{\partial w}{\partial y_N} \right\}. \tag{2.18}$$

One typical example of f is: $f(u) = u^p - au^q$, where $a \geqslant 0$, $1 < q < p < (\frac{N+2}{N-2})_+$. For the uniqueness of w, see [39] and [40]. The proof of nondegeneracy is given in [58].

Without loss of generality, we may assume that $0 \in \Omega$. By the following rescaling:

$$x = \epsilon z, \quad z \in \Omega_\epsilon := \{z \mid |\epsilon z \in \Omega\}, \tag{2.19}$$

equation (2.16) becomes

$$\begin{cases} \Delta u - u + f(u) = 0 \quad \text{in } \Omega_\epsilon, \\ u > 0 \quad \text{in } \Omega_\epsilon, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega_\epsilon. \end{cases} \tag{2.20}$$

For $u \in H^2(\Omega_\epsilon)$, we put

$$S_\epsilon[u] = \Delta u - u + f(u). \tag{2.21}$$

Then (2.20) is equivalent to

$$S_\epsilon[u] = 0, \quad u \in H^2(\Omega_\epsilon), \quad u > 0 \quad \text{in } \Omega_\epsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.22}$$

Associated with problem (2.20) is the following energy functional

$$\tilde{J}_\epsilon[u] = \frac{1}{2} \int_{\Omega_\epsilon} \big(|\nabla u|^2 + u^2\big) - \int_{\Omega_\epsilon} F(u), \quad u \in H^1(\Omega_\epsilon). \tag{2.23}$$

We define two inner products:

$$\langle u, v\rangle_\epsilon = \int_{\Omega_\epsilon} uv, \quad \text{for } u, v \in L^2(\Omega_\epsilon); \tag{2.24}$$

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv), \quad \text{for } u, v \in H^1(\Omega_\epsilon). \tag{2.25}$$

Let σ be the Hölder exponent of f' and

$$M > \frac{6+2\sigma}{\sigma} K \tag{2.26}$$

be a fixed positive constant. Now we define a configuration space:

$$\Lambda := \left\{(Q_1, \ldots, Q_K) \in \Omega^K \;\middle|\; \varphi_K(Q_1, \ldots, Q_K) \geqslant M\epsilon|\ln\epsilon|\right\} \tag{2.27}$$

where φ_K is defined at (2.14).

Let w be the unique solution of (2.17). By the well-known result of Gidas, Ni and Nirenberg [22], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w'(r) < 0$ for $r > 0, r = |y|$. Moreover, we have the following asymptotic behavior of w:

$$\begin{aligned} w(r) &= A_N r^{-\frac{N-1}{2}} e^{-r}\left(1 + O\left(\frac{1}{r}\right)\right), \\ w'(r) &= -A_N r^{-\frac{N-1}{2}} e^{-r}\left(1 + O\left(\frac{1}{r}\right)\right), \end{aligned} \tag{2.28}$$

for r large, where $A_N > 0$ is a constant. Let $K(r)$ be the fundamental solution of $-\Delta + 1$ centered at 0. Then we have

$$\begin{aligned} w(r) &= \left(A_0 + O\left(\frac{1}{r}\right)\right) K(r), \\ w'(r) &= \left(-A_0 + O\left(\frac{1}{r}\right)\right) K(r), \quad \text{for } r \geqslant 1, \end{aligned} \tag{2.29}$$

where A_0 is a positive constant.

The idea of *LEM* is to look for solutions of (2.16) of the following type:

$$u = \sum_{j=1}^{K} w\left(z - \frac{Q_j}{\epsilon}\right) + \phi \tag{2.30}$$

where ϕ is solved first by Lyapunov–Schmidt reduction process, and $(Q_1, \ldots, Q_K)$ are adjusted so as to achieve a solution. *LEM* is a method of reducing the infinite-dimensional problem of finding a critical point of $\tilde{J}_\epsilon$ to a finite-dimensional problem of $(Q_1, \ldots, Q_K)$. In general, it consists of the following five steps:

STEP 1. Find out good approximate functions.

This step contains most of the important computations. The idea is to choose good approximate functions such that the error S_ϵ is small.

For $Q \in \Omega$, we define $w_{\epsilon,Q}$ to be the unique solution of

$$\Delta v - v + f\left(w\left(\cdot - \frac{Q}{\epsilon}\right)\right) = 0 \quad \text{in } \Omega_\epsilon, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.31}$$

Let $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$. We then define the approximate solution as

$$w_{\epsilon,\mathbf{Q}} = \sum_{j=1}^{K} w_{\epsilon,Q_j}. \tag{2.32}$$

We first analyze $w_{\epsilon,Q}$. To this end, set

$$\varphi_{\epsilon,Q}(x) = w\left(\frac{|x-Q|}{\varepsilon}\right) - w_{\epsilon,Q}\left(\frac{x}{\epsilon}\right).$$

We state the following useful lemmas on the properties of $\varphi_{\epsilon,Q}$, whose proof can be found in [44].

LEMMA 2.6. *Assume that* $\frac{M}{2}\epsilon|\ln\epsilon| \leqslant d(Q,\partial\Omega) \leqslant \delta$ *where* δ *is sufficiently small. We have*

$$\varphi_{\epsilon,Q} = -\big(A_0 + o(1)\big)K\left(\frac{|x-Q^*|}{\epsilon}\right) + O\big(\epsilon^{\sqrt{2}M+N+1}\big) \tag{2.33}$$

where $K(r)$ *is the* (*radially symmetric*) *fundamental solution of* $-\Delta + 1$ *in* $\mathbb{R}^N$, $Q^* = Q + 2d(Q,\partial\Omega)\nu_{\bar{Q}}$, $\nu_{\bar{Q}}$ *denotes the unit outer normal at* $\bar{Q} \in \partial\Omega$ *and* $\bar{Q}$ *is the unique point on* $\partial\Omega$ *such that* $d(\bar{Q}, Q) = d(Q, \partial\Omega)$.

The next lemma analyze $w_{\epsilon,\mathbf{Q}}$ in Ω_ϵ. To this end, we divide Ω_ϵ into $K+1$-parts:

$$\Omega_{\epsilon,j} = \left\{\left|z - \frac{Q_j}{\epsilon}\right| \leqslant \frac{1}{2\epsilon}\varphi_K(\mathbf{Q})\right\}, \quad j = 1, \ldots, K,$$

$$\Omega_{\epsilon,K+1} = \Omega_\epsilon \Big\backslash \bigcup_{j=1}^{K} \Omega_{\epsilon,j}. \tag{2.34}$$

LEMMA 2.7. *For* $z \in \Omega_{\epsilon,j}$, $j = 1, \ldots, K$, *we have*

$$w_{\epsilon,\mathbf{Q}} = w_{\epsilon,Q_j} + O\big(K\epsilon^{\frac{M}{2}}\big) = w\left(z - \frac{Q_j}{\epsilon}\right) + O\big(K\epsilon^{\frac{M}{2}}\big). \tag{2.35}$$

For $z \in \Omega_{\epsilon,K+1}$, *we have*

$$w_{\epsilon,\mathbf{Q}} = O\big(K\epsilon^{\frac{M}{2}}\big). \tag{2.36}$$

PROOF. For $k \neq j$ and $z \in \Omega_{\epsilon,j}$, we have

$$\begin{aligned} w_{\epsilon,Q_k} &= w\left(z - \frac{Q_k}{\epsilon}\right) - \varphi_{\epsilon,Q_k}(\epsilon z) \\ &= O\big(e^{-|z-\frac{Q_k}{\epsilon}|} + e^{-|z-\frac{Q_k^*}{\epsilon}|} + \epsilon^{M+N+1}\big) = O\big(\epsilon^{\frac{M}{2}}\big) \end{aligned}$$

and so

$$\sum_{k\neq j} w_{\epsilon,Q_k} = O\big(K\epsilon^{\frac{M}{2}}\big)$$

which proves (2.35). The proof of (2.36) is similar. □

Next we state a useful lemma about the interactions of two w's.

LEMMA 2.8. *For* $\frac{|Q_1-Q_2|}{\epsilon}$ *large, it holds*

$$\int_{\mathbb{R}^N} f\left(w\left(z - \frac{Q_1}{\epsilon}\right)\right) w\left(z - \frac{Q_2}{\epsilon}\right) = \big(\gamma_0 + o(1)\big) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) \tag{2.37}$$

where

$$\gamma_0 = \int_{\mathbb{R}^N} f\big(w(y)\big) e^{-y_1}\, dy. \tag{2.38}$$

REMARK. Note that $\gamma_0 > 0$. See Lemma 4.7 of [61].

PROOF. By (2.28), we have for $|\epsilon y| \ll |Q_1 - Q_2|$,

$$\begin{aligned} w\left(y + \frac{Q_1 - Q_2}{\epsilon}\right) &= \big(A_N + o(1)\big)\left(\frac{\epsilon}{|\epsilon y + Q_1 - Q_2|}\right)^{\frac{N-1}{2}} e^{-|y+\frac{Q_1-Q_2}{\epsilon}|} \\ &= w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) e^{-\langle y, \frac{Q_1-Q_2}{|Q_1-Q_2|}\rangle + o(|y|)}. \end{aligned}$$

Thus by Lebesgue's Dominated Convergence Theorem

$$\int_{\mathbb{R}^N} f\left(w\left(z - \frac{Q_1}{\epsilon}\right)\right) w\left(z - \frac{Q_2}{\epsilon}\right)$$

$$= \int_{\mathbb{R}^N} f\big(w(y)\big) w\left(y + \frac{Q_1 - Q_2}{\epsilon}\right)$$
$$= \big(1 + o(1)\big) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right) \int_{\mathbb{R}^N} f\big(w(y)\big) e^{-\langle y, \frac{Q_1 - Q_2}{|Q_1 - Q_2|}\rangle} \, dy$$
$$= \big(\gamma_0 + o(1)\big) w\left(\frac{|Q_1 - Q_2|}{\epsilon}\right). \qquad \square$$

Let us define several quantities for later use:

$$B_\epsilon(Q_j) = -\int_{\Omega_\epsilon} f(w_j)\varphi_{\epsilon, Q_j}, \; B_\epsilon(Q_i, Q_j) = \int_{\Omega_\epsilon} f(w_i) w_j. \tag{2.39}$$

Then we have

LEMMA 2.9. *For* $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$, *it holds*

$$B_\epsilon(Q_j) = \big(\gamma_0 + o(1)\big) w\left(\frac{2d(Q_j, \partial\Omega)}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big), \tag{2.40}$$

$$B_\epsilon(Q_i, Q_j) = \big(\gamma_0 + o(1)\big) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big). \tag{2.41}$$

PROOF. Note that

$$A_0 K\left(\frac{|x - Q^*|}{\epsilon}\right) = \big(1 + o(1)\big) w\left(\frac{|x - Q^*|}{\epsilon}\right)$$

and by Lemma 2.6

$$B_\epsilon(Q_j) = \big(1 + o(1)\big) \int_{\Omega_\epsilon} f(w_j) w\left(z - \frac{Q_j^* - Q_j}{\epsilon}\right) + O\big(\epsilon^{\sqrt{2}M + N + 1}\big)$$
$$= \big(\gamma + o(1)\big) w\left(\frac{|Q_j - Q_j^*|}{\epsilon}\right) + o\big(w(M|\ln\epsilon|)\big)$$
$$= \big(\gamma + o(1)\big) w\left(\frac{2d(Q_j, \partial\Omega)}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big).$$

(2.40) follows from Lemma 2.6. To prove (2.41), we note that

$$B_\epsilon(Q_i, Q_j) = \int_{\mathbb{R}^N} f(w) w\left(y - \frac{Q_i - Q_j}{\epsilon}\right)$$
$$- \int_{\mathbb{R}^N \setminus \Omega_{\epsilon, Q_i}} f(w) w\left(y - \frac{Q_i - Q_j}{\epsilon}\right)$$

$$= \big(\gamma + o(1)\big) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + O\big(e^{-(1+\frac{\sigma}{2})\frac{d(Q_i,\partial\Omega)}{\epsilon}} e^{-\frac{d(Q_j,\partial\Omega)}{\epsilon}}\big)$$

$$= \big(\gamma + o(1)\big) w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big). \qquad \square$$

We then have the following which provides the key estimates on the energy expansion and error estimates.

LEMMA 2.10. *For any* $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Lambda$ *and* ϵ *sufficiently small we have*

$$\tilde{J}_\epsilon\left[\sum_{i=1}^{K} w_{\epsilon,Q_j}\right] = K I[w] - \frac{1}{2}\sum_{i=1}^{K} B_\epsilon(Q_i) - \frac{1}{2}\sum_{i,j=1,\ldots,K,\, i\neq j} B_\epsilon(Q_i, Q_j) + o\big(w\big(M|\ln\epsilon|\big)\big), \tag{2.42}$$

and

$$\left\| S_\epsilon\left[\sum_{j=1}^{K} w_{\epsilon,Q_j}\right]\right\|_{L^q(\Omega_\epsilon)} \leqslant C K^{\frac{q+1}{q}+\sigma} \epsilon^{\frac{M(1+\sigma)}{2}} \tag{2.43}$$

for any $q > \frac{N}{2}$.

The proof of Lemma 2.10 is technical and tedious. We refer to [44] for the computations.

STEP 2. Obtain a priori estimates for a linear problem.

This is the fundamental step in reducing an infinite-dimensional problem to finite-dimensional one. The key result we need here is the nondegeneracy assumption (f2).

Fix $\mathbf{Q} \in \Lambda$. We define the following functions

$$Z_{i,j} = (\Delta - 1)\left[\frac{\partial w_i}{\partial z_j}\chi_i(z)\right], \quad \text{where } \chi_i(z) = \chi\left(\frac{2|\epsilon z - Q_i|}{(M-1)\epsilon|\ln\epsilon|}\right),$$
$$i = 1, \ldots, K,\ j = 1, \ldots, N, \tag{2.44}$$

where $\chi(t)$ is a smooth cut-off function such that $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > \frac{M^2}{M^2-1}$. Note that the support of $Z_{i,j}$ belongs to $B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

In this step, we consider the following linear problem: Given $h \in L^2(\Omega_\epsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\epsilon[\phi] := \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = h + \sum_{k,l} c_{k,l} Z_{k,l}; \\ \langle \phi, Z_{i,j} \rangle_\epsilon = 0, \quad i = 1, \ldots, K, \ j = 1, \ldots, N, \quad \text{and} \\ \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon, \end{cases} \tag{2.45}$$

for some constants $c_{k,l}$, $k = 1, \ldots, K$, $l = 1, \ldots, N$.

To this purpose, we define two norms

$$\|\phi\|_* = \|\phi\|_{W^{2,q}(\Omega_\epsilon)}, \qquad \|f\|_{**} = \|f\|_{L^q(\Omega_\epsilon)}, \tag{2.46}$$

where $q > \frac{N}{2}$ is a fixed number.

We have the following result:

PROPOSITION 2.11. *Let ϕ satisfy* (2.45). *Then for ϵ sufficiently small and $\mathbf{Q} \in \Lambda$, we have*

$$\|\phi\|_* \leqslant C\|h\|_{**} \tag{2.47}$$

where C is a positive constant independent of ϵ, K and $\mathbf{Q} \in \Lambda$.

PROOF. Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_{**} = o(1). \tag{2.48}$$

We multiply (2.45) by $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ and integrate over Ω_ϵ to obtain

$$\sum_{k,l} c_{k,l}\left\langle Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = -\left\langle h, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon + \left\langle \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon. \tag{2.49}$$

From the exponential decay of w one finds

$$\left\langle h, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = o(1).$$

Observe that $\frac{\partial w_i}{\partial z_j}\chi_i(z)$ satisfies

$$\Delta\left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) - \left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) + f'(w_i)\left(\frac{\partial w_i}{\partial z_j}\chi_i(z)\right) = 2\nabla_z \frac{\partial w_i}{\partial z_j}\nabla_z \chi_i + (\Delta\chi_i)\frac{\partial w_i}{\partial z_j}. \tag{2.50}$$

Integrating by parts and using Lemma 2.7, we deduce

$$\begin{aligned}
&\left\langle \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon \\
&\quad = \left\langle \left(f'(w_{\epsilon,\mathbf{Q}}) - f'(w_i)\right)\frac{\partial w_i}{\partial z_j}\chi_i(z), \phi\right\rangle_\epsilon + O\left(\epsilon^{\frac{M-1}{2}}\|\phi\|_*\right) \\
&\quad = O\left(K^\sigma \epsilon^{\frac{M\sigma}{2}}\|\phi\|_*\right) = o\left(\|\phi\|_*\right) = o(1)
\end{aligned}$$

where we have used the fact that $M > \frac{6+2\sigma}{\sigma}N$ and that

$$\left\| \left(f'(w_{\epsilon,\mathbf{Q}}) - f'(w_i)\right)\frac{\partial w_i}{\partial z_j}\chi_i \right\|_{**} \leqslant C \left\| |w_{\epsilon,\mathbf{Q}} - w_i|^\sigma \left| \frac{\partial w_i}{\partial z_j}\chi_i \right| \right\|_* \leqslant K^\sigma \epsilon^{\frac{M\sigma}{2}}.$$

It is easy to see that

$$\left\langle Z_{i,j}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = -\int_{\mathbb{R}^N} f'(w)\left(\frac{\partial w}{\partial y_j}\right)^2 dy + o(1). \tag{2.51}$$

On the other hand, for $k \neq i$ we have

$$\left\langle Z_{k,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = 0 \tag{2.52}$$

and for $k = i$ and $l \neq j$, we have

$$\left\langle Z_{i,l}, \frac{\partial w_i}{\partial z_j}\chi_i(z)\right\rangle_\epsilon = O\left(\epsilon^M\right). \tag{2.53}$$

The left hand side of (2.49) becomes

$$c_{i,j} + \sum_{l\neq j} O\left(\epsilon^M c_{i,l}\right) = o(1)$$

and hence

$$c_{i,j} = o(1), \quad i = 1, \ldots, K, \ j = 1, \ldots, N. \tag{2.54}$$

To obtain a contradiction, we define the following cut-off functions:

$$\phi_i = \phi\chi_i', \quad \text{where } \chi_i' = \chi\left(\frac{2|\epsilon z - Q_i|}{(M - M^{-1})\epsilon|\ln\epsilon|}\right), \ i = 1, \ldots, K. \tag{2.55}$$

Note that $\chi_i' = 1$ for $z \in B_{\frac{M^2-1}{2M}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$ and the support of ϕ belongs to $B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q_i}{\epsilon})$.

Then the conditions $\langle \phi, Z_{i,j}\rangle_\epsilon = 0$ is equivalent to

$$\langle \phi_i, Z_{i,j}\rangle_\epsilon = 0. \tag{2.56}$$

The equation for ϕ_i becomes

$$\Delta\phi_i - \phi_i + f'(w_{\epsilon,\mathbf{Q}})\phi_i = \sum_j c_{i,j} Z_{i,j} + h\chi_i' + 2\nabla\phi\nabla\chi_i' + \big(\Delta\chi_i'\big)\phi. \tag{2.57}$$

Lemma 2.7 yields

$$f'(w_{\epsilon,\mathbf{Q}})\phi_i = \big(f(w_i) + o\big(\epsilon^{M/2-N}\big)\big)\phi_i. \tag{2.58}$$

Using (2.56) and (2.58), a contradiction argument similar to that of Proposition 3.2 of [27] gives

$$\|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} \leqslant C\big\|h\chi_i'\big\|^q_{L^q(\Omega_\epsilon)} + C\big\|2\nabla\phi\nabla\chi_i' + \big(\Delta\chi_i'\big)\phi\big\|^q_{L^q(\Omega_\epsilon)}. \tag{2.59}$$

Next, we decompose

$$\phi = \sum_{i=1}^K \phi_i + \Phi \tag{2.60}$$

where $\Phi = \phi(1 - \sum_{i=1}^K \chi_i')$. Then the equation for Φ becomes

$$\begin{aligned}&\Delta\Phi - \Phi + f'(w_{\epsilon,\mathbf{Q}})\Phi\\ &\quad = h\left(1 - \sum_{i=1}^K \chi_i'\right) - 2\sum_{i=1}^K \nabla\phi\nabla\chi_i' - \sum_{i=1}^K \big(\Delta\chi_i'\big)\phi.\end{aligned} \tag{2.61}$$

By Lemma 2.7, $f'(w_{\epsilon,\mathbf{Q}})\Phi = o(1)\Phi$. Standard regularity theorem gives

$$\begin{aligned}\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)} &\leqslant C\left\|h\left(1 - \sum_{i=1}^K \chi_i'\right)\right\|^q_{L^q(\Omega_\epsilon)}\\ &\quad + C\left\|2\sum_{i=1}^K \nabla\phi\nabla\chi_i' + \sum_{i=1}^K (\Delta\chi_i')\phi\right\|^q_{L^q(\Omega_\epsilon)}.\end{aligned} \tag{2.62}$$

(Observe that the constant C in the L^p-regularity is independent of $\epsilon < 1$. The case of Dirichlet boundary condition has been proved in Lemma 6.4 of [61]. The case of Neumann boundary condition can be proved similarly.)

Combining (2.60), (2.59) and (2.62), we obtain

$$\begin{aligned}\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)} &\leqslant C\left\|\sum_{i=1}^K \phi_i\right\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)}\\ &\leqslant C\sum_{i=1}^K \|\phi_i\|^q_{W^{2,q}(\Omega_\epsilon)} + C\|\Phi\|^q_{W^{2,q}(\Omega_\epsilon)}\\ &\leqslant C\left(\sum_{i=1}^K \|h\chi_i'\|^q_{L^q(\Omega_\epsilon)} + \left\|h\left(1-\sum_{i=1}^K \chi_i'\right)\right\|^q_{L^q(\Omega_\epsilon)}\right)\\ &\quad + C\sum_{i=1}^K \|2\nabla\phi\nabla\chi_i' + (\Delta\chi_i')\phi\|^q_{L^q(\Omega_\epsilon)}\\ &\leqslant C\|h\|^q_{L^q(\Omega_\epsilon)} + O\left(|\ln\epsilon|^{-1}\right)\|\phi\|^q_{W^{2,q}(\Omega_\epsilon)}\end{aligned}$$

since

$$\sum_{i=1}^K (\chi_i')^q + \left(1-\sum_{i=1}^K \chi_i'\right)^q \leqslant 2, \qquad |\nabla\chi'| + |\Delta\chi'| \leqslant C\left(|\ln\epsilon|\right)^{-1}. \tag{2.63}$$

This gives

$$\|\phi\|_{W^{2,q}(\Omega_\epsilon)} = o(1). \tag{2.64}$$

A contradiction to (2.48). □

From Proposition 2.11, we derive the following existence result:

PROPOSITION 2.12. *There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the following property holds true. Given $h \in W^{2,q}(\Omega_\epsilon)$, there exist s a unique pair $(\phi, \mathbf{c}) = (\phi, \{c_{i,j}\}_{i=1,\ldots,K,j=1,\ldots,N})$ such that*

$$L_\epsilon[\phi] = h + \sum_{i,j} c_{i,j} Z_{i,j}, \tag{2.65}$$

$$\langle \phi, Z_{i,j}\rangle_\epsilon = 0, \quad i = 1,\ldots,K,\ j = 1,\ldots,N, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.66}$$

Moreover, we have

$$\|\phi\|_* \leqslant C\|h\|_{**} \tag{2.67}$$

for some positive constant C.

PROOF. The bound in (2.67) follows from Proposition 2.11 and (2.54). Let us now prove the existence part. Set

$$\mathcal{H} = \left\{ u \in H^1(\Omega_\epsilon) \;\middle|\; \left(u, (\Delta - 1)^{-1} Z_{i,j}\right)_\epsilon = 0 \right\}$$

where we define the inner product on $H^1(\Omega_\epsilon)$ as

$$(u, v)_\epsilon = \int_{\Omega_\epsilon} (\nabla u \nabla v + uv).$$

Note that, integrating by parts, one has

$$\psi \in \mathcal{H} \quad \text{if and only if} \quad \langle \psi, Z_{i,j} \rangle_\epsilon = 0, \quad i = 1, \ldots, K,\ j = 1, \ldots, N.$$

Observe that ϕ solves (2.65) and (2.66) if and only if $\phi \in \mathcal{H}$ satisfies

$$\int_{\Omega_\epsilon} (\nabla \phi \nabla \psi + \phi \psi) - \left\langle f'(w_{\epsilon,\mathbf{Q}})\phi, \psi \right\rangle_\epsilon = \langle h, \psi \rangle_\epsilon, \quad \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as

$$\phi + \mathcal{S}(\phi) = \bar{h} \quad \text{in } \mathcal{H}, \tag{2.68}$$

where $\bar{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator.

Using Fredholm's alternative, showing that equation (2.68) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 2.11 and our proof is complete. □

In the following, if ϕ is the unique solution given in Proposition 2.12, we set

$$\phi = \mathcal{A}_\epsilon(h). \tag{2.69}$$

Note that (2.67) implies

$$\left\| \mathcal{A}_\epsilon(h) \right\|_* \leqslant C \|h\|_{**}. \tag{2.70}$$

STEP 3. A nonlinear Lyapunov–Schmidt reduction.

For ϵ small and for $\mathbf{Q} \in \Lambda$, we are going to find a function $\phi_{\epsilon,\mathbf{Q}}$ such that for some constants $c_{i,j}$, $j = 1, \ldots, N$, the following equation holds true

$$\begin{cases} \Delta(w_{\epsilon,\mathbf{Q}} + \phi) - (w_{\epsilon,\mathbf{Q}} + \phi) + f(w_{\epsilon,\mathbf{Q}} + \phi) = \sum_{k,l} c_{k,l} Z_{k,l} & \text{in } \Omega_\epsilon, \\ \langle \phi, Z_{i,j} \rangle_\epsilon = 0, \quad j = 1, \ldots, N, \quad \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2.71}$$

The first equation in (2.71) can be written as

$$\Delta\phi - \phi + f'(w_{\epsilon,\mathbf{Q}})\phi = \big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi] + \sum_{i,j} c_{i,j} Z_{i,j},$$

where

$$N_\epsilon[\phi] = -\big[f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi\big]. \tag{2.72}$$

LEMMA 2.13. *For $\mathbf{Q} \in \Lambda$ and ϵ sufficiently small, we have for $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \leqslant 1$,*

$$\big\|N_\epsilon[\phi]\big\|_{**} \leqslant C\|\phi\|_*^{1+\sigma}; \tag{2.73}$$

$$\big\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\big\|_{**} \leqslant C\big(\|\phi_1\|_*^\sigma + \|\phi_2\|_*^\sigma\big)\|\phi_1 - \phi_2\|_*. \tag{2.74}$$

PROOF. Inequality (2.73) follows from the mean-value theorem. In fact, for all $z \in \Omega_\epsilon$ there holds

$$f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) = f'(w_{\epsilon,\mathbf{Q}} + \theta\phi)\phi.$$

Since f' is Hölder continuous with exponent σ, we deduce

$$\big|f(w_{\epsilon,\mathbf{Q}} + \phi) - f(w_{\epsilon,\mathbf{Q}}) - f'(w_{\epsilon,\mathbf{Q}})\phi\big| \leqslant C|\phi|^{1+\sigma},$$

which implies (2.73). The proof of (2.74) goes along the same way. □

PROPOSITION 2.14. *For $\mathbf{Q} \in \Lambda$ and ϵ sufficiently small, there exists a unique $\phi = \phi_{\epsilon,\mathbf{Q}}$ such that* (2.71) *holds. Moreover, $\mathbf{Q} \mapsto \phi_{\epsilon,\mathbf{Q}}$ is of class C^1 as a map into $W^{2,q}(\Omega_\epsilon) \cap \mathcal{H}$, and we have*

$$\|\phi_{\epsilon,\mathbf{Q}}\|_* \leqslant rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}} \tag{2.75}$$

for some constant $r > 0$.

PROOF. Let $\mathcal{A}_\epsilon$ be as defined in (2.69). Then (2.71) can be written as

$$\phi = \mathcal{A}_\epsilon\big[\big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi]\big]. \tag{2.76}$$

Let r be a positive (large) number, and set

$$\mathcal{F}_r = \Big\{\phi \in \mathcal{H} \cap W^{2,q}(\Omega_\epsilon)\colon\ \|\phi\|_* < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}}\Big\}.$$

Define now the map $\mathcal{G}_\epsilon : \mathcal{F}_r \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon)$ as

$$\mathcal{G}_\epsilon[\phi] = \mathcal{A}_\epsilon\big[\big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big) + N_\epsilon[\phi]\big].$$

Solving (2.71) is equivalent to finding a fixed point for $\mathcal{G}_\epsilon$. By Lemmas 2.10 and 2.13, for ϵ sufficiently small and r large we have

$$\left\|\mathcal{G}_\epsilon[\phi]\right\|_* \leqslant C\left\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\right\|_{**} + C\left\|N_\epsilon[\phi]\right\|_{**} < rK^{\frac{q+1}{q}+\sigma}\epsilon^{\frac{M(1+\sigma)}{2}},$$

$$\left\|\mathcal{G}_\epsilon[\phi_1] - \mathcal{B}_\epsilon[\phi_2]\right\|_* \leqslant C\left\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\right\|_* < \frac{1}{2}\|\phi_1 - \phi_2\|_*,$$

which shows that $\mathcal{G}_\epsilon$ is a contraction mapping on $\mathcal{F}_r$. Hence there exists a unique $\phi = \phi_{\epsilon,\mathbf{Q}} \in \mathcal{F}_r$ such that (2.71) holds.

Now we come to the differentiability of $\phi_{\epsilon,\mathbf{Q}}$. Consider the following map $H_\epsilon : \Lambda \times \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$ of class C^1

$$H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = \begin{pmatrix} (\Delta - 1)^{-1}(S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi]) - \sum_{i,j} c_{i,j}(\Delta - 1)^{-1}Z_{i,j} \\ (\phi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\phi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix}. \tag{2.77}$$

Equation (2.71) is equivalent to $H_\epsilon(\mathbf{Q}, \phi, \mathbf{c}) = 0$. We know that, given $\mathbf{Q} \in \Lambda$, there is a unique local solution $\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}}$ obtained with the above procedure. We prove that the linear operator

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q},\phi_{\epsilon,\mathbf{Q}},\mathbf{c}_{\epsilon,\mathbf{Q}})} : \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK} \to \mathcal{H} \cap W^{2,q}(\Omega_\epsilon) \times R^{NK}$$

is invertible for ϵ small. Then the C^1-regularity of $\mathbf{Q} \mapsto (\phi_{\epsilon,\mathbf{Q}}, c_{\epsilon,\mathbf{Q}})$ follows from the Implicit Function Theorem. Indeed we have

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q},\phi_{\epsilon,\mathbf{Q}},\mathbf{c}_{\epsilon,\mathbf{Q}})}[\psi, \mathbf{d}]$$
$$= \begin{pmatrix} (\Delta - 1)^{-1}(S'[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}](\psi)) - \sum_{i,j} d_{ij}(\Delta - 1)^{-1}Z_{i,j} \\ (\psi, (\Delta - 1)^{-1}Z_{1,1})_\epsilon \\ \vdots \\ (\psi, (\Delta - 1)^{-1}Z_{K,N})_\epsilon \end{pmatrix}.$$

Since $\|\phi_{\epsilon,\mathbf{Q}}\|_*$ is small, the same proof as in that of Proposition 2.11 shows that

$$\left.\frac{\partial H_\epsilon(\mathbf{Q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{Q},\phi_{\epsilon,\mathbf{Q}},\mathbf{c}_{\epsilon,\mathbf{Q}})}$$

is invertible for ϵ small.

This concludes the proof of Proposition 2.14. □

In some cases (e.g., critical or nearly critical exponent problems), we need to obtain further differentiability of $\phi_{\epsilon,\mathbf{Q}}$ (e.g., C^2 in $\mathbf{Q}$). This will be achieved by further reduction. See [13,65] and [66] for such arguments.

STEP 4. A reduction lemma.

Fix $\mathbf{Q} \in \Lambda$. Let $\phi_{\epsilon,\mathbf{Q}}$ be the solution given by Proposition 2.14. We define a new functional

$$\mathcal{M}_\epsilon(\mathbf{Q}) = \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] : \Lambda \to R. \tag{2.78}$$

Then we have the following reduction lemma

LEMMA 2.15. *If $\mathbf{Q}_\epsilon$ is critical point of $\mathcal{M}_\epsilon(\mathbf{Q})$ in Λ, then $u_\epsilon = w_{\epsilon,\mathbf{Q}_\epsilon} + \phi_{\epsilon,\mathbf{Q}_\epsilon}$ is a critical point of $\tilde{J}_\epsilon[u]$.*

PROOF. By Proposition 2.14, there exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$ we have a C^1 map which, to any $\mathbf{Q} \in \Lambda$, associates $\phi_{\epsilon,\mathbf{Q}}$ such that

$$\begin{aligned} &S_\epsilon[w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}}] = \sum_{k=1,\dots,K;\, l=1,\dots,N} c_{kl} Z_{k,l}, \\ &\langle \phi_{\epsilon,\mathbf{Q}}, Z_{i,j} \rangle_\epsilon = 0 \end{aligned} \tag{2.79}$$

for some constants $c_{kl} \in R^{KN}$.

Let $\mathbf{Q}^\epsilon \in \Lambda$ be a critical point of $\mathcal{M}_\epsilon$. Set $u_\epsilon = w_{\epsilon,\mathbf{Q}^\epsilon} + \phi_{\epsilon,\mathbf{Q}^\epsilon}$. Then we have

$$D_{Q_{i,j}}|_{Q_i = Q_i^\epsilon} \mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) = 0, \quad i = 1, \dots, K,\ j = 1, \dots, N.$$

Hence we have

$$\begin{aligned} \int_{\Omega_\epsilon} \Bigg[&\nabla u_\epsilon \nabla \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} + u_\epsilon \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} \\ &- f(u_\epsilon) \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} \Bigg] = 0, \end{aligned}$$

which gives

$$\sum_{k=1,\dots,K;\ l=1,\dots,N} c_{kl} \int_{\Omega_\epsilon} Z_{k,l} \frac{\partial (w_{\epsilon,\mathbf{Q}} + \phi_{\epsilon,\mathbf{Q}})}{\partial Q_{i,j}} \bigg|_{Q_i = Q_i^\epsilon} = 0. \tag{2.80}$$

We claim that (2.80) is a diagonally dominant system. In fact, since $\langle \phi_{\epsilon,\mathbf{Q}}, Z_{i,j}\rangle_\epsilon = 0$, we have that

$$\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial \phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q^\epsilon_{i,j}} = -\int_{\Omega_\epsilon} \phi_{\epsilon,\mathbf{Q}^\epsilon} \frac{\partial Z_{k,l}}{\partial Q^\epsilon_{i,j}} = 0 \quad \text{if } k \neq i.$$

If $k = i$, we have

$$\begin{aligned}\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial \phi_{\epsilon,\mathbf{Q}^\epsilon}}{\partial Q^\epsilon_{k,j}} &= -\int_{\Omega_\epsilon} \frac{\partial Z_{k,l}}{\partial Q^\epsilon_{k,j}} \phi_{\epsilon,\mathbf{Q}^\epsilon} = \left\| \frac{\partial Z_{k,l}}{\partial Q^\epsilon_{k,j}} \right\|_{**} \|\phi_{\epsilon,\mathbf{Q}^\epsilon}\|_{**} \\ &= O\left(K^{\frac{q+1}{q}+\sigma} \epsilon^{\frac{M(1+\sigma)}{2}-1}\right) = O\left(\epsilon^{\frac{M(1+\sigma)}{2}-(\frac{q+1}{q}+\sigma)N-1}\right) \\ &= O\left(\epsilon^{\frac{M}{2}}\right).\end{aligned}$$

For $k \neq i$, we have

$$\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial w_{\epsilon,Q^\epsilon_i}}{\partial Q^\epsilon_{i,j}} = \int_{\Omega_\epsilon \cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q^\epsilon_k}{\epsilon})} Z_{k,l} \frac{\partial w_{\epsilon,Q^\epsilon_i}}{\partial Q^\epsilon_{i,j}} = O\left(\epsilon^M\right).$$

For $k = i$, we have

$$\begin{aligned}\int_{\Omega_\epsilon} Z_{k,l} \frac{\partial w_{\epsilon,Q^\epsilon_k}}{\partial Q^\epsilon_{k,j}} &= \int_{\Omega_\epsilon \cap B_{\frac{M}{2}|\ln\epsilon|}(\frac{Q^\epsilon_k}{\epsilon})} Z_{k,l} \frac{\partial w_{\epsilon,Q^\epsilon_k}}{\partial Q^\epsilon_{k,j}} \\ &= -\epsilon^{-1}\delta_{lj} \int_{\mathbb{R}^N} f'(w)\left(\frac{\partial w}{\partial y_j}\right)^2 + O(1).\end{aligned}$$

For each (k,l), the off-diagonal term gives

$$O\left(\epsilon^{\frac{M}{2}}\right) + \sum_{k\neq i} \epsilon^M + \sum_{k=i, l\neq j} O(\epsilon) = O\left(\epsilon^{\frac{M}{2}} + K\epsilon^M + \epsilon\right) = o(1)$$

by our choice of $M > \frac{6+2\sigma}{\sigma} N$.

Thus equation (2.80) becomes a system of homogeneous equations for c_{kl} and the matrix of the system is nonsingular. So $c_{kl} \equiv 0$, $k = 1, \ldots, K$, $l = 1, \ldots, N$.

Hence $u_\epsilon = \sum_{i=1}^K w_{\epsilon,Q^\epsilon_i} + \phi_{\epsilon,Q^\epsilon_1,\ldots,Q^\epsilon_K}$ is a solution of (2.20). □

STEP 5. Using variational arguments to find critical points for the finite-dimensional reduced problem.

By Lemma 2.15, we just need to find a critical point for the reduced energy functional $\mathcal{M}_\epsilon(\mathbf{Q})$. Depending on the asymptotic behavior of the reduced energy functional, one can use either local minimization, or local maximization [29], or saddle point techniques [66]. Here there is no compactness problem since the reduced problem is already finite-dimensional.

We first obtain an asymptotic formula for $\mathcal{M}_\epsilon(\mathbf{Q})$. In fact for any $\mathbf{Q} \in \Lambda$, we have

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}) &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + \int_{\Omega_\epsilon} (\nabla w_{\epsilon,\mathbf{Q}} \nabla \phi_{\epsilon,\mathbf{Q}} + w_{\epsilon,\mathbf{Q}} \phi_{\epsilon,\mathbf{Q}}) \\ &\quad - \int_{\Omega_\epsilon} f(w_{\epsilon,\mathbf{Q}})\phi_{\epsilon,\mathbf{Q}} + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + \int_{\Omega_\epsilon} \big(-S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big)\phi_{\epsilon,\mathbf{Q}} + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + O\big(\big\|S_\epsilon[w_{\epsilon,\mathbf{Q}}]\big\|_{**}\|\phi_{\epsilon,\mathbf{Q}}\|_*\big) + O\big(\|\phi_{\epsilon,\mathbf{Q}}\|_*^2\big) \\ &= \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + O\big(K^{2+\frac{2}{q}+2\sigma}\epsilon^{M(1+\sigma)}\big) = \tilde{J}_\epsilon[w_{\epsilon,\mathbf{Q}}] + o\big(w\big(M|\ln\epsilon|\big)\big)\end{aligned}$$

by Lemma 2.10, Proposition 2.14 and the choice of M at (2.26).

By Lemma 2.10, we obtain

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}) &= K I[w] - \frac{1}{2}\big(\gamma_0 + o(1)\big)\sum_{i=1}^{K} w\left(\frac{2d(Q_i, \partial\Omega)}{\epsilon}\right) \\ &\quad - \frac{1}{2}\big(\gamma_0 + o(1)\big)\sum_{i\neq j} w\left(\frac{|Q_i - Q_j|}{\epsilon}\right) + o\big(w\big(M|\ln\epsilon|\big)\big).\end{aligned} \tag{2.81}$$

We shall prove

PROPOSITION 2.16. *For ϵ small, the following maximization problem*

$$\max\{\mathcal{M}_\epsilon(\mathbf{Q}) \colon \mathbf{Q} \in \Lambda\} \tag{2.82}$$

has a solution $\mathbf{Q}^\epsilon \in \Lambda^\circ$—the interior of Λ.

PROOF. First, we obtain a lower bound for $\mathcal{M}_\epsilon$: Recall that $K_\Omega(r)$ is the maximum number of nonoverlapping balls with equal radius r packed in Ω. Now we choose K such that

$$1 \leqslant K \leqslant K_\Omega\left(\frac{M+2N}{2}\epsilon|\ln\epsilon|\right). \tag{2.83}$$

Let $\mathbf{Q}^0 = (Q_1^0, \dots, Q_K^0)$ be the centers of arbitrary K balls among those $K_\Omega(\frac{M+2N}{2} \times \epsilon|\ln\epsilon|)$ balls. Certainly $\mathbf{Q}^0 \in \Lambda$. Then we have

$$w\left(\frac{2d(Q_i^0, \partial\Omega)}{\epsilon}\right) \leqslant e^{-\frac{2d(Q_i^0,\partial\Omega)}{\epsilon}} \leqslant \epsilon^{M+2N}, \qquad w\left(\frac{|Q_i^0 - Q_j^0|}{\epsilon}\right) \leqslant \epsilon^{M+2N}$$

and hence

$$\begin{aligned}\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) \geqslant \mathcal{M}_\epsilon(\mathbf{Q}^0) \geqslant KI[w] - \frac{K}{2}(\gamma_0 + o(1))\epsilon^{M+2N} \\ - \frac{K^2}{2}(\gamma_0 + o(1))\epsilon^{M+2N} + o(w(M|\ln\epsilon|)) \\ \geqslant KI[w] - K^2(\gamma_0 + o(1))\epsilon^{M+2N} + o(w(M|\ln\epsilon|)).\end{aligned} \tag{2.84}$$

On the other hand, if $\mathbf{Q}^\epsilon \in \partial\Lambda$, then either there exists (i, j) such that $|Q_i^\epsilon - Q_j^\epsilon| = M\epsilon|\ln\epsilon|$, or there exists a k such that $d(Q_k^\epsilon, \partial\Omega) = \frac{M}{2}\epsilon|\ln\epsilon|$. In both cases we have

$$\mathcal{M}_\epsilon(\mathbf{Q}^\epsilon) \leqslant KI[w] - \frac{1}{2}(\gamma_0 + o(1))w(M|\ln\epsilon|) + o(w(M|\ln\epsilon|)). \tag{2.85}$$

Combining (2.85) and (2.84), we obtain

$$w(M|\ln\epsilon|) \leqslant 2K^2\epsilon^{M+2N} \leqslant C\epsilon^M(|\ln\epsilon|)^{-2N} \tag{2.86}$$

which is impossible.

We conclude that $\mathbf{Q}^\epsilon \in \Lambda$. This completes the proof of Proposition 2.16. □

COMPLETION OF PROOF OF THEOREM 2.5. Theorem 2.5 follows from Proposition 2.16 and the reduction Lemma 2.15. □

2.4. *Bubbles to (2.4): the critical case*

Let $p = \frac{N+2}{N-2}$. By suitable scaling, (2.4) becomes the following problem

$$\begin{cases} \Delta u - \mu u + u^{\frac{N+2}{N-2}} = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.87}$$

where $\mu = \frac{1}{\epsilon^2}$ is large.

It is well known that the solutions to

$$\Delta U + U^{\frac{N+2}{N-2}} = 0 \tag{2.88}$$

are given by the following

$$U_{\Lambda,\xi} = c_N\left(\frac{1}{\Lambda^2 + |x-\xi|^2}\right)^{\frac{N-2}{2}}, \quad \text{where } \Lambda > 0,\ \xi \in \mathbb{R}^N. \tag{2.89}$$

A notable difference here is that the linearized operator $\Delta + (\frac{N+2}{N-2})U_{\Lambda,\xi}^{\frac{4}{N-2}}$ has $(N+1)$-dimensional kernels. Namely,

$$\text{Kernel}\left(\Delta + \frac{N+2}{N-2}U_{\Lambda,\xi}^{\frac{4}{N-2}}\right) = \text{span}\left\{\frac{\partial U_{\lambda,\xi}}{\partial \Lambda}, \frac{\partial U_{\Lambda,\xi}}{\partial \xi_1}, \ldots, \frac{\partial U_{\lambda,\xi}}{\partial \xi_N}\right\}. \tag{2.90}$$

Thus when we apply LEM, we need also to take care of the scaling parameters. See [13,43,65,66] and the references therein.

Concerning boundary bubbles, the existence of mountain-pass solutions was first proved in Wang [69] and Adimurthi and Mancini [1]. Ni, Takagi and Pan [55] showed the least energy solutions develop a bubble at the maximum point of the mean curvature (thereby establishing results similar to Theorem 2.1). Local mountain-pass solutions concentrating on one or separated boundary points are established in [23]. At nondegenerate critical points of the positive mean curvature, single boundary bubbles exist [2]. Lin, Wang and Wei [43] established results similar to Theorem 2.2 for dimension $N \geqslant 7$, at a nondegenerate local minimum point of the mean curvature with positive value:

THEOREM 2.17. *Suppose the following two assumptions hold*:

(H1) $N \geqslant 7$,

(H2) $Q_0 = 0$ *is a nondegenerate minimum point of* $H(Q)$ *and* $H(Q_0) > 0$.

Let $K \geqslant 2$ *be a fixed integer. Then there exists a* $\mu_K > 0$ *such that for* $\mu > \mu_K$, *problem* (2.87) *has a nontrivial solution* u_μ *with the following properties*

(1)

$$u(x) = \sum_{j=1}^{K} U_{\frac{1}{\mu}\Lambda_j, Q_0 + \mu^{\frac{3-N}{N}}\hat{Q}_j^\mu} + O\left(\mu^{\frac{N-4}{2}}\right),$$

where $\Lambda_j \to \Lambda_0 := A_0 H(Q_0) > 0$, $j = 1, \ldots, K$, *and*

(2) $\hat{\mathbf{Q}}^\mu := (\hat{Q}_1^\mu, \ldots, \hat{Q}_K^\mu)$ *approach an optimal configuration in the following problem*:

$(*)$ *Find out the optimal configuration* $(\hat{Q}_1, \ldots, \hat{Q}_K)$ *that minimizes the functional* $R[\hat{Q}_1, \ldots, \hat{Q}_K]$.

Here for $\hat{\mathbf{Q}} = (\hat{Q}_1, \ldots, \hat{Q}_K) \in R^{(N-1)K}$, $\hat{Q}_i \neq \hat{Q}_j$, *we define*

$$R[\hat{Q}_1, \ldots, \hat{Q}_K] := c_1 \sum_{j=1}^{K} \varphi(\hat{Q}_j) + c_2 \sum_{i \neq j} \frac{1}{|\hat{Q}_i - \hat{Q}_j|^{N-2}} \tag{2.91}$$

where $\varphi(Q) = \sum_{k,l} \partial_k \partial_l H(Q_0) Q_k Q_l$, c_1 *and* c_2 *are two generic constants.*

Theorem 2.17 is proved by *LEM*. Here the computation is more complicated, since the interaction between bubbles is very involved.

Concerning interior bubbles, under some assumptions, it is proved in [24] and [64] that there are *no* interior bubble solutions. However interior bubble solutions can be recovered if one add the boundary layers. (The boundary layer solution has been constructed in [50] (see Section 2.6).) The following result establishes the existence of multiple interior bubbles in dimension $N = 3, 4, 5$.

THEOREM 2.18. (See [71,91].) *Let $N = 3, 4, 5$. For any fixed integer k, then problem* (2.87) *has a solution* (*at least along a subsequence $\epsilon_k \to 0$*) *with k interior bubbles and one boundary layer.*

2.5. *Bubbles to (2.4): slightly supercritical case*

In the slightly supercritical case, we let $p = \frac{N+2}{N-2} + \delta$ where $\delta > 0$. Consider

$$\begin{cases} \Delta u - \mu u + u^p = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} & \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.92}$$

The following result was proved by [66] and [14] through the use of LEM.

THEOREM 2.19. *Let $N \geqslant 3$. Then $\delta > 0$ sufficiently small, problem* (2.92) *admits a boundary bubble solution.*

In fact, in the slightly supercritical case, there is also the phenomena of *bubble-towers*. A bubble-tower is a sum of bubbles centered at the same point

$$\sum_{j=1}^{K} U_{\Lambda_j,\xi}, \quad \text{where } \Lambda_1, \ \frac{\Lambda_{j+1}}{\Lambda_j} \to +\infty, \ j = 1, \ldots, K-1. \tag{2.93}$$

This has been discussed in [15] and [25].

It is completely open whether or not point condensation solutions exist for (2.92) when $p > \frac{N+2}{N-2} + \delta$. In fact, let Ω be the unit ball. Using Pohozaev's identity, it is not difficult to show that *there exists a positive constant c_0, independent of $\epsilon \leqslant 1$, such that*

$$\inf_{\Omega} u \geqslant c_0 \tag{2.94}$$

for all radial solution u of (2.4). This marks a *basic* difference between the behavior of solutions of these two cases $p \leqslant \frac{N+2}{N-2}$ and $p > \frac{N+2}{N-2}$. *It eliminates the possibility of the existence of a radial spiky solution which approaches zero in measure as ϵ approaches zero* in the supercritical case $p > \frac{N+2}{N-2}$.

2.6. *Concentration on higher-dimensional sets*

The following conjecture has been made by Ni [53,54].

CONJECTURE. *Given any integer* $0 \leqslant k \leqslant n-1$, *there exists* $p_k \in (1, \infty)$ *such that for* $1 < p < p_k$, (2.4) *possesses a solution with* k*-dimensional concentration set, provided that* ϵ *is sufficiently small.*

Progress in this direction has only been made very recently. In [49] and [50], Malchiodi and Montenegro proved that for $N \geqslant 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (2.4) has a solution u_{ε_k} which concentrates at boundary of $\partial\Omega$ (or any component of $\partial\Omega$). Such a solution has the following energy bound

$$J_{\varepsilon_k}[u_{\varepsilon_k}] \sim \varepsilon_k^{N-1}. \tag{2.95}$$

In [48], Malchiodi showed the concentration phenomena for (2.4) along a closed nondegenerate geodesic of $\partial\Omega$ in three-dimensional smooth bounded domain Ω. F. Mahmoudi and A. Malchiodi in [51] prove a full general concentration of solutions along k-dimensional $(1 \leqslant k \leqslant n-1)$ nondegenerate minimal sub-manifolds of the boundary for $n \geqslant 3$ and $1 < p < \frac{n-k+2}{n-k-2}$. When $\Omega = B_1(0)$, there are also multiple (radially symmetric) clustered interfaces near the boundary [52].

For concentrations on lines intersecting with the boundary, Wei and Yang [92] made the first attempt in the two-dimensional case. Let $\Gamma \subset \Omega \subset \mathbb{R}^2$ be a curve satisfying the following assumptions: The curvature of Γ is zero and Γ intersects $\partial\Omega$ at exactly two points, saying, γ_1, γ_0 and at these points $\Gamma \perp \partial\Omega$. Let $-k_1$ and k_0 are the curvatures of the boundary $\partial\Omega$ at the points γ_1 and γ_0 respectively. A picture of Γ and Ω is as in Figure 1.

We define a geometric eigenvalue problem

$$\begin{aligned}
&-f''(\theta) = \lambda f(\theta), \quad 0 < \theta < 1,\\
&f'(1) + k_1 f(1) = 0,\\
&f'(0) + k_0 f(0) = 0.
\end{aligned} \tag{2.96}$$

We say that Γ is nondegenerate if (2.96) does not have a zero eigenvalue. This is equivalent to the following condition:

$$k_0 - k_1 + k_0 k_1 |\Gamma| \neq 0, \tag{2.97}$$

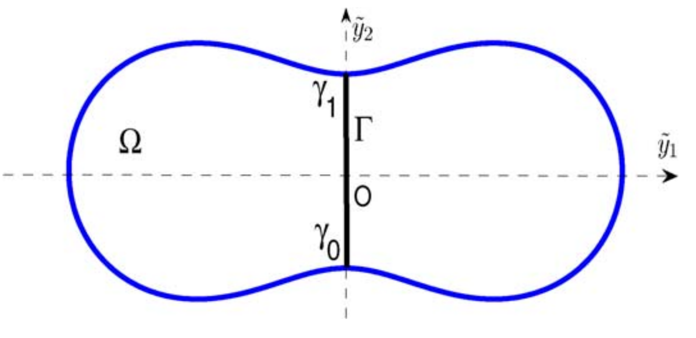

Fig. 1. Lines intersecting with $\partial\Omega$ orthogonally.

where $|\Gamma|$ denotes the length of Γ.

Moreover, we set up the gap condition that there exists a small constant $c > 0$

$$\left|\lambda_0 - \frac{k^2\pi^2}{|\Gamma|^2}\varepsilon^2\right| \geqslant c\varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.98}$$

In [92], the following result was proved

THEOREM 2.20. *We assume that the line segment Γ satisfies the nondegenerate condition* (2.97). *Given a small constant c, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition* (2.98), *problem* (2.4) *has a positive solution u_ε concentrating along a curve Γ_ϵ near Γ. Moreover, there exists some number c_0 such that u_ε satisfies globally,*

$$u_\varepsilon(x) \leqslant \exp\left[-c_0\varepsilon^{-1}\operatorname{dist}(x, \Gamma_\epsilon)\right]$$

and the curve Γ_ϵ will collapse to Γ as $\varepsilon \to 0$.

REMARK 2.6.1. The geometric eigenvalue problem (2.96) was first introduced by M. Kowalczyk in [37] where he constructed layered solution concentrating on a line for the Allen–Cahn equation.

REMARK 2.6.2. Theorem 2.20 is proved using the *infinite-dimensional Lyapunov–Schmidt reduction* technique introduced in [18].

REMARK 2.6.3. One can also construct multiple clustered line concentrating solutions, using the Toda system. See [93]. This follows from earlier work in [19], where multiple clustered interfaces are constructed at nonminimizing lines for the Allen–Cahn equation. It is quite interesting to see the connection between Toda system

$$q_j'' + e^{q_j - q_{j+1}} - e^{q_{j-1} - q_j} = 0 \tag{2.99}$$

and clustered interfaces.

REMARK 2.6.4. It will be interesting to construct solutions concentrating on surfaces which intersect with $\partial\Omega$ orthogonally.

2.7. *Robin boundary condition*

Robin boundary conditions are particularly interesting in biological models where they often arise. We refer the reader to [10] for this aspect.

In [3], Berestycki and Wei discussed the existence and asymptotic behavior of least energy solution for following singularly perturbed problem with Robin boundary condition:

$$\begin{cases} \epsilon^2\Delta u - u + u^p = 0, u > 0 & \text{in } \Omega, \\ \epsilon\frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.100}$$

where $\lambda > 0$. Similar to [57], we can define the following energy functional associated with (2.100):

$$J_\epsilon[u] := \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega F(u) + \frac{\epsilon\lambda}{2}\int_{\partial\Omega} u^2, \tag{2.101}$$

where $F(u) = \int_0^u f(s)\,ds$, $f(s) = s^p$, $u \in H^1(\Omega)$.

Similarly, for $\epsilon \in (0,1)$, we can define the so-called mountain-pass value

$$c_{\epsilon,\lambda} = \inf_{h\in\Gamma}\max_{0\leqslant t\leqslant 1} J_\epsilon\big[h(t)\big] \tag{2.102}$$

where $\Gamma = \{h\colon [0,1] \to H^1(\Omega) \mid h(t) \text{ is continuous}, h(0) = 0, h(1) = e\}$.

For fixed ϵ small, as λ moves from 0 (which is Neumann BC) to $+\infty$ (which is Dirichlet BC), by the results of [57,58] and [61], the asymptotic behavior of $u_{\epsilon,\lambda}$ changes dramatically: a boundary spike is displaced to become an interior spike. The question we shall answer is: where is the borderline of λ for spikes to move inwards?

Note that when $N = 1$, by ODE analysis, it is easy to see that the borderline is exactly at $\lambda = 1$. In fact, we may assume that $\Omega = (0,1)$, and a s $\epsilon \to 0$, the least energy solution converges to a homoclinic solution of the following ODE:

$$w'' - w + w^p = 0 \quad \text{in } \mathbb{R}^1, \qquad w(y) \to 0 \quad \text{as } |y| \to +\infty. \tag{2.103}$$

Then it follows that

$$(w')^2 = w^2 - \frac{2}{p+1} w^{p+1}, \quad |w'| < w. \tag{2.104}$$

As $\epsilon \to 0$, the limiting boundary condition (2.100) becomes $w'(0) - \lambda w(0) = 0$. We see from (2.104) that this is possible if and only if $\lambda < 1$.

When $N = 2$, the situation changes dramatically. To understand the location of the spikes at the boundary, an essential role is played by the analogous problem in a half space with Robin boundary condition on the boundary. Thus we first consider

$$\begin{cases} \Delta u - u + f(u) = 0, u > 0 & \text{in } \mathbb{R}^N_+, \\ u \in H^1(\mathbb{R}^N_+), \quad \frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{2.105}$$

where $\mathbb{R}^N_+ = \{(y', y_N) \mid y_N > 0\}$ and ν is the outer normal on $\partial\mathbb{R}^N_+$.

Let

$$I_\lambda[u] = \int_{\mathbb{R}^+_N}\left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2\right) - \int_{\mathbb{R}^+_N} F(u) + \frac{\lambda}{2}\int_{\partial\mathbb{R}^+_N} u^2. \tag{2.106}$$

As before, we define a mountain-pass vale for I_λ:

$$c_\lambda = \inf_{v\not\equiv 0,\, v\in H^1(\mathbb{R}^+_N)}\ \sup_{t>0} I_\lambda[tv]. \tag{2.107}$$

Our first result deals with the half space problem:

THEOREM 2.21.

(1) *For* $\lambda \leqslant 1$, c_λ *is achieved by some function* w_λ, *which is a solution of* (2.105).
(2) *For* λ *large enough*, c_λ *is never achieved*.
(3) *Set*

$$\lambda_* = \inf\{\lambda \mid c_\lambda \text{ is achieved}\}. \tag{2.108}$$

Then $\lambda_* > 1$ *and for* $\lambda \leqslant \lambda_*$, c_λ *is achieved, and for* $\lambda > \lambda_*$, c_λ *is not achieved*.

The proof of Theorem 2.21 is by the method of *concentration-compactness*, and the method of *vanishing viscosity*.

Now consider the problem in a bounded domain.

THEOREM 2.22. *Let* $\lambda \leqslant \lambda_*$ *and* $u_{\epsilon,\lambda}$ *be a least energy solution of* (2.100). *Let* $x_\epsilon \in \Omega$ *be a point where* $u_{\epsilon,\lambda}$ *reaches its maximum value. Then after passing to a subsequence,* $x_\epsilon \to x_0 \in \partial\Omega$ *and*

(1) $d(x_\epsilon, \partial\Omega)/\epsilon \to d_0$, *for some* $d_0 > 0$,
(2) $v_{\epsilon,\lambda}(y) = u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w_\lambda(y)$ *in* C^1 *locally, where* w_λ *attains* c_λ *of* (2.107) (*and thus is a solution of* (2.105)),
(3) *the associated critical value can be estimated as follows*:

$$c_{\epsilon,\lambda} = \epsilon^N \big\{ c_\lambda - \epsilon \bar{H}(x_0) + o(\epsilon) \big\} \tag{2.109}$$

where c_λ *is given by* (2.107), *and* $\bar{H}(x_0)$ *is given by the following*

$$\bar{H}(x_0) = \max_{w_\lambda \in \mathcal{S}_\lambda} \left[- \int_{\mathbb{R}_N^+} y' \cdot \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} H(x_0) \right] \tag{2.110}$$

where $\mathcal{S}_\lambda$ *is the set of all solutions of* (2.105) *attaining* c_λ, *and* $y' = (y_1, \ldots, y_{N-1})$, $\nabla' = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}})$,
(4) $\bar{H}(x_0) = \max_{x \in \partial\Omega} \bar{H}(x)$.

On the other hand, when $\lambda > \lambda_*$, a different asymptotic behavior appears.

THEOREM 2.23. *Let* $\lambda > \lambda_*$ *and* $u_{\epsilon,\lambda}$ *be a least energy solution of* (2.100). *Let* $x_\epsilon \in \Omega$ *be a point where* $u_{\epsilon,\lambda}$ *reaches its maximum value. Then after passing a subsequence, we have*

(1) $d(x_\epsilon, \partial\Omega) \to \max_{x \in \Omega} d(x, \partial\Omega)$,
(2) $v_{\epsilon,\lambda}(y) := u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w(y)$ *in* C^1 *locally, where* w *is the unique solution of* (2.8),
(3) *the associated critical value can be estimated as follows*:

$$c_{\epsilon,\lambda} = \epsilon^N \left[I[w] + \exp\left(-\frac{2d(x_\epsilon, \partial\Omega)}{\epsilon} \big(1 + o(1)\big) \right) \right]. \tag{2.111}$$

3. Stability and instability in the shadow system case

As we have already seen in Section 2 that there are *many* single and multiple spike solutions for the shadow system (2.2). The question is: are they all stable with respect to the shadow system (2.2)? Unfortunately, as we will show below, only one of them is stable.

Let u_ϵ be a (boundary or interior) spike solution. Then it is easy to see that $(a_\epsilon, \xi_\epsilon)$ defined by the following

$$a_\epsilon = \xi_\epsilon^{q/(p-1)} u_\epsilon, \qquad \xi_\epsilon = \left(\frac{1}{|\Omega|} \int_\Omega u_\epsilon^r \, dx \right)^{-(p-1)/(qr-(p-1)(s+1))} \tag{3.1}$$

is a solution pair of the stationary problem to the shadow system (2.2).

In this section, we analyze the following linearized eigenvalue problem

$$\begin{cases} \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + p \frac{a_\epsilon^{p-1}}{\xi_\epsilon^q} \phi_\epsilon - q \frac{a_\epsilon^p}{\xi_\epsilon^{q+1}} \eta = \alpha_\epsilon \phi_\epsilon, \quad \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \frac{r}{\tau|\Omega|} \int_\Omega \frac{a_\epsilon^{r-1} \phi_\epsilon}{\xi_\epsilon^s} \, dx - \frac{1+s}{\tau} \eta = \alpha_\epsilon \eta. \end{cases} \tag{3.2}$$

By using (3.1), it is easy to see that the eigenvalues of problem (3.2) in $H^2(\Omega) \times L^\infty(\Omega)$ are the same as the eigenvalues of the following eigenvalue problem

$$\epsilon^2 \Delta \phi - \phi + p u_\epsilon^{p-1} \phi - \frac{qr}{s+1+\tau\alpha_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1} \phi}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \alpha_\epsilon \phi,$$
$$\phi \in H^2(\Omega). \tag{3.3}$$

A simple argument [8] shows that

THEOREM 3.1. *Any multiple-spike solution is linearly unstable for the shadow system* (2.2).

Let

$$L_\epsilon(\phi) = \epsilon^2 \Delta \phi - \phi + p u_\epsilon^{p-1} \phi,$$
$$\mathcal{L}_\epsilon(\phi) = L_\epsilon(\phi) - \frac{qr}{s+1+\tau\lambda} \frac{\int_\Omega u_\epsilon^{r-1} \phi}{\int_\Omega u_\epsilon^r} u_\epsilon^p. \tag{3.4}$$

Thus we can only concentrate on the study of stability for single-spike solutions. The study of stability and instability of single spike solutions can be divided into two parts: *small eigenvalues* and *large eigenvalues*.

3.1. *Small eigenvalues for L_ϵ*

In [72], it was proved that single boundary spike must concentrate at a critical point of the mean curvature function $H(P)$. On the other hand, at a nondegenerate critical point of $H(P)$, there is also a single boundary spike. Furthermore, in [75], it is proved that the single boundary spike at a nondegenerate critical point of $H(P)$ is actually nondegenerate.

Next we study the eigenvalue estimates associated with the linearized operator at u_ϵ: $L_\epsilon = \epsilon^2\Delta - 1 + pu_\epsilon^{p-1}$. (Here the domain of L_ϵ is $H^2(\Omega)$.) We first note the following result.

LEMMA 3.2. *The following eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi = \mu\phi \quad \text{in } \mathbb{R}^N,\ \phi \in H^1(\mathbb{R}^N) \tag{3.5}$$

admits the following set of eigenvalues:

$$\mu_1 > 0, \quad \mu_2 = \cdots = \mu_{N+1} = 0, \quad \mu_{N+2} < 0, \ldots. \tag{3.6}$$

Moreover, the eigenfunction corresponding to μ_1 is radial and of constant sign.

PROOF. This follows from Theorem 2.12 of [42] and Lemma 4.2 of [58]. □

The small eigenvalues for L_ϵ were characterized completely in [75].

THEOREM 3.3. (See [75].) *For ϵ sufficiently small, the following eigenvalue problem*

$$\begin{cases} \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + pu_\epsilon^{p-1}\phi_\epsilon = \tau_\epsilon\phi_\epsilon & \text{in } \Omega, \\ \frac{\partial\phi_\epsilon}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

admits exactly $(N-1)$ eigenvalues $\tau_\epsilon^1 \leqslant \tau_\epsilon^2 \leqslant \cdots \leqslant \tau_\epsilon^{N-1}$ in the interval $[\frac{\mu_{N+1}}{2}, \frac{\mu_1}{2}]$, where μ_1 and μ_{N+1} are given by Lemma 3.2.

Moreover, we have the following asymptotic behavior of τ_ϵ^j:

$$\frac{\tau_\epsilon^j}{\epsilon^2} \to \eta_0\lambda_j, \quad j = 1, \ldots, N-1, \tag{3.8}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ are the eigenvalues of the matrix $G_b(P_0) := (\partial_i\partial_j H(P_0))$, and

$$\eta_0 = \frac{N-1}{N+1}\frac{\int_{R_+^N}(w'(|z|))^2 z_N\,dz}{\int_{R_+^N}(\frac{\partial w}{\partial z_1})^2\,dz} > 0. \tag{3.9}$$

(*Here $w'(|z|)$ denotes the radial derivative of w with respect to $|z|$.*)

Furthermore the eigenfunction corresponding to τ_ϵ^j, $j = 1, \ldots, N-1$, is given by the following:

$$\phi_j^\epsilon = \sum_{i=1}^{N-1}\big(a_{ij} + o(1)\big)\frac{\partial w_{\epsilon,P_\epsilon}}{\partial\tau_i(P_\epsilon)} \tag{3.10}$$

where P_ϵ *is the local maximum point of* u_ϵ, $\vec{a}_j = (a_{1j}, \dots, a_{(N-1)j})^T$ *is the eigenvector corresponding to* λ_j, *namely*

$$G_b(P_0)\vec{a}_j = \lambda_j \vec{a}_j, \quad j = 1, \dots, N-1. \tag{3.11}$$

For single interior spikes, we obtain similar results. But it becomes more involved since now the error is exponentially small.

The existence of interior spike solutions depends highly on the geometry of the domain. In [73] and [74], the author first constructed a single interior spike solution. To state the result, we need to introduce some notations. Let

$$d\mu_{P_0}(z) = \lim_{\varepsilon \to 0} \frac{e^{-\frac{2|z-P_0|}{\varepsilon}}\, dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_0|}{\varepsilon}}\, dz}. \tag{3.12}$$

It is easy to see that the support of $d\mu_{P_0}(z)$ is contained in $\bar{B}_{d(P_0,\partial\Omega)}(P_0) \cap \partial\Omega$.

A point P_0 is called "nondegenerate peak point" if the followings hold: there exists $a \in \mathbb{R}^N$ such that

$$\int_{\partial\Omega} e^{\langle z-P_0, a\rangle}(z-P_0)\, d\mu_{P_0}(z) = 0 \tag{H1}$$

and

$$\left(\int_{\partial\Omega} e^{\langle z-P_0, a\rangle}(z-P_0)_i (z-P_0)_j\, d\mu_{P_0}(z)\right) := G_i(P_0) \quad \text{is nonsingular.} \tag{H2}$$

Such a vector a is unique. Moreover, $G_i(P_0)$ is a positive definite matrix. A geometric characterization of a nondegenerate peak point P_0 is the following:

$$P_0 \in \text{ interior (convex hull of support } \big(d\mu_{P_0}(z)\big).$$

For a proof of the above facts, see Theorem 5.1 of [73].

In [74] and [73], the author proved the following theorem.

THEOREM 3.4. *Suppose that* P_0 *is a nondegenerate peak point. Then for* $\epsilon \ll 1$, *there exists a single interior spike solution* u_ϵ *concentrating at* P_0. *Furthermore,* u_ϵ *is locally unique. Namely, if there are two families of single interior spike solutions* $u_{\epsilon,1}$ *and* $u_{\epsilon,2}$ *of* (2.4) *such that* $P_\epsilon^1 \to P_0$, $P_\epsilon^2 \to P_0$ *where*

$$u_{\epsilon,1}\big(P_\epsilon^1\big) = \max_{P \in \bar{\Omega}} u_\epsilon(P), \qquad u_{\epsilon,2}\big(P_\epsilon^2\big) = \max_{P \in \bar{\Omega}} u_{\epsilon,2}(P),$$

then $P_\epsilon^1 = P_\epsilon^2$, $u_{\epsilon,1} = u_{\epsilon,2}$. *Moreover,*

$$P_\epsilon^1 = P_\epsilon^2 = P_0 + \epsilon\left(\frac{1}{2} d(P_0, \partial\Omega) a + o(1)\right) \quad \text{as } \epsilon \to 0.$$

Let $w_{\epsilon,P}$ and $\varphi_{\epsilon,P}$ be defined as in Section 2.3. (It was proved in [74] and [73] that $-\epsilon\log[-\varphi_{\epsilon,P}(P)] \to 2d(P,\partial\Omega)$ as $\epsilon \to 0$.)

Similarly, we obtain the following eigenvalue estimates for u_ϵ

THEOREM 3.5. *The following eigenvalue problem*

$$\epsilon^2\Delta\phi - \phi + pu_\epsilon^{p-1}\phi = \tau^\epsilon\phi \quad \text{in } \Omega, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{3.13}$$

admits the following set of eigenvalues:

$$\tau_1^\epsilon = \mu_1 + o(1), \quad \tau_j^\epsilon = \big(c_0 + o(1)\big)\varphi_{\epsilon,P_0}(P_0)\lambda_{j-1}, \quad j = 2,\ldots,N+1,$$
$$\tau_l^\epsilon = \mu_l + o(1), \quad l \geqslant N+2,$$

where λ_j, $j = 1,\ldots,N$, are the eigenvalues of $G_i(P_0)$ and

$$c_0 = 2d^{-2}(P_0,\partial\Omega)\frac{\int_{\mathbb{R}^N} pw^{p-1}w'u_*'(r)}{\int_{\mathbb{R}^N}(\frac{\partial w}{\partial y_1})^2\,dy} < 0, \tag{3.14}$$

where $u_*(r)$ is the unique radial solution of the following problem

$$\Delta u - u = 0, \qquad u(0) = 1, \qquad u = u(r) \quad \text{in } \mathbb{R}^N. \tag{3.15}$$

Furthermore, the eigenfunction (*suitably normalized*) *corresponding to τ_j^ϵ, $j = 2,\ldots,N+1$, is given by the following*:

$$\phi_j^\epsilon = \sum_{l=1}^N \big(a_{j-1,l} + o(1)\big)\epsilon\frac{\partial w_{\epsilon,P}}{\partial P_l}\bigg|_{P=P_\epsilon}, \tag{3.16}$$

where $\vec{a}_j = (a_{j,1},\ldots,a_{j,N})^t$ is the eigenvector corresponding to λ_j, namely

$$G_i(P_0)\vec{a}_j = \lambda_j\vec{a}_j, \quad j = 1,\ldots,N.$$

3.2. *A reduction lemma*

Let α_ϵ be an eigenvalue of (3.3). Then the following holds. (The proof of it is routine. See Appendix of [76].)

LEMMA A.

(1) $\alpha_\epsilon = o(1)$ *if and only if* $\alpha_\epsilon = (1+o(1))\tau_j^\epsilon$ *for some* $j = 2,\ldots,N+1$, *where* τ_j^ϵ *is given by Theorem* 3.3 *or Theorem* 3.5.

(2) *If* $\alpha_\epsilon \to \alpha_0 \neq 0$. *Then* α_0 *is an eigenvalue of the following eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi - \frac{qr}{s+1+\tau\alpha_0}\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p = \alpha_0\phi,$$
$$\phi \in H^2(\mathbb{R}^N). \tag{3.17}$$

A direct application of Theorem 3.5 is the following corollary.

COROLLARY 3.6. *For* $\epsilon \ll 1$, $(a_\epsilon, \xi_\epsilon)$ *is unstable with respect to the shadow system* (2.2).

3.3. *Large eigenvalues: NLEP method*

This section is devoted to the study of the nonlocal eigenvalue problem (3.17). By [76] and [77], if problem (3.17) admits an eigenvalue λ with positive real part, then all single point-condensation solutions are unstable, while if all eigenvalues of problem (3.17) have negative real part, then all single point-condensation solutions are either stable or metastable. (Here we say that a solution is metastable if the eigenvalues of the associated linearized operator either are exponentially small or have strictly negative real parts.) Therefore it is vital to study problem (3.17).

We first consider the simple case when $\tau = 0$. Namely, we study the following NLEP:

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p = \lambda\phi, \quad \phi \in H^2(\mathbb{R}^N), \tag{3.18}$$

where

$$\gamma := \frac{qr}{(s+1)(p-1)},$$
$$\lambda \in \mathcal{C}, \quad \lambda \neq 0, \quad \phi(x) = \phi(|x|). \tag{3.19}$$

For problem (3.18), it is known that when $\gamma = 0$, there exists an eigenvalue $\lambda = \mu_1 > 0$ (Lemma 3.2). An important property of (3.18) is that nonlocal term can push the eigenvalues of problem (3.18) to become negative so that the point-condensation solutions of the Gierer–Meinhardt system become stable or metastable.

A major difficulty in studying problem (3.18) is that the left-hand side operator is *not self-adjoint* if $r \neq p+1$. (In the classical Gierer–Meinhardt system, $r = 2$, $p = 2$.) Therefore it may have complex eigenvalues or Hopf bifurcations. Many traditional techniques do not work here.

In [77] and [76], the eigenvalues of problem (3.18) in the following two cases

$$r = 2, \quad \text{or} \quad r = p+1$$

are studied and the following results are proved.

THEOREM 3.7.

(1) *If* (p,q,r,s) *satisfies*

$$\text{(A)}\quad \gamma=\frac{qr}{(s+1)(p-1)}>1,$$

and

$$\text{(B)}\quad r=2,\quad 1<p\leqslant 1+\frac{4}{N}\quad \text{or } r=p+1,\ 1<p<\left(\frac{N+2}{N-2}\right)_+,$$

where $(\frac{N+2}{N-2})_+=\frac{N+2}{N-2}$ *when* $N\geqslant 3$ *and* $(\frac{N+2}{N-2})_+=+\infty$ *when* $N=1,2$.

Then $Re(\lambda)<-c_1<0$ *for some* $c_1>0$, *where* $\lambda\neq 0$ *is an eigenvalue of problem* (3.18).

(2) *If* $\gamma<1$, *problem* (3.18) *has an eigenvalue* $\lambda_1>0$.

(3) *If*

$$\text{(C)}\quad r=2,\quad p>1+\frac{4}{N}\quad \text{and}\quad 1<\gamma<1+c_0,$$

for some $c_0>0$. *Then problem* (3.18) *has an eigenvalue* $\lambda_1>0$.

We give a complete proof of Theorem 3.7 since this is the key element in all the stability result later on.

The proof of Theorem 3.7 is based on the following important inequalities which are new and interesting.

LEMMA 3.8. *Let* w *be the unique solution to* (2.8).

(1) *If* $1<p<1+\frac{4}{N}$, *then there exists a positive constant* $a_1>0$ *such that*

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2\right)+\frac{2(p-1)\int_{\mathbb{R}^N}w\phi\int_{\mathbb{R}^N}w^p\phi}{\int_{\mathbb{R}^N}w^2}\\&\quad-(p-1)\frac{\int_{\mathbb{R}^N}w^{p+1}}{(\int_{\mathbb{R}^N}w^2)^2}\left(\int_{\mathbb{R}^N}w\phi\right)^2\\&\quad\geqslant a_1 d^2_{L^2(\mathbb{R}^N)}(\phi,X_1),\end{aligned}\tag{3.20}$$

for all $\phi\in H^1(\mathbb{R}^N)$, *where* $X_1:=\operatorname{span}\{w,\ \frac{\partial w}{\partial y_j},\ j=1,\dots,N\}$.

(2) *If* $p=1+\frac{4}{N}$, *then there exists a positive constant* $a_2>0$ *such that*

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2\right)+\frac{2(p-1)\int_{\mathbb{R}^N}w\phi\int_{\mathbb{R}^N}w^p\phi}{\int_{\mathbb{R}^N}w^2}\\&\quad-(p-1)\frac{\int_{\mathbb{R}^N}w^{p+1}}{(\int_{\mathbb{R}^N}w^2)^2}\left(\int_{\mathbb{R}^N}w\phi\right)^2\end{aligned}$$

$$\geqslant a_2 d^2_{L^2(\mathbb{R}^N)}(\phi, X_2), \tag{3.21}$$

for all $\phi \in H^1(\mathbb{R}^N)$, *where* $X_2 := \operatorname{span}\{w,\ \frac{1}{p-1}w + \frac{1}{2}y\nabla w(y),\ \frac{\partial w}{\partial y_j},\ j = 1, \ldots, N\}$.

(3) *There exists a positive constant* $a_3 > 0$ *such that*

$$\begin{aligned} &\int_{\mathbb{R}^N} \bigl(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2\bigr) + \frac{(p-1)(\int_{\mathbb{R}^N} w^p\phi)^2}{\int_{\mathbb{R}^N} w^{p+1}} \\ &\quad \geqslant a_3 d^2_{L^2(\mathbb{R}^N)}(\phi, X_1), \quad \forall\phi \in H^1\bigl(\mathbb{R}^N\bigr). \end{aligned} \tag{3.22}$$

PROOF OF LEMMA 3.8. To this end, we first introduce some notations and make some preparations. Set

$$L\phi := L_0\phi - \gamma(p-1)\frac{\int_{\mathbb{R}^N} w^{r-1}\phi}{\int_{\mathbb{R}^N} w^r} w^p, \quad \phi \in H^2\bigl(\mathbb{R}^N\bigr)$$

where $\gamma = \frac{qr}{(p-1)(s+1)}$ and $L_0 := \Delta - 1 + pw^{p-1}$. Note that L is not selfadjoint if $r \neq p+1$.

Let

$$X_0 := \operatorname{kernel}(L_0) = \operatorname{span}\left\{\frac{\partial w}{\partial y_j} \;\middle|\; j = 1, \ldots, N\right\}.$$

Then

$$L_0 w = (p-1)w^p, \qquad L_0\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) = w \tag{3.23}$$

and

$$\int_{\mathbb{R}^N} \bigl(L_0^{-1}w\bigr)w = \int_{\mathbb{R}^N} w\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) = \left(\frac{1}{p-1} - \frac{N}{4}\right)\int_{\mathbb{R}^N} w^2, \tag{3.24}$$

$$\begin{aligned} \int_{\mathbb{R}^N} \bigl(L_0^{-1}w\bigr)w^p &= \int_{\mathbb{R}^N} w^p\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) \\ &= \int_{\mathbb{R}^N} \bigl(L_0^{-1}w\bigr)\frac{1}{p-1}L_0 w = \frac{1}{p-1}\int_{\mathbb{R}^N} w^2. \end{aligned} \tag{3.25}$$

Since L is not selfadjoint, we introduce a new operator as follows:

$$\begin{aligned} L_1\phi := {}& L_0\phi - (p-1)\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2}w^p - (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}w \\ &+ (p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w\phi}{(\int_{\mathbb{R}^N} w^2)^2}w. \end{aligned} \tag{3.26}$$

By (3.26), L_1 is selfadjoint. Next we compute the kernel of L_1. It is easy to see that $w, \frac{\partial w}{\partial y_j}, j = 1, \dots, N, \in \text{kernel}(L_1)$. On the other hand, if $\phi \in \text{kernel}(L_1)$, then by (3.23)

$$\begin{aligned} L_0\phi &= c_1(\phi)w + c_2(\phi)w^p \\ &= c_1(\phi)L_0\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) + c_2(\phi)L_0\left(\frac{w}{p-1}\right) \end{aligned}$$

where

$$\begin{aligned} c_1(\phi) &= (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2} - (p-1)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w\phi}{(\int_{R^N} w^2)^2}, \\ c_2(\phi) &= (p-1)\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2}. \end{aligned}$$

Hence

$$\phi - c_1(\phi)\left(\frac{1}{p-1}w + \frac{1}{2}x\nabla w\right) - c_2(\phi)\frac{1}{p-1}w \in \text{kernel}(L_0). \tag{3.27}$$

Note that

$$\begin{aligned} c_1(\phi) &= (p-1)c_1(\phi)\frac{\int_{\mathbb{R}^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{\int_{\mathbb{R}^N} w^2} \\ &\quad - (p-1)c_1(\phi)\frac{\int_{\mathbb{R}^N} w^{p+1}\int_{\mathbb{R}^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{(\int_{\mathbb{R}^N} w^2)^2} \\ &= c_1(\phi) - c_1(\phi)\left(\frac{1}{p-1} - \frac{N}{4}\right)\frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2} \end{aligned}$$

by (3.24) and (3.25). This implies that $c_1(\phi) = 0$. By (3.27) and Lemma 3.2, this shows that the kernel of L_1 is exactly X_1.

Now we prove (3.20). Suppose (3.20) is not true, then there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \dots, N$, and (iii) $L_1\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}w. \tag{3.28}$$

We first claim that $\int_{\mathbb{R}^N} w^p\phi \neq 0$. In fact if $\int_{\mathbb{R}^N} w^p\phi = 0$, then $\alpha > 0$ is an eigenvalue of L_0. By Lemma 3.2, $\alpha = \mu_1$ and ϕ has constant sign. This contradicts with the fact that $\phi \perp w$. Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^{\perp}$. So (3.28) implies

$$\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}(L_0 - \alpha)^{-1}w.$$

Thus

$$
\begin{aligned}
&\int_{\mathbb{R}^N} w^p\phi = (p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^2}\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w^p,\\
&\int_{\mathbb{R}^N} w^2 = (p-1)\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w^p,\\
&\int_{\mathbb{R}^N} w^2 = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)\big((L_0-\alpha)w+\alpha w\big),\\
&0 = \int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)w. \qquad (3.29)
\end{aligned}
$$

Let $h_1(\alpha)=\int_{\mathbb{R}^N}((L_0-\alpha)^{-1}w)w$, then

$$
\begin{aligned}
h_1(0) &= \int_{\mathbb{R}^N}\big(L_0^{-1}w\big)w = \int_{\mathbb{R}^N}\left(\frac{1}{p-1}w+\frac{1}{2}x\cdot\nabla w\right)w\\
&= \left(\frac{1}{p-1}-\frac{N}{4}\right)\int_{\mathbb{R}^N} w^2 > 0
\end{aligned}
$$

since $1<p<1+\frac{4}{N}$. Moreover

$$
h_1'(\alpha)=\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-2}w\big)w=\int_{\mathbb{R}^N}\big((L_0-\alpha)^{-1}w\big)^2>0.
$$

This implies $h_1(\alpha)>0$ for all $\alpha\in(0,\mu_1)$. Clearly, also $h_1(\alpha)<0$ for $\alpha\in(\mu_1,\infty)$ (since $\lim_{\alpha\to+\infty} h_1(\alpha)=0$). This is a contradiction to (3.29)!

This proves the inequality (3.20).

The proof of (3.21) is similar. In this case we have

$$
\int_{\mathbb{R}^N}\big(L_0^{-1}w\big)w=\int_{\mathbb{R}^N} w\left(\frac{1}{p-1}w+\frac{1}{2}x\nabla w\right)=0. \qquad (3.30)
$$

Thus the kernel of L_1 is X_2. The rest of the proof is exactly the same as before.

To prove (3.22), we introduce

$$
L_3\phi := L_0\phi-(p-1)\frac{\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}w^p. \qquad (3.31)
$$

Similar as before, the kernel of L_3 is exactly X_1.

Suppose (3.22) is not true, then there exists (α,ϕ) such that (a) α is real and positive, (b) $\phi\perp w, \phi\perp\frac{\partial w}{\partial y_j}$, $j=1,\dots,N$, and (c) $L_3\phi=\alpha\phi$.

We show that this is impossible. From (a) and (c), we have

$$
(L_0-\alpha)\phi=\frac{(p-1)\int_{\mathbb{R}^N} w^p\phi}{\int_{\mathbb{R}^N} w^{p+1}}w^p. \qquad (3.32)
$$

Similar to the proof of (3.20), we have that $\int_{\mathbb{R}^N} w^p \phi \neq 0, \alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^{\perp}$. So (3.32) implies

$$\phi = \frac{(p-1)\int_{\mathbb{R}^N} w^p \phi}{\int_{\mathbb{R}^N} w^{p+1}} (L_0 - \alpha)^{-1} w^p.$$

Thus

$$\int_{\mathbb{R}^N} w^p \phi = (p-1) \frac{\int_{\mathbb{R}^N} w^p \phi}{\int_{\mathbb{R}^N} w^{p+1}} \int_{\mathbb{R}^N} \big((L_0 - \alpha)^{-1} w^p\big) w^p,$$

$$\int_{\mathbb{R}^N} w^{p+1} = (p-1) \int_{\mathbb{R}^N} \big((L_0 - \alpha)^{-1} w^p\big) w^p. \tag{3.33}$$

Let

$$h_3(\alpha) = (p-1) \int_{\mathbb{R}^N} \big((L_0 - \alpha)^{-1} w^p\big) w^p - \int_{\mathbb{R}^N} w^{p+1},$$

then

$$h_3(0) = (p-1) \int_{\mathbb{R}^N} (L_0^{-1} w^p) w^p - \int_{\mathbb{R}^N} w^{p+1} = 0.$$

Moreover

$$h_3'(\alpha) = (p-1) \int_{\mathbb{R}^N} \big((L_0 - \alpha)^{-2} w^p\big) w^p = (p-1) \int_{\mathbb{R}^N} \big((L_0 - \alpha)^{-1} w^p\big)^2 > 0.$$

This implies $h_3(\alpha) > 0$ for all $\alpha \in (0, \mu_1)$. Clearly, also $h_3(\alpha) < 0$ for $\alpha \in (\mu_1, \infty)$. A contradiction to (3.33)! □

Using Lemma 3.8, we can prove Theorem 3.7(i).

PROOF OF THEOREM 3.7(I). We divide the proof into three cases.

CASE 1. $r = 2, 1 < p < 1 + \frac{4}{N}$.

Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then we obtain two equations

$$L_0 \phi_R - (p-1)\gamma \frac{\int_{\mathbb{R}^N} w \phi_R}{\int_{\mathbb{R}^N} w^2} w^p = \alpha_R \phi_R - \alpha_I \phi_I, \tag{3.34}$$

$$L_0 \phi_I - (p-1)\gamma \frac{\int_{\mathbb{R}^N} w \phi_I}{\int_{\mathbb{R}^N} w^2} w^p = \alpha_R \phi_I + \alpha_I \phi_R. \tag{3.35}$$

Multiplying (3.34) by ϕ_R and (3.35) by ϕ_I and adding them together, we obtain

$$
\begin{aligned}
&-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) \\
&\quad = L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\
&\qquad + (p-1)(\gamma-2) \frac{\int_{\mathbb{R}^N} w\phi_R \int_{\mathbb{R}^N} w^p \phi_R + \int_{\mathbb{R}^N} w\phi_I \int_{\mathbb{R}^N} w^p \phi_I}{\int_{\mathbb{R}^N} w^2} \\
&\qquad + (p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2} \left[\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right].
\end{aligned}
$$

Multiplying (3.34) by w and (3.35) by w we obtain

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w^p \phi_R - \gamma (p-1) \frac{\int_{\mathbb{R}^N} w\phi_R}{\int_{\mathbb{R}^N} w^2} \int_{\mathbb{R}^N} w^{p+1} \\
&\quad = \alpha_R \int_{\mathbb{R}^N} w\phi_R - \alpha_I \int_{\mathbb{R}^N} w\phi_I,
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w^p \phi_I - \gamma (p-1) \frac{\int_{\mathbb{R}^N} w\phi_I}{\int_{\mathbb{R}^N} w^2} \int_{\mathbb{R}^N} w^{p+1} \\
&\quad = \alpha_R \int_{\mathbb{R}^N} w\phi_I + \alpha_I \int_{\mathbb{R}^N} w\phi_R.
\end{aligned} \tag{3.37}
$$

Multiplying (3.36) by $\int_{\mathbb{R}^N} w\phi_R$ and (3.37) by $\int_{\mathbb{R}^N} w\phi_I$ and adding them together, we obtain

$$
\begin{aligned}
&(p-1) \int_{\mathbb{R}^N} w\phi_R \int_{\mathbb{R}^N} w^p \phi_R + (p-1) \int_{\mathbb{R}^N} w\phi_I \int_{\mathbb{R}^N} w^p \phi_I \\
&\quad = \left(\alpha_R + \gamma (p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2} \right) \left(\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) \\
&\quad = L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\
&\qquad + (p-1)(\gamma-2) \left(\frac{1}{p-1} \alpha_R + \gamma \frac{\int_{\mathbb{R}^N} w^{p+1}}{\int_{\mathbb{R}^N} w^2} \right) \frac{(\int_{\mathbb{R}^N} w\phi_R)^2 + (\int_{\mathbb{R}^N} w\phi_I)^2}{\int_{\mathbb{R}^N} w^2} \\
&\qquad + (p-1) \frac{\int_{\mathbb{R}^N} w^{p+1}}{(\int_{\mathbb{R}^N} w^2)^2} \left[\left(\int_{\mathbb{R}^N} w\phi_R \right)^2 + \left(\int_{\mathbb{R}^N} w\phi_I \right)^2 \right].
\end{aligned}
$$

Set

$$\phi_R = c_R w + \phi_R^{\perp}, \phi_R^{\perp} \perp X_1, \quad \phi_I = c_I w + \phi_I^{\perp}, \quad \phi_I^{\perp} \perp X_1.$$

Then

$$\int_{\mathbb{R}^N} w\phi_R = c_R \int_{\mathbb{R}^N} w^2, \qquad \int_{\mathbb{R}^N} w\phi_I = c_I \int_{\mathbb{R}^N} w^2,$$
$$d^2_{L^2(\mathbb{R}^N)}(\phi_R, X_1) = \|\phi_R^{\perp}\|^2_{L^2}, \qquad d^2_{L^2(\mathbb{R}^N)}(\phi_I, X_1) = \|\phi_I^{\perp}\|^2_{L^2}.$$

By some simple computations we have

$$\begin{aligned} &L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\ &\quad + (\gamma - 1)\alpha_R\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^2 + (p-1)(\gamma-1)^2\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^{p+1} \\ &\quad + \alpha_R\big(\|\phi_R^{\perp}\|^2_{L^2} + \|\phi_I^{\perp}\|^2_{L^2}\big) = 0. \end{aligned}$$

By Lemma 3.8 (1)

$$\begin{aligned} &(\gamma - 1)\alpha_R\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^2 \\ &\quad + (p-1)(\gamma-1)^2\big(c_R^2 + c_I^2\big) \int_{\mathbb{R}^N} w^{p+1} \\ &\quad + (\alpha_R + a_1)\big(\|\phi_R^{\perp}\|^2_{L^2} + \|\phi_I^{\perp}\|^2_{L^2}\big) \leqslant 0. \end{aligned}$$

Since $\gamma > 1$, we must have $\alpha_R < 0$, which proves Theorem 3.7 in Case 1.

CASE 2. $r = 2$, $p = 1 + \frac{4}{N}$.

Set

$$w_0 = \frac{1}{p-1} w + \frac{1}{2} x \nabla w. \tag{3.38}$$

We just need to take care of w_0.

Suppose that $\alpha_0 \neq 0$ is an eigenvalue of L. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then similar to Case 1, we obtain two equations (3.34) and (3.35). We now decompose

$$\phi_R = c_R w + b_R w_0 + \phi_R^{\perp}, \quad \phi_R^{\perp} \perp X_1,$$
$$\phi_I = c_I w + b_I w_0 + \phi_I^{\perp}, \quad \phi_I^{\perp} \perp X_1.$$

Similar to Case 1, we obtain

$$\begin{aligned}
&L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)\\
&\quad+(\gamma-1)\alpha_R\big(c_R^2+c_I^2\big)\int_{\mathbb{R}^N} w^2+(p-1)(\gamma-1)^2\big(c_R^2+c_I^2\big)\int_{\mathbb{R}^N} w^{p+1}\\
&\quad+\alpha_R\left(b_R^2\left(\int_{\mathbb{R}^N} w_0^2\right)^2+b_I^2\left(\int_{\mathbb{R}^N} w_0^2\right)^2+\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2\right)\leqslant 0
\end{aligned}$$

By Lemma 3.8(2)

$$\begin{aligned}
&(\gamma-1)\alpha_R\big(c_R^2+c_I^2\big)\int_{\mathbb{R}^N} w^2+(p-1)(\gamma-1)^2\big(c_R^2+c_I^2\big)\int_{\mathbb{R}^N} w^{p+1}\\
&\quad+\alpha_R\left(b_R^2\left(\int_{\mathbb{R}^N} w_0^2\right)^2+b_I^2\left(\int_{\mathbb{R}^N} w_0^2\right)^2\right)+(\alpha_R+a_2)\big(\|\phi_R^\perp\|_{L^2}^2+\|\phi_I^\perp\|_{L^2}^2\big)\\
&\quad\leqslant 0.
\end{aligned}$$

If $\alpha_R\geqslant 0$, then necessarily we have

$$c_R=c_I=0,\quad \phi_R^\perp=0,\quad \phi_I^\perp=0.$$

Hence $\phi_R=b_Rw_0, \phi_I=b_Iw_0$. This implies that

$$b_RL_0w_0=(b_R-b_I)w_0,\quad b_IL_0w_0=(b_R+b_I)w_0,$$

which is impossible unless $b_R=b_I=0$. A contradiction!

CASE 3. $r=p+1$, $1<p<(\frac{N+2}{N-2})_+$.

Let $r=p+1$. L becomes

$$L=L_0-\frac{qr}{s+1}\frac{\int_{\mathbb{R}^N} w^p\cdot}{\int_{\mathbb{R}^N} w^{p+1}}w^p.$$

We will follow the proof of Case 1.

Let $\alpha_0=\alpha_R+i\alpha_I$ and $\phi=\phi_R+i\phi_I$. Since $\alpha_0\neq 0$, we can choose $\phi\perp \text{kernel}(L_0)$. Then similarly we obtain two equations

$$L_0\phi_R-(p-1)\gamma\frac{\int_{\mathbb{R}^N} w^p\phi_R}{\int_{\mathbb{R}^N} w^{p+1}}w^p=\alpha_R\phi_R-\alpha_I\phi_I, \tag{3.39}$$

$$L_0\phi_I-(p-1)\gamma\frac{\int_{\mathbb{R}^N} w^p\phi_I}{\int_{\mathbb{R}^N} w^{p+1}}w^p=\alpha_R\phi_I+\alpha_I\phi_R. \tag{3.40}$$

Multiplying (3.39) by ϕ_R and (3.40) by ϕ_I and adding them together, we obtain

$$-\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) = L_3(\phi_R, \phi_R) + L_3(\phi_I, \phi_I) + (p-1)(\gamma-1)\frac{(\int_{\mathbb{R}^N} w^p \phi_R)^2 + (\int_{\mathbb{R}^N} w^p \phi_I)^2}{\int_{\mathbb{R}^N} w^{p+1}}.$$

By Lemma 3.8(3)

$$\alpha_R \int_{\mathbb{R}^N} (\phi_R^2 + \phi_I^2) + a_3 d_{L^2}^2(\phi, X_1) + (p-1)(\gamma-1)\frac{(\int_{\mathbb{R}^N} w^p \phi_R)^2 + (\int_{\mathbb{R}^N} w^p \phi_I)^2}{\int_{\mathbb{R}^N} w^{p+1}} \leqslant 0$$

which implies $\alpha_R < 0$ since $\gamma > 1$.

Theorem 3.7(i) in Case 3 is thus proved. □

PROOF OF THEOREM 3.7(II). Assume that $\gamma < 1$. To prove Theorem 3.7(ii), we introduce the following function:

$$h_4(\lambda) := \int_{\mathbb{R}^N} w^r - \gamma(p-1)\int_{\mathbb{R}^N} \big((L_0-\lambda)^{-1} w^p\big) w^{r-1}. \tag{3.41}$$

Note that $h_4(\lambda)$ is well defined in $(0, \mu_1)$, where μ_1 is the unique positive eigenvalue of L_0. Let us denote the corresponding eigenfunction by Φ_0. Since μ_1 is a principal eigenvalue, we may assume that $\Phi_0 > 0$.

It is easy to see that to prove Theorem 3.7(ii), it is enough to find a positive zero of $h_4(\lambda)$.

First we have

$$h_4(0) = \int_{\mathbb{R}^N} w^r - \gamma(p-1)\int_{\mathbb{R}^N} L_0^{-1} w^p w^{r-1} = (1-\gamma)\int_{\mathbb{R}^N} w^r > 0. \tag{3.42}$$

Set $\Phi_\lambda = (L_0-\lambda)^{-1} w^p$. Then Φ_λ satisfies

$$(L_0 - \lambda)\Phi_\lambda = w^p. \tag{3.43}$$

Multiplying (3.43) by Φ_0 and integrating by parts, we have

$$(\mu_1 - \lambda)\int_{\mathbb{R}^N} \Phi_\lambda \Phi_0 = \int_{\mathbb{R}^N} \Phi_0 w^p,$$

which implies that

$$\int_{\mathbb{R}^N} \Phi_\lambda \Phi_0 = \frac{1}{\mu_1 - \lambda}\int_{\mathbb{R}^N} \Phi_0 w^p.$$

Let

$$\Phi_\lambda = \left(\frac{1}{(\mu_1-\lambda)\int_{\mathbb{R}^N}\Phi_0^2}\int_{\mathbb{R}^N}\Phi_0 w^p\right)\Phi_0 + \Phi_\lambda^\perp, \quad \Phi_\lambda^\perp \perp \Phi_0. \tag{3.44}$$

Then as $\lambda \to \mu_1, \lambda < \mu_1$, we have that $\|\Phi_\lambda^\perp\|_{L^2(\mathbb{R}^N)}$ is uniformly bounded and by (3.44)

$$\int_{\mathbb{R}^N}\Phi_\lambda w^{r-1} \to +\infty,$$

which implies that

$$h_4(\lambda) \to -\infty \quad \text{as } \lambda \to \mu_1,\ \lambda < \mu_1. \tag{3.45}$$

By (3.42) and (3.45), there is a $\lambda_0 \in (0, \mu_1)$ such that $h_4(\lambda_0) = 0$.

This proves (ii) of Theorem 3.7. □

PROOF OF THEOREM 3.7(III). Similarly, we just need to find a zero of

$$h_5(\lambda) := \int_{\mathbb{R}^N} w^2 - \gamma(p-1)\int_{\mathbb{R}^N} w(L_0-\lambda)^{-1}w^p. \tag{3.46}$$

We write it as

$$\begin{aligned} h_5(\lambda) &= (1-\gamma)\int_{\mathbb{R}^N} w^2 - \gamma(p-1)\lambda\int_{\mathbb{R}^N} w\big[(L_0-\lambda)^{-1}(w)\big] \\ &= (1-\gamma)\int_{\mathbb{R}^N} w^2 - \gamma(p-1)\lambda\int_{\mathbb{R}^N} wL_0^{-1}(w) + O\big(\lambda^2\big). \end{aligned}$$

Since $\int_{\mathbb{R}^N} wL_0^{-1}(w) < 0$, we see that for $0 < \gamma - 1$ small, there is a small $\lambda_0 > 0$ such that $h_5(\lambda_0) > 0$. □

For general r, the author in [79] proved the following:

THEOREM 3.9.

(1) *If*

$$D(r) := \frac{(p-1)\int_{\mathbb{R}^N} L_0^{-1}w^{r-1}w^{r-1}\int_{\mathbb{R}^N} w^2}{(\int_{\mathbb{R}^N} w^r)^2} > 0 \tag{3.47}$$

where $L_0 = \Delta - 1 + pw^{p-1}$ *(*L_0^{-1} *exists in* $H_r^2(\mathbb{R}^N) = \{u \in H^2(\mathbb{R}^N) \mid u(x) = u(|x|)\}$*) and*

$$1 + \frac{1}{\sqrt{1+\rho_0}} < \gamma < 1 + \frac{1}{\sqrt{1-\rho_0}}, \tag{3.48}$$

where $\rho_0 > 0$ *is given by*

$$\rho_0 := \frac{\int_{\mathbb{R}^N} w^{p+1}}{\sqrt{\int_{\mathbb{R}^N} w^{2p} \int_{\mathbb{R}^N} w^2}} < 1. \tag{3.49}$$

Then for any nonzero eigenvalue λ *of problem* (3.18), *we have* $\mathrm{Re}(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$.

(2) *If* (p, q, r, s) *satisfies*

$$1 + \frac{2r}{N} < p < \left(\frac{N+2}{N-2}\right)_+ \quad \text{and} \quad \gamma < 1 + c_0, \tag{3.50}$$

for some $c_0 > 0$. *Then problem* (3.18) *has a real eigenvalue* $\lambda_1 > 0$.

Generally speaking, $D(r)$ is very difficult to compute. A recent result of the author and L. Zhang partially solved this problem and moreover we obtained more general and explicit result. For example the following result are proved [80].

THEOREM 3.10. *Let*

$$F(r) = 1 - \frac{p-1}{2r} N.$$

Suppose $2 < r < p + 1$, $1 < p < 1 + \frac{2r}{N}$ *and*

$$F(r) \geqslant \frac{\gamma - 2}{\gamma} F(p+1) + \frac{|\gamma - 2|}{\gamma} \sqrt{F(p+1)(F(p+1) - F(2))}, \tag{3.51}$$

then for any nonzero eigenvalue λ *of problem* (3.18), *we have* $\mathrm{Re}(\lambda) < -c_1 < 0$ *for some* $c_1 > 0$.

REMARK. Condition (3.51) holds if $2 < r < p + 1$, $F(r) \geqslant 0$ (i.e., $1 < p \leqslant 1 + \frac{2r}{N}$) and $1 < \gamma \leqslant 2$. Thus in this case we obtain the stability of the nonzero eigenvalues of (3.18). This is the first explicit result for the case when $r \notin \{2, p + 1\}$. For $\gamma > 2$, we need

$$F(r) \geqslant \frac{\gamma - 2}{\gamma} \left[F(p+1) - \sqrt{F(p+1)(F(p+1) - F(2))}\right].$$

Going back to the shadow system case, the following result was proved in [76].

THEOREM 3.11. *Assume that* $\epsilon \ll 1$ *and* τ *is small. If* (p, q, r, s) *satisfy* (A) *and* (B) *in Theorem* 3.7, *then*

(1) *single boundary spike solution at a nondegenerate local maximum point of mean curvature is stable, and*
(2) *single interior spike solution is metastable.*

Related work can also be found in [59] and [60].

3.4. *Uniqueness of Hopf bifurcations*

In Section 3.3, we have discussed the NLEP (3.17) when $\tau = 0$. It is easy to see that when τ small, results in Theorem 3.7 still hold. On the other hand, for τ large, it is easy to see that there is an unstable eigenvalue [8] to (3.17). (In fact, as $\tau \to +\infty$, there is a positive eigenvalue near $\mu_1 > 0$.) Therefore, as τ varies from 0 to ∞, Hopf bifurcation may occur. In this section, we show that in some special cases, Hopf bifurcation is actually unique.

We consider the following nonlocal eigenvalue problem (putting $r = p = 2, s = 0$ in (3.17))

$$L\phi := \Delta\phi - \phi + 2w\phi - \frac{\gamma}{1+\tau\lambda_0}\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi, \quad \phi \in H^2(\mathbb{R}^N). \tag{3.52}$$

THEOREM 3.12. *Let L be defined by* (3.52). *Assume that $N \leqslant 3$ and $\gamma > 1$. Then there exists a unique $\tau = \tau_1 > 0$ such that for $\tau < \tau_1$,* (3.52) *admits a positive eigenvalue, and for $\tau > \tau_1$, all nonzero eigenvalues of problem* (3.52) *satisfy* $\mathrm{Re}(\lambda) < 0$. *At $\tau = \tau_1$, L has a Hopf bifurcation.*

PROOF OF THEOREM 3.12. Let $\gamma > 1$. As in [8], we may consider radially symmetric functions only. By Theorem 1.4 of [76], for $\tau = 0$ (and by perturbation, for τ small), all eigenvalues lie on the left half plane. By [8], for τ large, there exist unstable eigenvalues.

Note that the eigenvalues will not cross through zero: in fact, if $\lambda_0 = 0$, then we have

$$L_0\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w^2 = 0$$

which implies that

$$L_0\left(\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w\right) = 0$$

and hence by Lemma 3.2

$$\phi - \gamma\frac{\int_{\mathbb{R}^N} w\phi}{\int_{\mathbb{R}^N} w^2} w \in X_0.$$

This is impossible since ϕ is radially symmetric and $\phi \neq cw$ for all $c \in R$.

Thus there must be a point τ_1 at which L has a Hopf bifurcation, i.e., L has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To prove Theorem 3.12, all we need to show is that τ_1 is unique. That is

LEMMA 3.13. *Let $\gamma > 1$. Then there exists a unique $\tau_1 > 0$ such that L has a Hopf bifurcation.*

PROOF. Let $\lambda_0 = \sqrt{-1}\alpha_I$ be an eigenvalue of L. Without loss of generality, we may assume that $\alpha_I > 0$. (Note that $-\sqrt{-1}\alpha_I$ is also an eigenvalue of L.) Let $\phi_0 = (L_0 - \sqrt{-1}\alpha_I)^{-1} w^2$. Then (3.52) becomes

$$\frac{\int_{\mathbb{R}^N} w\phi_0}{\int_{\mathbb{R}^N} w^2} = \frac{1+\tau\sqrt{-1}\alpha_I}{\gamma}. \tag{3.53}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (3.53), we obtain the two equations

$$\frac{\int_{\mathbb{R}^N} w\phi_0^R}{\int_{\mathbb{R}^2} w^N} = \frac{1}{\gamma}, \tag{3.54}$$

$$\frac{\int_{\mathbb{R}^N} w\phi_0^I}{\int_{\mathbb{R}^2} w^N} = \frac{\tau\alpha_I}{\gamma}. \tag{3.55}$$

Note that (3.54) is independent of τ.

Let us now compute $\int_{\mathbb{R}^N} w\phi_0^R$. Observe that (ϕ_0^R, ϕ_0^I) satisfies

$$L_0\phi_0^R = w^2 - \alpha_I\phi_0^I, \quad L_0\phi_0^I = \alpha_I\phi_0^R.$$

So $\phi_0^R = \alpha_I^{-1} L_0 \phi_0^I$ and

$$\phi_0^I = \alpha_I\big(L_0^2 + \alpha_I^2\big)^{-1} w^2, \qquad \phi_0^R = L_0\big(L_0^2 + \alpha_I^2\big)^{-1} w^2. \tag{3.56}$$

Substituting (3.56) into (3.54) and (3.55), we obtain

$$\frac{\int_{\mathbb{R}^N} [wL_0(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathbb{R}^N} w^2} = \frac{1}{\gamma}, \tag{3.57}$$

$$\frac{\int_{\mathbb{R}^N} [w(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{\mathbb{R}^2} w^2} = \frac{\tau}{\gamma}. \tag{3.58}$$

Let

$$h_6(\alpha_I) = \frac{\int_{\mathbb{R}^N} wL_0(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathbb{R}^2} w^2}.$$

Then integration by parts gives

$$h_6(\alpha_I) = \frac{\int_{\mathbb{R}^N} w^2(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{\mathbb{R}^N} w^2}.$$

Note that

$$h_6'(\alpha_I) = -2\alpha_I\frac{\int_{\mathbb{R}^N} w^2(L_0^2+\alpha_I^2)^{-2}w^2}{\int_{\mathbb{R}^N} w^2} < 0.$$

So since

$$h_6(0) = \frac{\int_{\mathbb{R}^N} w(L_0^{-1} w^2)}{\int_{\mathbb{R}^N} w^2} > 0,$$

$h_6(\alpha_I) \to 0$ as $\alpha_I \to \infty$, and $\gamma > 1$, there exists a unique $\alpha_I > 0$ such that (3.57) holds. Substituting this unique α_I into (3.58), we obtain a unique $\tau = \tau_1 > 0$.

Lemma 3.13 is thus proved. □

Theorem 3.12 now follows from Lemma 3.13. □

3.5. *Finite ϵ case*

In all the previous sections, it is always assumed that ϵ is small. However, in practical applications, it is vital to know how small ϵ should be. The finite ϵ case has been completely characterized in one-dimensional case by Wei and Winter [88]. We summarize the results here.

Without loss of generality, we may assume that $\Omega = (0, 1)$. That is, we consider

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{\xi^q}, & 0 < x < 1,\ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 a^r \, dx, \\ a > 0, \quad a_x(0, t) = a_x(1, t) = 0. \end{cases} \tag{3.59}$$

The steady-state problem of (3.59) is equivalent to the following problem for the transformed function u_ϵ given by $u_\epsilon(x) = \xi^{-\frac{q}{p-1}} a(x)$:

$$\xi^{1+s-\frac{qr}{p-1}} = \int_0^1 u^r(x) \, dx$$

and

$$\begin{aligned} &\epsilon^2 u_{xx} - u + u^p = 0, \\ &u_x(x) < 0,\ 0 < x < 1, \quad u_x(0) = u_x(1) = 0. \end{aligned} \tag{3.60}$$

Letting

$$L := \frac{1}{\epsilon} \tag{3.61}$$

and rescaling $u(x) = w_L(y)$, where $y = Lx$, we see that w_L satisfies the following ODE:

$$\begin{aligned} &w_L'' - w_L + w_L^p = 0, \\ &w_L'(y) < 0,\ 0 < y < L, \quad w_L'(0) = w_L'(L) = 0. \end{aligned} \tag{3.62}$$

Since (3.62) is an autonomous ODE, it is easy to see that a nontrivial solution exists if and only if

$$\epsilon < \frac{\sqrt{p-1}}{\pi} \quad \left(\text{or } L > \frac{\pi}{\sqrt{p-1}}\right). \tag{3.63}$$

The stability of steady-state solutions to (3.59) has been a subject of study in the last few years. A recent result of [56] (see Theorem 1.1 of [56]) says that a stable solution to (3.59) must be asymptotically monotone. More precisely, if $(A(x,t), \xi(t)), t \geqslant 0$ is a solution to (3.59) that is linearly neutrally stable, then there is a $t_0 > 0$ such that

$$a_x(x, t_0) \neq 0 \quad \text{for all } (x,t) \in (0,1) \times [t_0, +\infty). \tag{3.64}$$

Thus all *nonmonotone* steady-state solutions are linearly unstable. Therefore we focus our attention on *monotone solutions*. There are two monotone solutions—the monotone increasing one and the monotone decreasing one. Since these two solutions differ by reflection, we consider the monotone decreasing function only. This solution is then called u_ϵ and it has the least energy among all positive solutions of (3.60), see [60]. If $L \leqslant \frac{\pi}{\sqrt{p-1}}$, then $w_L = 1$. We also denote the corresponding solutions to (3.59) by

$$a_L(x) = \xi_L^{\frac{q}{p-1}} w_L(Lx), \qquad \xi_L^{1+s-\frac{qr}{p-1}} = \int_0^1 w_L^r(Lx)\,dx. \tag{3.65}$$

Before stating our results, we first introduce some notation. Let $I = (0, L)$ and $\phi \in H^2(I)$. We define the following operator:

$$\mathcal{L}[\phi] = \phi'' - \phi + p w_L^{p-1} \phi. \tag{3.66}$$

It is proved [88] that $\mathcal{L}$ has the spectrum

$$\lambda_1 > 0, \quad \lambda_j < 0, \quad j = 2, 3, \ldots. \tag{3.67}$$

Hence for the map $\mathcal{L}$ from $H^2(I)$ to $L^2(I)$ we know that $\mathcal{L}^{-1}$ exists, where $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$. This implies that $\mathcal{L}^{-1} w_L$ is well defined.

Then we have the following theorem

THEOREM 3.14. *Assume that* $L > \frac{\pi}{\sqrt{p-1}}$ *and either*

$$r = 2, \quad \int_0^L w_L \mathcal{L}^{-1} w_L \,dy > 0 \tag{3.68}$$

or

$$r = p + 1. \tag{3.69}$$

Then (a_L, ξ_L) (*given by* (3.65)) *is a linearly stable steady state to* (3.59) *for* τ *small.*

This theorem reduces the issue of stability to the computation of the integral

$$\int_0^L w_L \mathcal{L}^{-1} w_L \, dy.$$

This integral is quite difficult to compute for general L.

For τ finite, we have the following theorem.

THEOREM 3.15. *Let* (3.68) *be true and* $L > \frac{\pi}{\sqrt{p-1}}$. *Then there exists a unique* $\tau_c > 0$ *such that for* $\tau < \tau_c$, (a_L, ξ_L) *is stable and for* $\tau > \tau_c$, (a_L, ξ_L) *is unstable. At* $\tau = \tau_c$, *there exists a unique Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal, namely, we have*

$$\left.\frac{d\lambda_R}{d\tau}\right|_{\tau=\tau_c} > 0, \tag{3.70}$$

where λ_R *is the real part of the eigenvalue.*

Using Weierstrass $p(z)$ functions and Jacobi elliptic integrals, one can show that $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$ for all $L > \pi$ in the cases $r = 2$, $p = 2, 3$. The original Gierer–Meinhardt system $((p, q, r, s) = (2, 1, 2, 0))$ falls into this class. Thus for the shadow system of the original Gierer–Meinhardt system, we have a complete picture of the stability of (a_L, ξ_L) for any $\tau > 0$ and any $L > 0$, by the following theorem

THEOREM 3.16. *Assume that* $L > \frac{\pi}{\sqrt{p-1}}$ *and* $r = 2$, $p = 2$ *or* 3. *Then there exists a unique* $\tau_c > 0$ *such that for* $\tau < \tau_c$, (a_L, ξ_L) *is stable and for* $\tau > \tau_c$, (A_L, ξ_L) *is unstable. At* $\tau = \tau_c$, *there exists a Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal.*

Theorem 3.16 gives a complete picture of the stability of nontrivial monotone solutions in terms of L since for $L \leqslant \frac{\pi}{\sqrt{p-1}}$ we necessarily have $w_L \equiv 1$. Combining this with the results of [56], we have *completely* classified stability and instability of all steady-state solutions for *all* $\epsilon > 0$ for the shadow system of the original Gierer–Meinhardt system.

We do not know if the Hopf bifurcation in Theorem 3.15 is subcritical or supercritical. This is related to another interesting question: is there time-periodic solution $(a(x,t), \xi(x,t))$ to (3.59) at the Hopf bifurcation point $\tau = \tau_c$? If so, is it stable or unstable?

We can also extend this idea to general domains in $\mathbb{R}^N$, $N \geqslant 2$. Namely we consider

$$\begin{cases} a_t = \Delta a - a + \frac{a^p}{\xi^q}, \quad x \in \Omega_L, \quad t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \frac{1}{|\Omega_L|} \int_{\Omega_L} a^r, \\ a > 0, \quad \frac{\partial a}{\partial \nu} = 0 \quad \text{on } \partial\Omega_L, \end{cases} \tag{3.71}$$

where we have scaled the ϵ into the domain through $\Omega_L = \frac{1}{\epsilon}\Omega$. In this case, let us assume that $\Omega_L \subset \mathbb{R}^N$ is a smooth and bounded domain, and the exponents (p, q, r, s) satisfy the following condition

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geqslant 0, \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1,$$

and p is subcritical:

$$1 < p < \frac{N+2}{N-2} \quad \text{if } N \geqslant 3; \quad 1 < p < +\infty \quad \text{if } N = 2.$$

The steady state solution of (3.71) is given by

$$a = \xi^{\frac{q}{p-1}} u, \quad \xi^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} u^r \tag{3.72}$$

where u is a solution of the following problem:

$$\begin{cases} \Delta u - u + u^p = 0, \quad u > 0 & \text{in } \Omega_L, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{3.73}$$

We again consider the minimizer solution $w_L(x)$ which satisfies (3.73) and

$$E[w_L] = \inf_{u \in H^1(\Omega_L),\ u \not\equiv 0} E[u] \tag{3.74}$$

where

$$E[u] = \frac{\int_{\Omega_L} (|\nabla u|^2 + u^2)}{(\int_{\Omega_L} u^{p+1})^{\frac{2}{p+1}}}.$$

The corresponding steady-state solution to the shadow system (3.71) is denoted by

$$a_L = \xi_L^{\frac{q}{p-1}} w_L, \quad \xi_L^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} w_L^r. \tag{3.75}$$

Let

$$\mathcal{L}[\phi] = \Delta\phi - \phi + p w_L^{p-1} \phi.$$

Then we have the following lemma whose proof is similar to Lemma 3.2.

LEMMA 3.17. *Consider the following eigenvalue problem*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, & \textit{in } \Omega_L, \\ \frac{\partial \phi}{\partial \nu} = 0 & \textit{on } \partial\Omega_L. \end{cases} \tag{3.76}$$

Then $\lambda_1 > 0$ *and* $\lambda_2 \leqslant 0$.

We now put two important assumptions:
We first assume that

$$\text{(A1)} \quad \mathcal{L}^{-1} \text{ exists.}$$

Under (A1), we assume that

$$\text{(A2)} \quad \int_{\Omega_L} w_L(\mathcal{L}^{-1} w_L) > 0.$$

We can now state the following theorem

THEOREM 3.18. *Assume that either*

$$r = p + 1, \quad \textit{and } \text{(A1)} \textit{ holds},$$

or

$$r = 2, \quad \textit{and } \text{(A1)} \textit{ and } \text{(A2)} \textit{ hold}.$$

Then (a_L, ξ_L) *is linearly stable for* τ *small.*

In the case of $r = 2$, *there exists a unique* $\tau = \tau_c$ *such that* (a_L, ξ_L) *is stable for* $\tau < \tau_c$, *unstable for* $\tau > \tau_c$, *and there is a Hopf bifurcation at* $\tau = \tau_c$. *Furthermore, the Hopf bifurcation is transversal.*

The proof of Theorem (3.18) is similar to the one-dimensional case.

It remains an interesting and difficult question as to verify (A1) and (A2) analytically. If L is large, the assumption (A1) is verified in [75] and assumption (A2) holds true if

$$1 < p < 1 + \frac{4}{N}. \tag{3.77}$$

This recovers the results of [76].

It is difficult to verify (A1) and (A2) in general domains. One may ask: does (A1) hold true for *generic* domains?

3.6. *The stability of boundary spikes for the Robin boundary condition*

The stability of least energy solution in the Robin boundary condition case is quite complicated. We state the following result which deals with one-dimensional case only:

THEOREM 3.19. (See [45].) *Consider the following*

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{\xi^q}, \quad 0 < x < 1,\ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 a^r \, dx, \\ a > 0, \quad \epsilon a_x(0,t) + \lambda a(0,t) = \epsilon a_x(1,t) + \lambda a(1,t) = 0, \\ h_x(0,t) = h_x(1,t) = 0. \end{cases} \tag{3.78}$$

Assume that $r = 2, 1 < p \leqslant 3$ *or* $r = p+1, 1 < p < +\infty$. *Then for each* $\lambda \in (0,1)$ *the least energy solution is stable for* $\tau < \tau_1$ *and unstable for* $\tau > \tau_1$. *At* τ_1, *there is a Hopf bifurcation*.

The main idea of the proof is similar to that of Theorem 3.14. Here we have to study an NLEP on a half line with Robin boundary condition:

$$\begin{cases} \phi'' - \phi + p w_{x_0}^{p-1} \phi - \gamma(p-1) \frac{\int_0^\infty w_{x_0} \phi}{\int_0^\infty w_{x_0}^2} w_{x_0}^p = \alpha \phi, \quad 0 < y < +\infty, \\ \phi'(0) - \lambda \phi(0) = 0 \end{cases} \tag{3.79}$$

where $w_{x_0} = w(y - x_0)$ with $w'(-x_0) = \lambda w(-x_0)$. Let $L_{x_0}(\phi) = \phi'' - \phi + p w_{x_0}^{p-1} \phi$. Then we need to show that

$$\int_0^\infty w_{x_0} \left[L_{x_0}^{-1}(w_{x_0}) \right] > 0. \tag{3.80}$$

By some lengthy computations, we can show that the function $\int_0^\infty w_{x_0} [L_{x_0}^{-1}(w_{x_0})]$ is an increasing function in x_0 when $p < 3$, and a constant when $p = 3$, and an decreasing function when $p > 3$.

REMARK 3.6.1. An interesting phenomena is the case of $3 < p < 5$. In this case, one can show that there exists a $a_0 \in (0,1)$ such that the boundary spike is stable when $a \in (0, a_0)$ and unstable when $a \in (a_0, 1)$. It is quite interesting to see that the Robin boundary condition can also introduce some instability.

4. Full Gierer–Meinhardt system: One-dimensional case

In this section, we study the full Gierer–Meinhardt system in the one-dimensional case. Unlike the shadow system case, where one can reduce the existence of solutions to a variational elliptic problem, there is no variational structure for the full Gierer–Meinhardt system. This is the major problem, which is also the source of all interesting new phenomena.

We begin with the steady-state problem in the full space case.

4.1. *Bound states: the case of $\Omega = \mathbb{R}^1$*

Let $\Omega = \mathbb{R}^1$. By a change of variables the steady-state problem for (GM) can be conveniently written as follows

$$\begin{cases} \Delta a - a + \frac{a^p}{h^q} = 0,\ a > 0 & \text{in } \mathbb{R}^1, \\ \Delta h - \sigma^2 h + \frac{a^r}{h^s} = 0,\ h > 0 & \text{in } \mathbb{R}^1, \\ a(x),\ h(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{4.1}$$

where

$$\sigma^2 = \frac{\epsilon^2}{D} \ll 1.$$

The existence of multiple spikes solutions to (4.1) is referred to as "symmetry-breaking" phenomena. This was proved in [11] and [12] (by dynamical system techniques) and [7] (by PDE methods). We will sketch the PDE methods in Section 5.1.

THEOREM 4.1. (See [7,12].) *For each fixed positive integer k, there exists $\sigma_k > 0$ such that problem* (4.1) *has a solution (a_ϵ, h_ϵ) with the following properties*

$$a_\epsilon(x) \sim \frac{c_k}{\sigma}\left(\sum_{j=1}^{k} w\big(x - \xi_j^\sigma\big)\right) \tag{4.2}$$

where $c_k > 0$ is a generic constant and

$$\xi_j^\sigma = \left(j - \frac{k+1}{2}\right)\log\frac{1}{\sigma} + O\left(\log\log\frac{1}{\sigma}\right), \quad j = 1, \ldots, k. \tag{4.3}$$

4.2. *The bounded domain case: existence of symmetric K-spikes*

Without loss of generality, we may assume that $\Omega = (-1, 1)$. We consider the following elliptic system

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0, & -1 < x < 1, \\ D h'' - h + \frac{a^r}{h^s} = 0, & -1 < x < 1, \\ a'(\pm 1) = h'(\pm 1) = 0. \end{cases} \tag{4.4}$$

In this case, the existence of multiple-peaked solutions was first obtained by I. Takagi in [67].

THEOREM 4.2. (See [67].) *Fix any positive integer K. If $\frac{\epsilon}{\sqrt{D}}$ sufficiently small, there exists a K-peaked solution $(a_{\epsilon,K}, h_{\epsilon K})$ to* (4.4) *such that $(a_{\epsilon,K}, h_{\epsilon,K})$ has exactly K local* *maximum points* $-1 < x_1 < x_2 < \cdots < x_K < 1$ *which are equally distributed. In fact, we have*

$$x_j = -1 + \frac{2j-1}{K}, \quad j = 1, \ldots, K.$$

Takagi's proof uses the symmetry of the problems: by reflection, one can reduce the existence of multiple symmetric spikes solutions to studying the existence of one boundary spike solution. Namely, we just need to study the following system

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0, \quad 0 < x < \frac{1}{2K}, \\ Dh'' - h + \frac{a^r}{h^s} = 0, \quad 0 < x < \frac{1}{2K}, \\ a(x) \sim \xi^{\frac{q}{p-1}} w(\frac{x}{\epsilon}), \quad h(0) = \xi, \\ a'(0) = a'(\frac{1}{2K}) = h'(0) = h'(\frac{1}{2K}) = 0. \end{cases} \tag{4.5}$$

For the one boundary spike solution, one can use the Implicit Function Theorem, since the linearized operator is invertible in the space of functions with Neumann boundary conditions. (The last statement follows from the fact that the kernel of the operator $\Delta - 1 + pw^{p-1}$ consists exactly those of partial derivatives of w. See Lemma 3.2.)

4.3. *The bounded domain case: existence of asymmetric K-spikes*

In the bounded domain case, as D is getting smaller, more and more new solutions appear. By using the same matched asymptotic analysis in [34], M. Ward and Wei in [70] discovered that for $D < D_K$, where D_K is given by (4.67) below, problem (4.4) has *asymmetric* K-peaked steady-state solutions. Such asymmetric solutions are generated by two types of peaks-called type **A** and type **B**, respectively. Type **A** and type **B** peaks have *different heights*. They can be arranged in any given order

ABAABBB...**ABBBA**...**B**

to form an K-peaked solution. The existence of such solutions is surprising. It shows that the solution structure of (4.4) is much more complicated than one would expect. The stability of such asymmetric K-peaked solutions is also studied in [70], through a formal approach. We remark that asymmetric patterns can also be obtained for the Gierer–Meinhardt system on the real line, see [12].

In this and next section, we present a *rigorous and unified theoretic foundation* for the existence and stability of general K-peaked (*symmetric* or *asymmetric*) solutions. In particular, the results of [34] and [70] are rigorously established. Moreover, we show that if the K peaks are separated, then they are generated by peaks of type **A** and type **B**, respectively. This implies that there are only two kinds of K-peaked patterns: symmetric K-peaked solutions constructed in [67] and asymmetric K-peaked patterns constructed in [70].

The existence proof is based on Lyapunov–Schmidt reduction. Stability is proved by first separating the problem into the case of large eigenvalues which tend to a nonzero limit and the case of small eigenvalues which tend to zero in the limit $\epsilon \to 0$. Large eigenvalues are then explored by studying nonlocal eigenvalue problems using results in Section 3.3 and employing an idea of Dancer [8]. Small eigenvalues are calculated explicitly by an asymptotic analysis with rigorous error estimates.

In this section, we present the existence part.

Before we state our main results, we introduce some notation. Let $G_D(x,z)$ be the Green function of

$$\begin{cases} DG''_D(x,z) - G_D(x,z) + \delta_z i(x) = 0 \quad \text{in } (-1,1), \\ G'_D(-1,z) = G'_D(1,z) = 0. \end{cases} \tag{4.6}$$

We can calculate explicitly

$$G_D(x,z) = \begin{cases} \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)] \cosh[\theta(1-z)], & -1 < x < z, \\ \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1-x)] \cosh[\theta(1+z)], & z < x < 1 \end{cases} \tag{4.7}$$

where

$$\theta = D^{-1/2}.$$

We set

$$K_D\big(|x-z|\big) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x-z|}, \tag{4.8}$$

to be the singular part of $G_D(x,z)$ and by $G_D = K_D - H_D$ we define the regular part H_D of G_D. Note that H_D is C^∞ in both x and z.

Let $-1 < t_1^0 < \cdots < t_j^0 < \cdots < t_K^0 < 1$ be K points in $(-1,1)$ and w be the unique solution of (2.8).

Put

$$\xi_\epsilon := \left(\epsilon \int_R w^r(z)\, dz\right)^{\frac{p-1}{(p-1)(s+1)-qr}}. \tag{4.9}$$

We introduce several matrices for later use: For $\mathbf{t} = (t_1, \ldots, t_K) \in (-1,1)^K$, let

$$\mathcal{G}_D(\mathbf{t}) = \big(G_D(t_i, t_j)\big). \tag{4.10}$$

Let us denote $\frac{\partial}{\partial t_i}$ as ∇_{t_i}. When $i \neq j$, we can define $\nabla_{t_i} G(t_i, t_j)$ in the classical way. When $i = j$, $K_D(|t_i - t_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant and we define

$$\nabla_{t_i} G_D(t_i, t_i) := -\frac{\partial}{\partial x}\bigg|_{x=t_i} H(x, t_i).$$

Similarly, we define

$$\nabla_{t_i}\nabla_{t_j} G_D(t_i,t_j) = \begin{cases} \frac{\partial}{\partial x}\Big|_{x=t_i} \frac{\partial}{\partial y}\Big|_{y=t_i} H_D(x,y) & \text{if } i=j, \\ \nabla_{t_i}\nabla_{t_j} G_D(t_i,t_j) & \text{if } i\neq j. \end{cases} \tag{4.11}$$

Now the derivatives of $\mathcal{G}$ are defined as follows:

$$\nabla \mathcal{G}_D(\mathbf{t}) = \big(\nabla_{t_i} G_D(t_i,t_j)\big), \tag{4.12}$$

$$\nabla^2 \mathcal{G}_D(\mathbf{t}) = \big(\nabla_{t_i}\nabla_{t_j} G_D(t_i,t_j)\big). \tag{4.13}$$

We now have our first assumption:
(H1) There exists a solution $(\hat{\xi}_1^0,\dots,\hat{\xi}_N^0)$ of the following equation

$$\sum_{j=1}^{N} G_D\big(t_i^0,t_j^0\big)\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s} = \hat{\xi}_i^0, \quad i=1,\dots,N. \tag{4.14}$$

Next we introduce the following matrix

$$b_{ij} = G_D\big(t_i^0,t_j^0\big)\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s-1}, \quad \mathcal{B}=(b_{ij}). \tag{4.15}$$

Our second assumption is the following:
(H2) It holds that

$$\frac{p-1}{qr-s(p-1)} \notin \sigma(\mathcal{B}), \tag{4.16}$$

where $\sigma(\mathcal{B})$ is the set of eigenvalues of $\mathcal{B}$.

REMARK 4.3.1. Since the matrix $\mathcal{B}$ is of the form $\mathcal{G}_D\mathcal{D}$, where $\mathcal{G}_D$ is symmetric and $\mathcal{D}$ is a diagonal matrix, it is easy to see that the eigenvalues of $\mathcal{B}$ are real.

By the assumption (H2) and the implicit function theorem, for $\mathbf{t}=(t_1,\dots,t_K)$ near $\mathbf{t}_0=(t_1^0,\dots,t_K^0)$, there exists a unique solution $\hat{\xi}(\mathbf{t})=(\hat{\xi}_1(\mathbf{t}),\dots,\hat{\xi}_K(\mathbf{t}))$ for the following equation

$$\sum_{j=1}^{K} G_D(t_i,t_j)\hat{\xi}_j^{\frac{qr}{p-1}-s} = \hat{\xi}_i, \quad i=1,\dots,K. \tag{4.17}$$

Set

$$\mathcal{H}(\mathbf{t}) = \big(\hat{\xi}_i(\mathbf{t})\delta_{ij}\big). \tag{4.18}$$

We define the following vector field:

$$F(\mathbf{t}) = \big(F_1(\mathbf{t}), \dots, F_K(\mathbf{t})\big),$$

where

$$\begin{aligned} F_i(\mathbf{t}) &= \sum_{l=1}^{K} \nabla_{t_i} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s} \\ &= -\nabla_{t_i} H_D(t_i, t_i) \hat{\xi}_i^{\frac{qr}{p-1}-s} + \sum_{l \neq i} \nabla_{t_i} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s}, \\ &\quad i = 1, \dots, K. \end{aligned} \tag{4.19}$$

Set

$$\mathcal{M}(\mathbf{t}) = \big(\hat{\xi}_i^{-1} \nabla_{t_j} F_i(\mathbf{t})\big). \tag{4.20}$$

Our final assumption concerns the vector field $F(\mathbf{t})$.
(H3) We assume that at $\mathbf{t}_0 = (t_1^0, \dots, t_K^0)$:

$$F(\mathbf{t}_0) = 0, \tag{4.21}$$

$$\det\big(\mathcal{M}(\mathbf{t}_0)\big) \neq 0. \tag{4.22}$$

Let us now calculate $\mathcal{M}(\mathbf{t}^0)$: Therefore we first compute the derivatives of $\hat{\xi}$. It is easy to see that $\hat{\xi}(\mathbf{t})$ is C^1 in $\mathbf{t}$. We can calculate:

$$\begin{aligned} \nabla_{t_j} \hat{\xi}_i &= \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{K} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_j} \hat{\xi}_l \\ &\quad + \sum_{l=1}^{K} \frac{\partial}{\partial t_j} \big(G_D(t_i, t_l)\big) \hat{\xi}_l^{\frac{qr}{p-1}-s}. \end{aligned}$$

For $i \neq j$, we have

$$\nabla_{t_j} \hat{\xi}_i = \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{N} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_j} \hat{\xi}_l + \nabla_{t_j} G_D(t_i, t_j) \hat{\xi}_j^{\frac{qr}{p-1}-s}.$$

For $i = j$, we have

$$\nabla_{t_j} \hat{\xi}_i = \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{K} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_i} \hat{\xi}_l + \sum_{l=1}^{K} \frac{\partial}{\partial t_i} \big(G_D(t_i, t_l)\big) \hat{\xi}_l^{\frac{qr}{p-1}-s}$$

$$= \left(\frac{qr}{p-1} - s\right) \sum_{l=1}^{K} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s-1} \nabla_{t_i} \hat{\xi}_l + \nabla_{t_i} G_D(t_i, t_i) \hat{\xi}_i^{\frac{qr}{p-1}-s}$$

$$+ \sum_{l=1}^{K} \nabla_{t_i} G_D(t_i, t_l) \hat{\xi}_l^{\frac{qr}{p-1}-s},$$

since $\frac{\partial}{\partial t_i} G_D(t_i, t_i) = 2\nabla_{t_i} G_D(t_i, t_i)$.

Note that

$$\left(\nabla_{t_j} G_D(t_i, t_j)\right) = (\nabla \mathcal{G}_D)^T.$$

Therefore if we denote the matrix

$$\nabla \xi = (\nabla_{t_j} \hat{\xi}_i) \tag{4.23}$$

then we have

$$\nabla \xi(\mathbf{t}) = \left(I - \left(\frac{qr}{p-1} - s\right) \mathcal{G}_D \mathcal{H}^{\frac{qr}{p-1}-s-1}\right)^{-1} (\nabla \mathcal{G}_D)^T \mathcal{H}^{\frac{qr}{p-1}-s}$$

$$+ O\left(\sum_{j=1}^{K} \left|F_j(\mathbf{t})\right|\right). \tag{4.24}$$

We can compute $\mathcal{M}(\mathbf{t}^0)$ by using (4.24):

$$\mathcal{M}\left(\mathbf{t}^0\right) = \mathcal{H}^{-1} \nabla^2 \mathcal{G}_D \mathcal{H}^{\frac{qr}{p-1}-s}$$

$$+ \mathcal{H}^{-1} \left(\frac{qr}{p-1} - s\right) \nabla \mathcal{G}_D \mathcal{H}^{\frac{qr}{p-1}-s-1}$$

$$\times \left(I - \left(\frac{qr}{p-1} - s\right) \mathcal{G}_D \mathcal{H}^{\frac{qr}{p-1}-s-1}\right)^{-1} (\nabla \mathcal{G}_D)^T \mathcal{H}^{\frac{qr}{p-1}-s}. \tag{4.25}$$

The existence result is as follows

THEOREM 4.3. (See [83].) *Assume that assumptions* (H1), (H2) *and* (H3) *are satisfied. Then for* $\epsilon \ll 1$, *problem* (4.4) *has an* K*-peaked solution which concentrates at* $t_1^\epsilon, \ldots, t_K^\epsilon$, *or more precisely*:

$$a_\epsilon(x) \sim \sum_{j=1}^{K} \xi_\epsilon^{\frac{q}{p-1}} \left(\hat{\xi}_j^0\right)^{\frac{q}{p-1}} w\left(\frac{x - t_j^\epsilon}{\epsilon}\right), \tag{4.26}$$

$$h_\epsilon\left(t_i^\epsilon\right) \sim \xi_\epsilon \hat{\xi}_i^0, \quad i = 1, \ldots, K, \tag{4.27}$$

$$t_i^\epsilon \to t_i^0, \quad i = 1, \ldots, K. \tag{4.28}$$

REMARK 4.3.2. In the case of symmetric K-peaked solutions, conditions (H2) and (H3) are not needed, as in the construction of solutions one can restrict the function space to the class of symmetric functions (see for example [67]). Note that for small ϵ (and not only in the limit $\epsilon \to 0$) the peaks are placed equidistantly.

REMARK 4.3.3. Our results here can be applied to give a *rigorous proof* for the existence and stability of K-peaked solutions consisting of peaks with *different heights.*

In [70], by using matched asymptotic analysis, Ward and the first author constructed such solutions and studied their stability. We now summarize their main ideas. First (4.4) is solved in a small interval $(-l, l)$:

$$\begin{cases} \epsilon^2 a'' - a + \frac{a^p}{h^q} = 0 & \text{in } (-l, l), \\ Dh'' - h + \frac{a^r}{h^s} = 0 & \text{in } (-l, l), \\ a(x) > 0, h(x) > 0 & \text{in } (-l, l), \\ a'(-l) = a'(l) = h'(-l) = h'(l) = 0. \end{cases} \tag{4.29}$$

Then the single interior symmetric spike solution is considered which was constructed by I. Takagi [67]. By some simple computations based on (4.6), we have that

$$h(l) \sim c(D) b\left(\frac{l}{\sqrt{D}}\right), \tag{4.30}$$

where $c(D)$ is some positive constant depending on D only and the function $b(z)$ is given by

$$b(z) := \frac{\tanh^{\alpha}(z)}{\cosh(z)}, \quad \alpha := \frac{(p-1)}{qr - (s+1)(p-1)}. \tag{4.31}$$

The idea now is that we fix l and try to find another $\bar{l} \neq l$ such that the following holds

$$b\left(\frac{l}{\sqrt{D}}\right) = b\left(\frac{\bar{l}}{\sqrt{D}}\right), \quad 0 < l < \bar{l} < 1, \tag{4.32}$$

which will imply that $h(l) = h(\bar{l})$. This shows that if there exists a solution to (4.32), we may match up $h(l)$ and $h(\bar{l})$. In other words, we may match up solutions of (4.29) in different intervals.

It turns out that for D small, (4.32) is always solvable. Now (4.32) has to be solved along with the following interval constraint:

$$K_1 l + K_2 \bar{l} = 1, \qquad K_1 + K_2 = K. \tag{4.33}$$

For a solution l of (4.60) and (4.33) and $j = 1, \ldots, K$ we define

$$l_j = l \quad \text{or} \quad l_j = \bar{l} \tag{4.34}$$

where the number of j's such that $l_j = l$ is K_1 (and consequently the number of j's such that $l_j = \bar{l}$ is K_2). We call the small spike with $l_j = l$ type **A** and the large spike with $l_j = \bar{l}$ type **B**.

Then we choose t_j^0 such that

$$\left|t_j^0 - t_{j+1}^0\right| = l_j + l_{j-1}, \quad j = 0, \ldots, K,$$

where $t_0^0 = -1, t_{K+1}^0 = 1$.

By using matched asymptotics, we now have K_1 type **A** and K_2 type **B** peaks. This ends the short review of the ideas in [70]. Let us now use Theorem 4.3 to give a rigorous proof of results of [70]. In order to apply Theorem 4.3, we have to check the three assumptions (H1), (H2) and (H3).

To this end, let us set

$$\hat{\xi}_j^0 = (2\sqrt{D}) \tanh(\theta_j), \quad j = 1, \ldots, K, \tag{4.35}$$

where

$$\theta_j = \frac{l_j}{\sqrt{D}}. \tag{4.36}$$

It is difficult to check (H1) directly. Instead we note that $\mathcal{G}_D^{-1}$ is a tridiagonal matrix. (See [34] and [70].) More precisely, we calculate

$$\mathcal{G}_D^{-1} = (a_{ij}) = 2\sqrt{D} \begin{pmatrix} \gamma_1 & \beta_1 & 0 & \ddots & \ddots & 0 \\ \beta_1 & \gamma_2 & \beta_2 & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \beta_{j-1} & \gamma_j & \beta_j & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \beta_{N-1} & \gamma_N \end{pmatrix}$$

where

$$\gamma_1 = \coth(\theta_1 + \theta_2) + \tanh(\theta_1),$$

$$\gamma_j = \coth(\theta_{j-1} + \theta_j) + \coth(\theta_j + \theta_{j+1}), \quad j = 2, \ldots, K-1,$$

$$\gamma_K = \coth(\theta_{K-1} + \theta_K) + \tanh(\theta_K),$$

$$\beta_j = -\operatorname{csch}(\theta_j + \theta_{j+1}), \quad j = 1, \ldots, N-1$$

and θ_j was defined in (4.36). Note that

$$a_{ij} = 2\sqrt{D}(\beta_j \delta_{i(j-1)} + \gamma_j \delta_{ij} + \beta_{j+1} \delta_{i(j+1)}). \tag{4.37}$$

Verifying (4.14) amounts to checking the following identity

$$\sum_{j=1}^{N} a_{ij}\hat{\xi}_j^0 = \left(\hat{\xi}_i^0\right)^{\frac{qr}{p-1}-s}, \tag{4.38}$$

which is an easy exercise.

It remains to verify (H2) and (H3).

To this end, we need to know the eigenvalues of $\mathcal{B}$ and $\mathcal{M}$. In the same way as for the matrix $\mathcal{G}_D$, one can show that $\mathcal{B}^{-1}$ is a tridiagonal matrix. However, it is almost impossible to obtain an explicit formula for the eigenvalues. Numerical software for solving eigenvalue problems of large matrices is indispensable. Then (H2) has to be checked explicitly. Numerical computations in [70] do suggest that assumption (H3) is always satisfied.

4.4. *Classification of asymmetric patterns*

A natural question is the following: Are all K-peaked solution generated by two types of peaks as the solutions which were constructed in [70]?

Our next theorem gives an affirmative answer. It completely classifies all K-peaked solutions, provided that the K peaks are separated.

THEOREM 4.4. (See [83].) *Suppose that for ϵ sufficiently small, there are solutions (a_ϵ, h_ϵ) of* (4.4) *such that*

$$a_\epsilon(x) \sim \sum_{j=1}^{K} \xi_\epsilon^{\frac{q}{p-1}} \left(\hat{\xi}_j^\epsilon\right)^{\frac{q}{p-1}} w\left(\frac{x - t_j^\epsilon}{\epsilon}\right), \tag{4.39}$$

and

$$h_\epsilon\left(t_i^\epsilon\right) \sim \xi_\epsilon \hat{\xi}_i^\epsilon, \quad i = 1, \dots, K, \tag{4.40}$$

where ξ_ϵ is given by (4.9),

$$\hat{\xi}_i^\epsilon \to \hat{\xi}_i^0 > 0, \quad t_i^\epsilon \to t_i^0, \quad t_i^0 \neq t_j^0, \quad i \neq j,\ i, j = 1, \dots, K. \tag{4.41}$$

Then necessarily, we have

$$l_i := t_i^0 - t_{i-1}^0 \in \{l, \bar{l}\}, \quad i = 1, \dots, K, \tag{4.42}$$

where $t_0^0 = -1$, l and $\bar{l}$ satisfy (4.32) *and* (4.33) *with K_1 being the number of i's for which $l_i = l$ and K_2 being the number of i's for which $l_i = \bar{l}$ (hence $K_1 + K_2 = K$).*

Theorem 4.4 shows that an K-peaked solution must be generated by exactly two types of peaks—type **A** with shorter length l and type **B** with larger length $\bar{l}$. This shows that the solutions constructed in [70] (through a formal approach) exhaust all possible separated K-peaked solutions. In particular, it shows that there are at most 2^K K-peaked solutions. If the assumptions (H1)–(H3) are met, then there are exactly 2^K K-peaked solutions.

PROOF OF THEOREM 4.4. First we make the following scaling

$$a_\epsilon = \xi_\epsilon^{\frac{q}{p-1}} \hat{a}_\epsilon, \quad h_\epsilon = \xi_\epsilon \hat{h}_\epsilon$$

where ξ_ϵ is defined at (4.9). Hence $(\hat{a}_\epsilon, \hat{h}_\epsilon)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \hat{a}_\epsilon - \hat{a}_\epsilon + \frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q} = 0, & -1 < x < 1, \\ D\Delta \hat{h}_\epsilon - \hat{h}_\epsilon + c_\epsilon \frac{\hat{a}_\epsilon^r}{\hat{h}_\epsilon^s} = 0, & -1 < x < 1, \end{cases} \tag{4.43}$$

where c_ϵ is defined as $c_\epsilon = (\epsilon \int_R w^r)^{-1}$.

Now (4.39) and (4.40) imply that

$$\hat{a}_\epsilon \sim \sum_{j=1}^{K} (\hat{\xi}_j^\epsilon)^{\frac{q}{p-1}} w\left(\frac{x - t_j^\epsilon}{\epsilon}\right), \quad \hat{h}_\epsilon(t_j^\epsilon) = \hat{\xi}_j^\epsilon. \tag{4.44}$$

Letting $\epsilon \to 0$, we assume that

$$\hat{\xi}_j^\epsilon \to \hat{\xi}_j^0, \quad t_j^\epsilon \to t_j^0, \quad j = 1, \ldots, K.$$

We see that $\hat{h}_\epsilon \to h_0(x)$ where $h_0(x)$ satisfies

$$\begin{cases} D\Delta h_0 - h_0 + \sum_{j=1}^K (\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} \delta(x - t_j^0) = 0, & -1 < x < 1, \\ h_0'(-1) = h_0'(1) = 0. \end{cases} \tag{4.45}$$

In other words, we have

$$h_0(x) = \sum_{j=1}^{K} (\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} G_D(x, t_j^0). \tag{4.46}$$

Since $h_0(t_j^0) = \hat{\xi}_j^0$, $j = 1, \ldots, K$, we have from (4.46) that $(\hat{\xi}_1^0, \ldots, \hat{\xi}_K^0)$ must satisfy the following identity:

$$\sum_{j=1}^{K} G_D(t_i^0, t_j^0)(\hat{\xi}_j^0)^{\frac{qr}{p-1} - s} = \hat{\xi}_i^0, \quad i = 1, \ldots, K. \tag{4.47}$$

This is the same as (4.14).

Define

$$\tilde{a}_{\epsilon,j} = \hat{a}_\epsilon \chi\left(\frac{x - t_j^0}{\tilde{r}_0}\right)$$

where $\tilde{r}_0$ is a very small number. Then $\tilde{a}_{\epsilon,j}$ is supported in the interval $I_j^\epsilon = (-\tilde{r}_0 + t_j^\epsilon, \tilde{r}_0 + t_j^\epsilon)$. We may choose $\tilde{r}_0$ so small that $I_i^\epsilon \cap I_j^\epsilon = \emptyset$ for $i \neq j$. Then

$$\hat{a}_\epsilon = \sum_{j=1}^{K} \tilde{a}_{\epsilon,j} + \text{e.s.t.}$$

Now we multiply the first equation in (4.43) by $\tilde{a}'_{\epsilon,j}$ and integrate over $(-1, 1)$. We obtain

$$\begin{aligned} 0 &= \int_{-1}^{1} \left[\left(\frac{\hat{a}_\epsilon{}^p}{\hat{h}_\epsilon{}^q}\right)\tilde{a}'_{\epsilon,j} - \left(\frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q}\right)'\tilde{a}_{\epsilon,j}\right] \\ &= -2\int_{I_j^\epsilon} \left(\frac{\hat{a}_\epsilon^p}{\hat{h}_\epsilon^q}\right)'\hat{a}_\epsilon + \text{e.s.t.} \\ &= -2\int_{I_j^\epsilon} \left[\frac{p\hat{a}_\epsilon^p \hat{a}'_\epsilon}{\hat{h}_\epsilon^q} - \frac{q\hat{a}_\epsilon^{p+1}\hat{h}'_\epsilon}{\hat{h}_\epsilon^{q+1}}\right] + \text{e.s.t.} \\ &= \frac{q(p+2)}{p+1}\int_{I_j^\epsilon} \frac{\hat{a}_\epsilon^{p+1}}{\hat{h}_\epsilon^{q+1}}\hat{h}'_\epsilon + \text{e.s.t.} \end{aligned} \tag{4.48}$$

By the equation for $\hat{h}_\epsilon$, we have that

$$\hat{h}_\epsilon(x) = c_\epsilon \int_{-1}^{1} G_D(x, z)\frac{\hat{a}_\epsilon^r}{\hat{h}_\epsilon^s}$$

and thus for $x \in I_j^\epsilon$,

$$\hat{h}_\epsilon(x) = \sum_{k=1}^{K} G_D\big(x, t_k^\epsilon\big)\big(\hat{\xi}_k^\epsilon\big)^{\frac{qr}{p-1} - s} + O(\epsilon)$$

and

$$\hat{H}'_\epsilon\big(t_j^\epsilon\big) = \sum_{k=1}^{K} \nabla_{t_j^\epsilon} G_D\big(t_j^\epsilon, t_k^\epsilon\big)\big(\hat{\xi}_k^\epsilon\big)^{\frac{qr}{p-1} - s} + O(\epsilon). \tag{4.49}$$

Substituting (4.49) into (4.48) and using (4.44), we obtain the following identity

$$\sum_{k=1}^{K} \nabla_{t_j^\epsilon} G_D\big(t_j^\epsilon, t_k^\epsilon\big)\big(\hat{\xi}_k^\epsilon\big)^{\frac{qr}{p-1}-s} = o(1) \tag{4.50}$$

and hence

$$\sum_{k=1}^{K} \nabla_{t_j^0} G_D\big(t_j^0, t_k^0\big)\big(\hat{\xi}_k^0\big)^{\frac{qr}{p-1}-s} = 0, \quad j = 1, \ldots, K, \tag{4.51}$$

which is the same as (4.21).

Note that by the expression for h_0 in (4.46), (4.51) is equivalent to the following

$$h_0'\big(t_j^0+\big) + h_0'\big(t_j^0-\big) = 0, \quad j = 1, \ldots, K, \tag{4.52}$$

where $h_0'(t_j^0+)$ is the right-hand derivative of h_0 at t_j^0 and $h_0'(t_j^0-)$ is the left-hand derivative of h_0 at t_j^0. On the other hand, from the equation for h_0, we have that

$$D\big(h_0'\big(t_j^0+\big) - h_0'\big(t_j^0-\big)\big) = -\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s}, \quad j = 1, \ldots, K. \tag{4.53}$$

Solving (4.52) and (4.53), we have that

$$h_0'\big(t_j^0+\big) = -h_0'\big(t_j^0-\big) = -\frac{1}{2D}\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s} < 0, \quad j = 1, \ldots, K. \tag{4.54}$$

Since h_0 satisfies $Dh_0'' = h_0 > 0$ in each interval (t_{j-1}^0, t_j^0), $j = 2, \ldots, K$, we see that there exists a unique point $s_{j-1} \in (t_{j-1}^0, t_j^0)$ such that $h_0'(s_{j-1}) = 0$. Since $h_0'(-1) = 0$, by using symmetry, we see that

$$\frac{s_{j-1} + s_j}{2} = t_j^0, \quad j = 1, \ldots, K, \tag{4.55}$$

where we take $s_0 = -1$, $s_K = 1$. Let $2l_j = s_j - s_{j-1}$, $j = 1, \ldots, K$.

Note that on each interval $(-l_j + t_j^0, l_j + t_j^0)$, h_0 satisfies

$$D\Delta h_0 - h_0 + \big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s}\delta\big(t - t_j^0\big) = 0$$

with Neumann boundary conditions at both ends. Thus from (4.6) it is easy to see that

$$\big(\hat{\xi}_j^0\big)^{\frac{qr}{p-1}-s-1} = 2\sqrt{D}\tanh\bigg(\frac{l_j}{\sqrt{D}}\bigg), \quad j = 1, \ldots, K, \tag{4.56}$$

$$h_0(l_j) = \frac{\hat{\xi}_j^0}{\cosh(\frac{l_j}{\sqrt{D}})}. \tag{4.57}$$

Since h_0 is continuous on $(-1,1)$, we have

$$h_0(l_1) = h_0(l_2) = \cdots = h_0(l_K). \tag{4.58}$$

Using (4.56) and (4.57), we see that (4.58) is equivalent to

$$b\left(\frac{l_1}{\sqrt{D}}\right) = b\left(\frac{l_2}{\sqrt{D}}\right) = \cdots = b\left(\frac{l_K}{\sqrt{D}}\right), \tag{4.59}$$

where the function b was defined in (4.31). Suppose without loss of generality that $l_1 \leqslant l_2$, then we take $l_1 = l$ and (4.59) implies that $l_2 \in \{l, \bar{l}\}$ and that $l_j \in \{l, \bar{l}\}$ for $j = 2, \ldots, K$. Thus l must satisfy (4.60) and (4.33).

This finishes the proof of Theorem 4.4. □

4.5. *The stability of symmetric and asymmetric K-spikes*

In this section, we present the stability of the K-peaked solutions constructed in Theorem 4.3.

To this end, we need to study the following linearized eigenvalue problem

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + p\frac{a_\epsilon^{p-1}}{H_\epsilon^q}\phi_\epsilon - q\frac{a_\epsilon^p}{h_\epsilon^{q+1}}\psi_\epsilon, \\ \frac{1}{\tau}\left(D\Delta\psi_\epsilon - \psi_\epsilon + r\frac{a_\epsilon^{r-1}}{h_\epsilon^s}\phi_\epsilon - s\frac{a_\epsilon^r}{h_\epsilon^{s+1}}\psi_\epsilon\right) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \tag{4.60}$$

where (a_ϵ, h_ϵ) is the solution constructed in Theorem 4.3 and $\lambda_\epsilon \in \mathcal{C}$—the set of complex numbers.

We say that (a_ϵ, h_ϵ) is *linearly stable* if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathcal{C}\colon \operatorname{Re}(\lambda) < 0\}$. (a_ϵ, h_ϵ) is called *linearly unstable* if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\operatorname{Re}(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable as defined above.)

THEOREM 4.5. *Let (a_ϵ, h_ϵ) be the solutions constructed in Theorem* 4.3. *Assume that $\epsilon \ll 1$.*

(1) (*Stability*) *If*

$$r = 2, \ < p < 5 \quad \text{or} \quad r = p + 1, \ < p < +\infty \tag{4.61}$$

and furthermore

$$\left(\frac{qr}{p-1} - s\right) \min_{\sigma \in \sigma(\mathcal{B})} \sigma > 1 \tag{4.62}$$

and

$$\sigma(\mathcal{M}) \subseteq \left\{\sigma \,\middle|\, \operatorname{Re}(\sigma) > 0\right\}, \tag{4.63}$$

there exists $\tau_0 > 0$ *such that* (a_ϵ, h_ϵ) *is linearly stable for* $0 \leqslant \tau < \tau_0$.

(2) (*Instability*) *If*

$$\left(\frac{qr}{p-1} - s\right) \min_{\sigma \in \sigma(\mathcal{B})} \sigma < 1, \tag{4.64}$$

there exists $\tau_0 > 0$ *such that* (a_ϵ, h_ϵ) *is linearly unstable for* $0 \geqslant \tau < \tau_0$.

(3) (*Instability*) *If there exists*

$$\sigma \in \sigma(\mathcal{M}), \quad \mathrm{Re}(\sigma) < 0, \tag{4.65}$$

then (a_ϵ, h_ϵ) *is linearly unstable for all* $\tau > 0$.

REMARK 4.5.1. In the original Gierer–Meinhardt model, $(p, q, r, s) = (2, 1, 2, 0)$ or $(p, q, r, s) = (2, 4, 2, 0)$. This means that condition (4.61) is satisfied.

REMARK 4.5.2. By Theorems 4.3 and 4.5, the existence and stability of K-peaked solutions are completely determined by the two matrices $\mathcal{B}$ and $\mathcal{M}$. They are related to the asymptotic behavior of large eigenvalues which tend to a nonzero limit and small eigenvalues which tend to zero as $\epsilon \to 0$, respectively. The computations of these two matrices are by no means easy. We refer to [34] and [70] for exact computations and numerics. Combining the results here and the computations in [34], the stability of symmetric K-peaked solutions is completely characterized and the following result is established rigorously.

THEOREM 4.6. (See [34,83].) *Let* $(a_{\epsilon,K}, h_{\epsilon,K})$ *be the symmetric* K*-peaked solutions constructed in* [67]. *Assume that* $\epsilon \ll 1$.

(a) (*Stability*) *Assume that* $0 < \tau < \tau_0$ *for some* τ_0 *small and that*

$$r = 2,\ 1 < p < 5 \quad \textit{or} \quad r = p + 1,\ 1 < p < +\infty \tag{4.66}$$

and

$$D < D_K := \frac{1}{K^2 (\log(\sqrt{\alpha} + \sqrt{\alpha + 1}))^2}, \tag{4.67}$$

where α *is given by* (4.31), *then the symmetric* K*-peaked solution is linearly stable.*

(b) (*Instability*) *If*

$$D > D_K, \tag{4.68}$$

where D_K *is given by* (4.67), *then the symmetric* K*-peaked solution is linearly unstable for all* $\tau > 0$.

The proof of Theorem 4.5 consists of two parts: we have to compute both *small* and *large* eigenvalues. For large eigenvalues, we will arrive at the following system of nonlocal eigenvalue problems (NLEPs)

$$\Phi'' - \Phi + pw^{p-1}\Phi - qr(I + s\mathcal{B})^{-1}\mathcal{B}\left(\int_{\mathbb{R}} w^{r-1}\Phi\right)\left(\int_{\mathbb{R}} w^r\right)^{-1} w^p = \lambda\Phi \tag{4.69}$$

where $\mathcal{B}$ is given by (4.15) and

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in \left(H^2(\mathbb{R})\right)^K.$$

By diagonalization, we may reduce it to K NLEPs of the form (3.17). Using the results of Theorem 3.7, we obtain the stability (or instability) of large eigenvalues.

For the study of small eigenvalues, we need to expand the eigenfunction up to the order $O(\epsilon^2)$ term. This computation is quite involved. In the end, the matrix $\mathcal{B}$ and $\mathcal{M}$ will appear.

A similar stability analysis for the Schnakenberg model has been carried out in [35].

5. The full Gierer–Meinhardt system: Two-dimensional case

Let us now consider the Gierer–Meinhardt system in a two-dimensional domain. The results are more complicated. To reduce the complexity and grasp the essential difficulties, we assume that $(p, q, r, s) = (2, 1, 2, 0)$ in this section.

We start with the bound states.

5.1. *Bound states: spikes on polygons*

We first consider the case when $\Omega = \mathbb{R}^2$:

$$\begin{cases} \Delta a - a + \frac{a^2}{h} = 0, \quad a > 0 & \text{in } \mathbb{R}^2, \\ \Delta h - \sigma^2 h + a^2 = 0, \quad h > 0 & \text{in } \mathbb{R}^2, \\ a(x), h(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{5.1}$$

As we will see, a notable feature of this ground-state problem in the plane is the presence of solutions with any prescribed number of bumps in the activator as the parameter σ gets smaller. These bumps are separated from each other at a distance $O(|\log\log\sigma|)$ and approach a single universal profile given by the unique radial solution of (2.8). The multi-bump solutions correspond respectively to bumps arranged at the vertices of a k-regular polygon and at those of two concentric regular polygons. These arrangements with one extra bump at the origin are also considered. This unveils a new side of the rich and complex structure of the solution set of the Gierer–Meinhardt system in the plane and gives rise to a number of questions.

Let us set

$$\tau_\sigma = \left(\frac{k}{2\pi} \log \frac{1}{\sigma} \int_{\mathbb{R}^2} w^2(y)\, dy \right)^{-1}. \tag{5.2}$$

THEOREM 5.1. (See [17].) *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (5.1) *admits a solution (a, h) with the following property*:

$$\lim_{\sigma \to 0} \left| \tau_\sigma a_\sigma(x) - \sum_{i=1}^{k} w(x - \xi_i) \right| = 0, \tag{5.3}$$

uniformly in $x \in \mathbb{R}^2$. Here the points ξ_i correspond to the vertices of a regular polygon centered at the origin, with sides of equal length l_σ satisfying

$$l_\sigma = \log\log \frac{1}{\sigma} + O\left(\log\log\log \frac{1}{\sigma} \right). \tag{5.4}$$

Finally, for each $1 \leqslant j \leqslant k$ we have

$$\lim_{\sigma \to 0} \left| \tau_\sigma h_\sigma(\xi_j + y) - 1 \right| = 0,$$

uniformly on compact sets in y.

Our second result gives existence of a solution with bumps at vertices of two concentric polygons.

THEOREM 5.2. (See [17].) *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (5.1) *admits a solution (a, h) with the following property*:

$$\lim_{\sigma \to 0} \left| \tau_\sigma a_\sigma(x) - \sum_{i=1}^{k} \left[w(x - \xi_i) + w(x - \xi_i^*) \right] \right| = 0, \tag{5.5}$$

uniformly in $x \in \mathbb{R}^2$. Here the points ξ_i and ξ_i^* are the vertices of two concentric regular polygons. They satisfy

$$\xi_j = \rho_\sigma e^{\frac{2j\pi}{k} i}, \quad \xi_j^* = \rho_\sigma^* e^{\frac{2\pi j}{k} i}, \quad j = 1, \ldots, k,$$

where

$$\rho_\sigma = \frac{1}{|1-e^{\frac{2\pi i}{k}}|}\log\log\frac{1}{\sigma} + O\left(\log\log\log\frac{1}{\sigma}\right),$$

and

$$\rho_\sigma^* = \left(1 + \frac{1}{|1-e^{\frac{2\pi i}{k}}|}\right)\log\log\frac{1}{\sigma} + O\left(\log\log\log\frac{1}{\sigma}\right).$$

A similar assertion to (5.4) *holds for* h_σ, *around each of the* ξ_i *and the* ξ_i^**'s.*

THEOREM 5.3. (See [17].) *Let* $k \geqslant 1$ *be given. Then there exists solutions which are exactly as those in Theorems* 5.1 *and* 5.2 *but with an additional bump at the origin. More precisely, with* $w(x)$ *added to* $\sum_{i=1}^k w(x-\xi_i)$ *in* (5.3) *and added to* $\sum_{i=1}^k [w(x-\xi_i) + w(x-\xi_i^*)]$ *in* (5.5).

The method employed in the proof of the above results consists of a Lyapunov–Schmidt type reduction. The basic idea of solving the second equation in (5.1) for h first and then working with a nonlocal elliptic PDE rather than directly with the system. Let $T(a^2)$ be the unique solution of the equation

$$\begin{aligned} &\Delta h - \sigma^2 h + a^2 = 0 \quad && \text{in } \mathbb{R}^2, \\ &h(x) \to 0 && \text{as } |x| \to +\infty, \end{aligned} \tag{5.6}$$

for $a^2 \in L^2(\mathbb{R}^2)$. Equation (5.3) can be solved via sub-super-solution method. Solving the second equation for h in (5.1) we get $h = T(a^2)$, which leads to the nonlocal PDE for a

$$\Delta a - a + \frac{a^2}{T(a^2)} = 0. \tag{5.7}$$

Fixing m points which satisfy the constraints

$$\frac{2}{3}\log\log\frac{1}{\sigma} \leqslant |\xi_j - \xi_i| \leqslant 2\log\log\frac{1}{\sigma}, \quad \text{for all } i \neq j.$$

We look for solutions to (5.7) of the form

$$a(x) = \frac{1}{\tau_\sigma}(W+\phi), \quad \text{where } W = \sum_{j=1}^K w(x-\xi_j). \tag{5.8}$$

By using finite-dimensional reduction method, we first solve an auxiliary problem

$$\begin{cases} \Delta(W+\phi) - (W+\phi) + \frac{(W+\phi)^2}{T(\frac{1}{\tau_\sigma}(W+\phi)^2)} = \sum_{i,\alpha} c_{i\alpha}\frac{\partial W}{\partial \xi_{i,\alpha}} \\ \int_{\mathbb{R}^2} \phi \frac{\partial W}{\partial \xi_{i,\alpha}} = 0, \quad i = 1,\dots,m, \ \alpha = 1,2. \end{cases} \tag{5.9}$$

Solutions satisfying the required conditions in Theorems 5.1–5.3 will be precisely those satisfying a nonlinear system of equations of the form

$$c_{i\alpha}(\xi_1, \xi_2, \ldots, \xi_m) = 0, \quad i = 1, \ldots, m, \; \alpha = 1, 2,$$

where for such a class of points the functions $c_{i\alpha}$ satisfy

$$c_{i\alpha}(\xi_1, \ldots, \xi_k) = \frac{\partial}{\partial \xi_{i\alpha}} \left[\sum_{i \neq j} F\big(|\xi_j - \xi_i|\big) \right] + \epsilon_{i\alpha}, \tag{5.10}$$

function $F : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$F(r) = \frac{c_7 \log r}{\log \frac{1}{\sigma}} + c_8 w(r),$$

c_7 and c_8 are universal constants and

$$\epsilon_{i\alpha} = O\left(\frac{1}{(\log \frac{1}{\sigma})^{1+\gamma}} \right),$$

for some $\gamma > 0$. Although (5.10) does not have a variational structure, solutions of the problem $c_{i\alpha} = 0$ are close to critical points of the functional $\sum_{i \neq j} F(|\xi_j - \xi_i|)$. In spite of the simple form of this functional, its critical points are highly degenerate because of the invariance under rotations and translations of the problem. Thus, to get solutions using degree theoretical arguments, we need to restrict ourselves to classes of points enjoying symmetry constraints. This is how Theorems 5.1–5.3 are established. On the other hand, we believe strongly that finer analysis may yield existence of more complex patterns, such as honey-comb patterns, or lattice patterns.

REMARK 5.1.1. Similar method can also be used to prove Theorem 4.1. In that case, we have

$$c_i(\xi_1, \ldots, \xi_k) = \frac{\partial}{\partial \xi_i} \left[\sum_{i \neq j} F_1\big(|\xi_j - \xi_i|\big) \right] + O\big(\sigma^{1+\gamma}\big), \tag{5.11}$$

function $F_1 : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$F_1(r) = c_9 \sigma r + c_{10} w(r),$$

c_9 and c_{10} are universal constants. It is easy to see that the critical points of $\sum_{i \neq j} F_1(|\xi_i - \xi_j|)$ is *nondegenerate* (in the class of points with $\sum_{j=1}^{K} \xi_j = 0$).

5.2. *Existence of symmetric K-spots*

We look for solutions to the stationary GM on a two-dimensional domain with the following form

$$a_\epsilon(x) \sim \sum_{j=1}^{K} \xi_{\epsilon,j} w\left(\frac{x - P_j}{\epsilon}\right) \tag{5.12}$$

where P_j are the locations of the K-spikes and $\xi_{\epsilon,j}$ is the height of the spike at P_j.

If all the heights are *asymptotically equal*, i.e.

$$\lim_{\epsilon \to 0} \frac{\xi_{\epsilon,i}}{\xi_{\epsilon,j}} = 1, \quad \text{for } i \neq j, \tag{5.13}$$

such solutions are called symmetric K-spots. Otherwise, they are called asymmetric K-spots.

In this section, we discuss the existence of symmetric K-spots. It turns out in two-dimensional case, we have to discuss two cases: the strong coupling case, $D \sim O(1)$, and the weak coupling case, $D \gg 1$.

We first have the following existence result in the strong coupling case

THEOREM 5.4. (See [84] and [85].) *Let $\Omega \subset R^2$ be a bounded smooth domain and D be a fixed positive constant. Let $G_D(x, y)$ be the Green function of $-D\Delta + 1$ in Ω (with Neumann boundary condition). Let $H_D(x, y)$ be the regular part of $G_D(x, y)$ and set $h_D(P) = H_D(P, P)$.*

Set

$$F_D(P_1, \ldots, P_K) = \sum_{i=1}^{K} H_D(P_i, P_i) - \sum_{j \neq l} G_D(P_j, P_l).$$

Assume that $(P_1, \ldots, P_K) \in \Omega^K$ is a nondegenerate critical point of $F_D(P_1, \ldots, P_K)$. Then for ϵ sufficiently small, problem (GM) *has a steady state solution (a_ϵ, h_ϵ) with the following properties*:

(1) $a_\epsilon(x) = \xi_\epsilon(\sum_{j=1}^{K} w(\frac{x-P_j^\epsilon}{\epsilon}) + o(1))$ *uniformly for $x \in \bar{\Omega}$, $P_j^\epsilon \to P_j^0$, $j = 1, \ldots, K$, as $\epsilon \to 0$, and w is the unique solution of the problem* (2.8).

(2) $h_\epsilon(x) = \xi_\epsilon(1 + O(\frac{1}{|\log \epsilon|}))$ *uniformly for $x \in \bar{\Omega}$, where*

(3) $\xi_\epsilon^{-1} = (\frac{1}{2\pi} + o(1))\epsilon^2 \log \frac{1}{\epsilon} \int_{\mathbb{R}^2} w^2$.

REMARK 5.2.1. Theorem 5.4 shows that interior peaks solutions are related to the Green function (contrast to shadow system case). Thus in the strong coupling case, the peaks are produced by a different mechanism. It seems that the equation for h controls everything.

REMARK 5.2.2. In a general domain, the function $F_D(\mathbf{P})$ always has a global maximum point $\mathbf{P}_0$ in $\Omega \times \cdots \times \Omega$. (A proof of this fact can be found in the Appendix of [85].)

The proof of Theorem 5.4 depends on fine estimates in the finite-dimensional reduction: the major problem is to sum up the errors of powers in terms of $\frac{1}{\log\frac{1}{\epsilon}}$.

Next, we discuss the *weak coupling* case. We assume that $\lim_{\epsilon\to 0} D = +\infty$. We first introduce a Green function G_0 which we need to formulate our main results.

Let $G_0(x,\xi)$ be the Green function given by

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{in } \Omega, \\ \int_\Omega G_0(x,\xi)\,dx = 0, \\ \frac{\partial G_0(x,\xi)}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.14}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi}\log\frac{1}{|x-\xi|} - G_0(x,\xi) \tag{5.15}$$

be the regular part of $G_0(x,\xi)$.

Denote $\mathbf{P} \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \ldots, P_K)$$

with

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \ldots, K.$$

For $\mathbf{P} \in \Omega^K$ we define

$$F_0(\mathbf{P}) = \sum_{k=1}^{K} H_0(P_k, P_k) - \sum_{i,j=1,\ldots,K,\ i\neq j} G_0(P_i, P_j) \tag{5.16}$$

and

$$M_0(\mathbf{P}) = \left(\nabla^2_{\mathbf{P}} F_0(\mathbf{P})\right). \tag{5.17}$$

Here $M_0(\mathbf{P})$ is a $(2K)\times(2K)$ matrix with components $\frac{\partial^2 F(\mathbf{P})}{\partial P_{i,j}\partial P_{k,l}}$, $i,k = 1,\ldots,K$, $j,l = 1,2$ (recall that $P_{i,j}$ is the jth component of P_i).

Set

$$D = \frac{1}{\beta^2}, \qquad \eta_\epsilon := \frac{\beta^2|\Omega|}{2\pi}\log\frac{1}{\epsilon}. \tag{5.18}$$

Then $D \to +\infty$ is equivalent to $\beta \to 0$.

The stationary system for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta a - a + \frac{a^2}{h} = 0, \quad a > 0 & \text{in } \Omega, \\ \Delta h - \beta^2 h + \beta^2 a^2 = 0, \quad h > 0 & \text{in } \Omega, \\ \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

The following concerns the existence of symmetric K-peaked solutions in a two-dimensional domain which generalizes the one-dimensional result Theorem 4.2.

THEOREM 5.5. (See [86].) *Let* $\mathbf{P}^0 = (P_1^0, P_2^0, \ldots, P_K^0)$ *be a nondegenerate critical point of* $F_0(\mathbf{P})$ (*defined by* (5.16)). *Moreover, we assume that the following technical condition holds*

$$\textit{if } K > 1, \ \textit{then } \lim_{\epsilon \to 0} \eta_\epsilon \neq K, \tag{5.20}$$

where η_ϵ *is defined by* (5.18).

Then for ϵ *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large, problem* (5.19) *has a solution* (a_ϵ, h_ϵ) *with the following properties*:

(1) $a_\epsilon(x) = \xi_\epsilon(\sum_{j=1}^K w(\frac{x - P_j^\epsilon}{\epsilon}) + O(k(\epsilon, \beta)))$ *uniformly for* $x \in \bar{\Omega}$. *Here* w *is the unique solution of* (2.8) *and*

$$\xi_\epsilon = \begin{cases} \frac{1}{K} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \textit{if } \eta_\epsilon \to 0, \\ \frac{1}{\eta_\epsilon} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \textit{if } \eta_\epsilon \to \infty, \\ \frac{1}{K + \eta_0} \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} & \textit{if } \eta_\epsilon \to \eta_0, \end{cases} \tag{5.21}$$

and

$$k(\epsilon, \beta) := \epsilon^2 \xi_\epsilon \beta^2. \tag{5.22}$$

(*By* (5.21), $k(\epsilon, \beta) = O(\min\{\frac{1}{\log \frac{1}{\epsilon}}, \beta^2\})$.)

Furthermore, $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, \ldots, K$.

(2) $h_\epsilon(x) = \xi_\epsilon(1 + O(k(\epsilon, \beta)))$ *uniformly for* $x \in \bar{\Omega}$.

5.3. *Existence of multiple asymmetric spots*

Similar to the on dimensional case, there are also multiple asymmetric spots in a two-dimensional domain. But the existence of such patterns is only restricted when

$$\lim_{\epsilon \to 0} \frac{D}{\log \frac{1}{\epsilon}} < +\infty. \tag{5.23}$$

We first derive the algebraic equations for the heights $(\xi_{\epsilon,1},\dots,\xi_{\epsilon,K})$.

For $\beta>0$ let $G_\beta(x,\xi)$ be the Green function given by

$$\begin{cases} \Delta G_\beta-\beta^2 G_\beta+\delta_\xi=0 & \text{in } \Omega,\\ \frac{\partial G_\beta}{\partial\nu}=0 & \text{on } \partial\Omega. \end{cases} \tag{5.24}$$

Recall that $\beta^2=\frac{1}{D}$ and therefore $\beta\sim\frac{1}{\sqrt{\log\frac{1}{\epsilon}}}$. Let $G_0(x,\xi)$ be the Green function defined in (5.14).

In Section 2 of [86] a relation between G_0 and G_β is derived as follows

$$G_\beta(x,\xi)=\frac{\beta^{-2}}{|\Omega|}+G_0(x,\xi)+O\big(\beta^2\big) \tag{5.25}$$

in the operator norm of $L^2\Omega)\to H^2(\Omega)$. (Note that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.)

We define cut-off functions as follows: Let $\mathbf{P}\in\Omega^K$. Introduce

$$\chi_{\epsilon,P_j}(x)=\chi\left(\frac{x-P_j}{\delta}\right),\quad x\in\Omega,\ j=1,\dots,K, \tag{5.26}$$

where χ is a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2\setminus\overline{B_2(0)}$.

Let us assume the following ansatz for a multiple-spike solution (a_ϵ,h_ϵ) of (GM):

$$\begin{cases} a_\epsilon\sim\sum_{i=1}^K\xi_{\epsilon,i}\,w\big(\frac{x-P_i^\epsilon}{\epsilon}\big)\chi_{\epsilon,P_i}(x),\\ h_\epsilon(P_i^\epsilon)\sim\xi_{\epsilon,i}, \end{cases} \tag{5.27}$$

where w is the unique solution of (2.8), $\xi_{\epsilon,i}$, $i=1,\dots,K$, are the heights of the peaks, to be determined later, and $\mathbf{P}^\epsilon=(P_1^\epsilon,\dots,P_K^\epsilon)$ are the locations of K peaks.

Then we can make the following calculations, which can be made rigorous with error terms of the order $O(\frac{1}{\log\frac{1}{\epsilon}})$ in $H^2(\Omega)$.

From the equation for h_ϵ,

$$\Delta h_\epsilon-\beta^2 h_\epsilon+\beta^2 a_\epsilon^2=0,$$

we get, using (5.25),

$$\begin{aligned} h_\epsilon\big(P_i^\epsilon\big)&=\int_\Omega G_\beta\big(P_i^\epsilon,\xi\big)\beta^2 a_\epsilon^2(\xi)\,d\xi\\ &=\int_\Omega\left(\frac{\beta^{-2}}{|\Omega|}+G_0\big(P_i^\epsilon,\xi\big)+O\big(\beta^2\big)\right)\beta^2 \end{aligned}$$

$$\times \left(\sum_{j=1}^{K} \xi_{\epsilon,j}^2 w^2 \left(\frac{\xi - P_j^\epsilon}{\epsilon} \right) \chi_{\epsilon,P_j}(\xi) \right) d\xi$$

$$= \int_\Omega \left(\frac{1}{|\Omega|} + \beta^2 G_0\big(P_i^\epsilon, \xi\big) + O\big(\beta^4\big) \right)$$

$$\times \left(\sum_{j=1}^{K} \xi_{\epsilon,j}^2 w^2 \left(\frac{\xi - P_j^\epsilon}{\epsilon} \right) \chi_{\epsilon,P_j}(\xi) \right) d\xi.$$

Thus

$$\begin{aligned} \xi_{\epsilon,i} &= \xi_{\epsilon,i}^2 \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2} w^2(y)\,dy + \xi_{\epsilon,i}^2 \beta^2 \int_\Omega G_0\big(P_i^\epsilon, \xi\big) w^2 \left(\frac{\xi - P_i^\epsilon}{\epsilon} \right) \chi_{\epsilon,P_i}(\xi)\,d\xi \\ &\quad + \sum_{j \neq i} \left(\frac{1}{|\Omega|} + \beta^2 G_0\big(P_i^\epsilon, P_j^\epsilon\big) \right) \xi_{\epsilon,j}^2 \epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy \\ &\quad + \sum_{j=1}^{K} \xi_{\epsilon,j}^2 \big(O\big(\beta^2 \epsilon^4\big) + O\big(\beta^4 \epsilon^2\big) \big). \end{aligned} \tag{5.28}$$

Here we have used that for $j \neq i$

$$\begin{aligned} &\int_\Omega G_0\big(P_i^\epsilon, \xi\big) w^2 \left(\frac{\xi - P_j^\epsilon}{\epsilon} \right) \chi_{\epsilon,P_j}(\xi)\,d\xi \\ &\quad = \epsilon^2 \int_{\mathbb{R}^2} G_0\big(P_i^\epsilon, \epsilon y + P_j^\epsilon\big) w^2(y)\,dy + \text{e.s.t.} \\ &\quad = \epsilon^2 G_0\big(P_i^\epsilon, P_j^\epsilon\big) \int_{\mathbb{R}^2} w^2(y)\,dy \\ &\qquad + \epsilon^3 \sum_{l=1}^{K} \frac{\partial G_0(P_i^\epsilon, P_j^\epsilon)}{\partial P_{j,l}^\epsilon} \int_{\mathbb{R}^2} w^2(y) y_l\,dy + O\big(\epsilon^4\big) \\ &\quad = \epsilon^2 G_0\big(P_i^\epsilon, P_j^\epsilon\big) \int_{\mathbb{R}^2} w^2(y)\,dy + O\big(\epsilon^4\big). \end{aligned}$$

(Note that we have set $y = \frac{\xi - P_j^\epsilon}{\epsilon}$ and we have used the relation

$$\int_{\mathbb{R}^2} w^2(y) y_l\,dy = 0$$

which holds since w is radially symmetric.)

Using (5.15) in (5.28) gives

$$\begin{aligned}
\xi_{\epsilon,i} &= \xi_{\epsilon,i}^2 \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2} w^2(y)\,dy \\
&\quad + \xi_{\epsilon,i}^2 \beta^2 \int_\Omega \left(\frac{1}{2\pi} \log \frac{1}{|P_i^\epsilon - \xi|} - H_0\big(P_i^\epsilon, \xi\big) \right) w^2\left(\frac{\xi - P_i^\epsilon}{\epsilon} \right) \chi_{\epsilon, P_i^\epsilon}(\xi)\,d\xi \\
&\quad + \sum_{j \neq i} \left(\frac{1}{|\Omega|} + \beta^2 G_0\big(P_i^\epsilon, P_j^\epsilon\big) \right) \xi_{\epsilon,j}^2 \epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy \\
&\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 \big(O\big(\beta^2 \epsilon^4\big) + O\big(\beta^4 \epsilon^2\big) \big) \\
&= \xi_{\epsilon,i}^2 \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2} w^2(y)\,dy + \xi_{\epsilon,i}^2 \frac{\beta^2}{2\pi} \epsilon^2 \log \frac{1}{\epsilon} \int_{\mathbb{R}^2} w^2(y)\,dy \\
&\quad + \xi_{\epsilon,i}^2 \frac{\beta^2}{2\pi} \left(\epsilon^2 \int_{\mathbb{R}^2} w^2(y) \log \frac{1}{|y|}\,dy - \epsilon^2 H_0\big(P_i^\epsilon, P_i^\epsilon\big) \int_{\mathbb{R}^2} w^2(y)\,dy \right) \\
&\quad + \sum_{j \neq i} \left(\frac{1}{|\Omega|} + \beta^2 G_0\big(P_i^\epsilon, P_j^\epsilon\big) \right) \xi_{\epsilon,j}^2 \epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy \\
&\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 \big(O\big(\beta^2 \epsilon^4\big) + O\big(\beta^4 \epsilon^2\big) \big). \qquad (5.29)
\end{aligned}$$

Recall that $H_0 \in C^2(\bar{\Omega} \times \Omega)$.

Considering only the leading terms in (5.29) we get following

$$\begin{aligned}
\xi_{\epsilon,i} &= \sum_{j=1}^K \xi_{\epsilon,j}^2 \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2} w^2(y)\,dy + \xi_{\epsilon,i}^2 \frac{\beta^2}{2\pi} \epsilon^2 \log \frac{1}{\epsilon} \int_{\mathbb{R}^2} w^2(y)\,dy \\
&\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 O\big(\beta^2 \epsilon^2\big). \qquad (5.30)
\end{aligned}$$

Let us rescale

$$\xi_{\epsilon,i} = \xi_\epsilon \hat{\xi}_{\epsilon,i}, \quad \text{where } \xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2}. \qquad (5.31)$$

Then from (5.30) we get

$$\xi_{\epsilon,i} = \left(\frac{1}{|\Omega|} + \frac{\eta_\epsilon}{|\Omega|} \right) \xi_{\epsilon,i}^2 \epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy$$

$$+\sum_{j\neq i}\xi_{\epsilon,j}^2\frac{\epsilon^2}{|\Omega|}\int_{\mathbb{R}^2}w^2(y)\,dy+\sum_{j=1}^{K}\xi_{\epsilon,j}^2 O\big(\beta^2\epsilon^2\big),$$

where η_ϵ was introduced in (5.18). Assuming that

$$\hat{\xi}_{\epsilon,i}\to\hat{\xi}_i,\quad \eta_\epsilon\to\eta_0, \tag{5.32}$$

we obtain the following system of algebraic equations

$$\hat{\xi}_{\epsilon,i}=\sum_{j=1}^{K}\hat{\xi}_{\epsilon,j}^2+\hat{\xi}_{\epsilon,i}^2\eta_0,\quad i=1,\dots,K, \tag{5.33}$$

which can be determined completely.

In fact, let

$$\rho(t)=t-\eta_0 t^2. \tag{5.34}$$

Then (5.33) is equivalent to

$$\rho(\hat{\xi}_i)=\sum_{j=1}^{K}\hat{\xi}_j^2,\quad i=1,\dots,K, \tag{5.35}$$

which implies that

$$\rho(\hat{\xi}_i)=\rho(\hat{\xi}_j)\quad\text{for } i\neq j. \tag{5.36}$$

That is

$$(\hat{\xi}_i-\hat{\xi}_j)\big(1-\eta_0(\hat{\xi}_i+\hat{\xi}_j)\big)=0. \tag{5.37}$$

Hence for $i\neq j$ we have

$$\hat{\xi}_i-\hat{\xi}_j=0\quad\text{or}\quad\hat{\xi}_i+\hat{\xi}_j=\frac{1}{\eta_0}. \tag{5.38}$$

The case of symmetric solutions ($\hat{\xi}_i=\hat{\xi}_1$, $i=2,\dots,N$) has been studied in [85] and [86]. Let us now consider asymmetric solutions, i.e., we assume that there exists an $i\in\{2,\dots,N\}$ such that $\hat{\xi}_i\neq\hat{\xi}_1$. Without loss of generality, let us assume that

$$\hat{\xi}_2\neq\hat{\xi}_1,$$

which implies that

$$\hat{\xi}_1+\hat{\xi}_2=\frac{1}{\eta_0}. \tag{5.39}$$

Let us calculate $\hat{\xi}_j,\ j=3,\ldots,K$. If $\hat{\xi}_j \neq \hat{\xi}_1$, then by (5.38), $\hat{\xi}_j + \hat{\xi}_1 = \frac{1}{\eta_0}$, which implies that $\hat{\xi}_j = \hat{\xi}_2$.

Thus for $j \geqslant 3$, we have either $\hat{\xi}_j = \hat{\xi}_1$ or $\hat{\xi}_j = \hat{\xi}_2$.

Let k_1 be the number of $\hat{\xi}_1$'s in $\{\hat{\xi}_1,\ldots,\hat{\xi}_K\}$ and k_2 be the number of $\hat{\xi}_2$'s in $\{\hat{\xi}_1,\ldots,\hat{\xi}_K\}$. Then we have $k_1 \geqslant 1$, $k_2 \geqslant 1$, $k_1+k_2=K$.

This gives

$$\hat{\xi}_1 - \eta_0\hat{\xi}_1^2 = \sum_{j=1}^{K}\hat{\xi}_j^2 = k_1\hat{\xi}_1^2 + k_2\hat{\xi}_2^2, \tag{5.40}$$

$$\hat{\xi}_2 = \frac{1}{\eta_0} - \hat{\xi}_1. \tag{5.41}$$

Substituting (5.41) into (5.40), we obtain

$$\hat{\xi}_1 - \eta_0\hat{\xi}_1^2 = k_1\hat{\xi}_1^2 + k_2\left(\frac{1}{\eta_0} - \hat{\xi}_1\right)^2$$

and therefore

$$(k_1+k_2+\eta_0)\hat{\xi}_1^2 - \frac{2k_2+\eta_0}{\eta_0}\hat{\xi}_1 + \frac{k_2}{\eta_0^2} = 0. \tag{5.42}$$

Equation (5.42) has a solution if and only if

$$\frac{(2k_2+\eta_0)^2}{\eta_0^2} \geqslant 4\frac{k_2}{\eta_0^2}(k_1+k_2+\eta_0). \tag{5.43}$$

The strict inequality of (5.43) is equivalent to

$$\eta_0 > 2\sqrt{k_1k_2}. \tag{5.44}$$

It is easy to see that if (5.44) holds, then there are two different solutions to (5.42) which are given by $(\rho_\pm, \eta_\pm)$.

Therefore we arrive at the following conclusion.

LEMMA 5.6. *Let $\eta_0 \geqslant 2\sqrt{k_1k_2}$. Then the solutions of* (5.33) *are given by* $(\hat{\xi}_1,\ldots,\hat{\xi}_N) \in (\{\rho_\pm,\eta_\pm\})^K$ *where the number of $\rho_\pm$'s is k_1 and the number of $\eta_\pm$'s is k_2.*

If $\eta_0 > 2\sqrt{k_1k_2}$, there exist two solutions $(\rho_\pm,\eta_\pm)$.

If $\eta_0 = 2\sqrt{k_1k_2}$, there exists one solution $(\rho_\pm,\rho_\pm)$.

If $\eta_0 < 2\sqrt{k_1k_2}$, there are no solutions $(\rho_\pm,\rho_\pm)$.

Let $\eta_0 > 2\sqrt{k_1k_2}$ where $k_1 + k_2 = K$, $k_1, k_2 \geqslant 1$. By Lemma 5.6, there are two solutions to (5.33). In fact, we can solve

$$\rho_+ = \frac{2k_2 + \eta_0 + \sqrt{\eta_0^2 - 4k_1k_2}}{2\eta_0(\eta_0 + K)}, \qquad \rho_- = \frac{2k_2 + \eta_0 - \sqrt{\eta_0^2 - 4k_1k_2}}{2\eta_0(\eta_0 + K)}, \tag{5.45}$$

$$\eta_+ = \frac{2k_1 + \eta_0 - \sqrt{\eta_0^2 - 4k_1k_2}}{2\eta_0(\eta_0 + K)}, \qquad \eta_- = \frac{2k_1 + \eta_0 + \sqrt{\eta_0^2 - 4k_1k_2}}{2\eta_0(\eta_0 + K)}. \tag{5.46}$$

Note that

$$\rho_+ + \eta_+ = \frac{1}{\eta_0}, \qquad \rho_- + \eta_- = \frac{1}{\eta_0}. \tag{5.47}$$

Let $(\rho, \eta) = (\rho_+, \eta_+)$ or $(\rho, \eta) = (\rho_-, \eta_-)$. We drop "±" if there is no confusion.

Let $(\hat{\xi}_1, \ldots, \hat{\xi}_K) \in R_+^K$ be such that

$$\hat{\xi}_j \in \{\rho, \eta\}, \text{ and the number of } \rho\text{'s in } (\hat{\xi}_1, \ldots, \hat{\xi}_K) \text{ is } k_1. \tag{5.48}$$

Then there are k_2 η's in $(\hat{\xi}_1, \ldots, \hat{\xi}_K)$.

Let $\mathbf{P} = (P_1, \ldots, P_K) \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \ldots, P_K)$$

with

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \ldots, K.$$

For $\mathbf{P} \in \Omega^K$ we define

$$\hat{F}_0(\mathbf{P}) = \sum_{k=1}^{K} H_0(P_k, P_k)\hat{\xi}_k^4 - \sum_{i,j,=1,\ldots,K,\, i \neq j} G_0(P_i, P_j)\hat{\xi}_i^2\hat{\xi}_j^2 \tag{5.49}$$

and

$$\hat{M}_0(\mathbf{P}) = \nabla_{\mathbf{P}}^2 \tilde{F}_0(\mathbf{P}). \tag{5.50}$$

Then we have the following theorem, which is on the existence of asymmetric K-peaked solutions.

THEOREM 5.7. (See [87].) *Let $K \geqslant 2$ be a positive integer. Let k_1, $k_2 \geqslant 1$ be two integers such that $k_1 + k_2 = K$. Let*

$$\beta^2 = \frac{1}{D}, \qquad \eta_\epsilon = \frac{\beta^2|\Omega|}{2\pi} \log \frac{\sqrt{|\Omega|}}{\epsilon},$$

where $|\Omega|$ denotes the area of Ω, Assume that $\eta_0 = \lim_{\epsilon\to 0}\eta_\epsilon > 2\sqrt{k_1 k_2}$,

(T1) $\eta_0 \neq K$

and that

(T2) $\mathbf{P}^0 = \left(P_1^0, P_2^0, \ldots, P_K^0\right)$ *is a nondegenerate critical point of* $\hat{F}_0(\mathbf{P})$

(*defined by* (5.49)).

Then for ϵ *sufficiently small the stationary* (GM) *has a solution* (a_ϵ, h_ϵ) *with the following properties*:

(1) $a_\epsilon(x) = \sum_{j=1}^K \xi_{\epsilon,j}(w(\frac{x-P_j^\epsilon}{\epsilon}) + O(\frac{1}{D}))$ *uniformly for* $x \in \bar{\Omega}$, *where* w *is the unique solution of* (2.8) *and*

$$\xi_{\epsilon,j} = \xi_\epsilon \hat{\xi}_{\epsilon,j}, \quad \xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2} w^2}. \tag{5.51}$$

Further, $(\hat{\xi}_{\epsilon,1}, \ldots, \hat{\xi}_{\epsilon,K}) \to (\hat{\xi}_1, \ldots, \hat{\xi}_K)$ *which is given by* (5.48).

(2) $h_\epsilon(P_j^\epsilon) = \xi_{\epsilon,j}(1 + \frac{1}{D})$ *in* $H^2(\Omega)$, $j = 1, \ldots, K$.

(3) $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, \ldots, K$.

5.4. *Stability of symmetric* K*-spots*

Next we study the stability and instability of the symmetric K-peaked solutions constructed in Theorems 5.4 and 5.5.

In the strong coupling case, it turns out all solutions are stable:

THEOREM 5.8. (See [85].) *Suppose* $D = O(1)$. *Let* $\mathbf{P}_0$ *and* (a_ϵ, h_ϵ) *be defined as in Theorem* 5.4. *Then for* ϵ *and* τ *sufficiently small* (a_ϵ, h_ϵ) *is stable if all eigenvalues of the matrix* $M_D(\mathbf{P}_0) = (\nabla^2_{\mathbf{P}_0} F_D(\mathbf{P}_0))$ *are negative.* (a_ϵ, h_ϵ) *is unstable if one of the eigenvalues of the matrix* $M_D(\mathbf{P}_0)$ *is positive.*

In the weak coupling case, the stability of symmetric K-peaked solutions in a bounded two-dimensional domain can be summarized as follows.

THEOREM 5.9. (See [86].) *Let* $\mathbf{P}^0$ *be a nondegenerate critical point of* $F_0(\mathbf{P})$ *and for* ϵ *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large let* (a_ϵ, h_ϵ) *be the* K*-peaked solutions constructed in Theorem* 5.5 *whose peaks approach* $\mathbf{P}^0$.

Assume (5.20) *holds and further that*

$(*)$ $\mathbf{P}^0$ *is a nondegenerate local maximum point of* $F_0(\mathbf{P})$.

Then we have

CASE 1. $\eta_\epsilon \to 0$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \gg \log\frac{1}{\epsilon}$).

If $K = 1$ then there exists a unique $\tau_1 > 0$ such that for $\tau < \tau_1$, (a_ϵ, h_ϵ) is linearly stable, while for $\tau > \tau_1$, (a_ϵ, h_ϵ) is linearly unstable.
If $K > 1$, (a_ϵ, h_ϵ) is linearly unstable for any $\tau \geqslant 0$.

CASE 2. $\eta_\epsilon \to +\infty$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \ll \log\frac{1}{\epsilon}$).

(a_ϵ, h_ϵ) is linearly stable for any $\tau > 0$.

CASE 3. $\eta_\epsilon \to \eta_0 \in (0, +\infty)$ (*i.e.*, $\frac{2\pi D}{|\Omega|} \sim \frac{1}{\eta_0}\log\frac{1}{\epsilon}$).

If $K > 1$ and $\eta_0 < K$, then (a_ϵ, h_ϵ) is linearly unstable for any $\tau > 0$.
If $\eta_0 > K$, then there exist $0 < \tau_2 \leqslant \tau_3$ such that (a_ϵ, h_ϵ) is linearly stable for $\tau < \tau_2$ and $\tau > \tau_3$.
If $K = 1$, $\eta_0 < 1$, then there exist $0 < \tau_4 \leqslant \tau_5$ such that (a_ϵ, h_ϵ) is linearly stable for $\tau < \tau_4$ and linearly unstable for $\tau > \tau_5$.

The statement of Theorem 5.9 is rather long. Let us therefore explain the results by the following remarks.

REMARK 5.4.1. Assuming that condition $(*)$ holds, then for ϵ small the stability behavior of (a_ϵ, h_ϵ) can be summarized in Table 1.

REMARK 5.4.2. The condition $(*)$ on the locations $\mathbf{P}^0$ arises in the study of small ($o(1)$) eigenvalues. For any bounded smooth domain Ω, the functional $F_0(\mathbf{P})$, defined by (5.16), always admits a global maximum at some $\mathbf{P}^0 \in \Omega^K$. The proof of this fact is similar to the Appendix in [86]. We believe that in *generic* domains, this global maximum point $\mathbf{P}^0$ is nondegenerate.

It is an interesting open question to numerically compute the critical points of $F_0(\mathbf{P})$ and link them explicitly to the geometry of the domain Ω.

We believe that for other types of critical points of $F_0(\mathbf{P})$, such as saddle points, the solution constructed in Theorem 5.5 should be linearly unstable. We are not able to prove this at the moment, since the operator $\mathcal{L}_\epsilon$ is *not self-adjoint*.

Table 1

	Case 1	Case 2	Case 3 ($\eta_0 < K$)	Case 3 ($\eta_0 > K$)
$K = 1$, τ small	stable	stable	stable	stable
$K = 1$, τ finite	?	stable	?	?
$K = 1$, τ large	unstable	stable	unstable	stable
$K > 1$, τ small	unstable	stable	unstable	stable
$K > 1$, τ finite	unstable	stable	unstable	?
$K > 1$, τ large	unstable	stable	unstable	stable

REMARK 5.4.3. Case 1 and Case 3 with $\eta_0 < K$ resemble the *shadow system* and Case 2 and Case 3 with $\eta_0 > K$ are similar to the *strong coupling* case.

From Case 2 and Case 3 of Theorem 5.9, we see that for multiple spikes ($K > 1$) large τ may increase stability, provided that $\eta_0 > K$. This is a *new* phenomenon in R^2. It is known that in R^1, large τ implies linear instability for multiple spikes [8,34,59,60].

REMARK 5.4.4. We conjecture that in Case 3, $\tau_2 = \tau_3$. This will imply that for any $\tau \geqslant 0$ and $\eta_0 > K$, multiple spikes are stable, provided condition $(*)$ is satisfied. (It is possible to obtain explicit values for τ_2 and τ_3.)

REMARK 5.4.5. Roughly speaking, assuming that condition $(*)$ holds and that τ is small, then for $\epsilon \ll 1$, $D_K(\epsilon) = \frac{|\Omega|}{2\pi K} \log \frac{1}{\epsilon}$ is the critical threshold for the asymptotic behavior of the diffusion coefficient of the inhibitor which determines the stability of K-peaked solutions.

The proof of Theorem 5.9 is again divided by two parts: large eigenvalues and small eigenvalues. For small eigenvalues, we relate them to the functional $F(\mathbf{P})$. For large eigenvalues, we obtain a system of NLEPs:

$$\begin{aligned} &\Delta\phi_i - \phi_i + 2w\phi_i \\ &\quad - \frac{2[(1+\eta_0(1+\tau\lambda_0))\int_{\mathbb{R}^2} w\phi_i + \sum_{j\neq i}\int_{\mathbb{R}^2} w\phi_j]}{(K+\eta_0)(1+\tau\lambda_0)\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi_i, \\ &\quad i = 1, \ldots, K. \end{aligned} \tag{5.52}$$

By diagonalization, we obtain two NELPs:

$$\Delta\phi - \phi + 2w\phi - \frac{2\eta_0}{(K+\eta_0)\int_{\mathbb{R}^2} w^2}\left[\int_{\mathbb{R}^2} w(y)\phi(y)\,dy\right] w^2 = \lambda\phi, \tag{5.53}$$

and

$$\begin{aligned} &\Delta\phi - \phi + 2w\phi - \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)} \frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi, \\ &\quad \phi \in H^2(\mathbb{R}^2), \end{aligned} \tag{5.54}$$

where $0 < \eta_0 < +\infty$ and $0 \leqslant \tau < +\infty$.

Problem (5.53) is the same as (3.7). For problem (5.54), we have the following result

THEOREM 5.10.

(1) *If $\eta_0 < K$, then for τ small problem* (5.54) *is stable while for τ large it is unstable.*

(2) *If $\eta_0 > K$, then there exists $0 < \tau_2 \leqslant \tau_3$ such that problem* (5.54) *is stable for $\tau < \tau_2$ or $\tau > \tau_3$.*

PROOF. Let us set

$$f(\tau\lambda) = \frac{2(K + \eta_0(1 + \tau\lambda))}{(K + \eta_0)(1 + \tau\lambda)}. \tag{5.55}$$

We note that

$$\lim_{\tau\lambda\to+\infty} f(\tau\lambda) = \frac{2\eta_0}{K + \eta_0} =: f_\infty.$$

If $\eta_0 < K$, then by Theorem 3.12(2), problem (3.52) with $\mu = f_\infty$ has a positive eigenvalue α_1. Now by perturbation arguments (similar to those in [8]), for τ large, problem (5.54) has an eigenvalue near $\alpha_1 > 0$. This implies that for τ large, problem (5.54) is unstable.

Now we show that problem (5.54) has no nonzero eigenvalues with nonnegative real part, provided that either τ is small or $\eta_0 > K$ and τ is large. (It is immediately seen that $f(\tau\lambda) \to 2$ as $\tau\lambda \to 0$ and $f(\tau\lambda) \to \frac{2\eta_0}{\eta_0+K} > 1$ as $\tau\lambda \to +\infty$ if $\eta_0 > K$. Then Theorem 3.12 should apply. The problem is that we do not have control on $\tau\lambda$. Here we provide a rigorous proof.)

We apply the following inequality (Lemma 3.8(1)): for any (real-valued function) $\phi \in H_r^2(R^2)$, we have

$$\int_{\mathbb{R}^2} \left(|\nabla\phi|^2 + \phi^2 - 2w\phi^2\right) + 2\frac{\int_{\mathbb{R}^2} w\phi \int_{\mathbb{R}^2} w^2\phi}{\int_{\mathbb{R}^2} w^2} - \frac{\int_{\mathbb{R}^2} w^3}{(\int_{\mathbb{R}^2} w^2)^2}\left(\int_{\mathbb{R}^2} w\phi\right)^2 \geqslant 0, \tag{5.56}$$

where equality holds if and only if ϕ is a multiple of w.

Now let $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$, $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (5.54). Then we have

$$L_0\phi - f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0\phi. \tag{5.57}$$

Multiplying (5.57) by $\bar\phi$—the conjugate function of ϕ—and integrating over R^2, we obtain that

$$\int_{\mathbb{R}^2} \left(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2\right) = -\lambda_0 \int_{\mathbb{R}^2} |\phi|^2 - f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} \int_{\mathbb{R}^2} w^2\bar\phi. \tag{5.58}$$

Multiplying (5.57) by w and integrating over $\mathbb{R}^2$, we obtain that

$$\int_{\mathbb{R}^2} w^2\phi = \left(\lambda_0 + f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\int_{\mathbb{R}^2} w\phi. \tag{5.59}$$

Taking the conjugate of (5.59) we have

$$\int_{\mathbb{R}^2} w^2\bar{\phi} = \left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\int_{\mathbb{R}^2} w\bar{\phi}. \tag{5.60}$$

Substituting (5.60) into (5.58), we have that

$$\begin{aligned}\int_{\mathbb{R}^2}&\left(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2\right)\\ &= -\lambda_0\int_{\mathbb{R}^2}|\phi|^2 - f(\tau\lambda_0)\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\frac{|\int_{\mathbb{R}^2} w\phi|^2}{\int_{\mathbb{R}^2} w^2}.\end{aligned} \tag{5.61}$$

We just need to consider the real part of (5.61). Now applying the inequality (5.56) and using (5.60) we arrive at

$$\begin{aligned}-\lambda_R \geqslant{}& \mathrm{Re}\left(f(\tau\lambda_0)\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right)\right)\\ &- 2\,\mathrm{Re}\left(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\right) + \frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2},\end{aligned}$$

where we recall $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ with $\lambda_R,\ \lambda_I \in R$.

Assuming that $\lambda_R \geqslant 0$, then we have

$$\frac{\int_{\mathbb{R}^2} w^3}{\int_{\mathbb{R}^2} w^2}\left|f(\tau\lambda_0) - 1\right|^2 + \mathrm{Re}\left(\bar{\lambda}_0\left(f(\tau\lambda_0) - 1\right)\right) \leqslant 0. \tag{5.62}$$

By the usual Pohozaev's identity for (2.8) (multiplying (2.8) by $y\cdot\nabla w(y)$ and integrating by parts), we obtain that

$$\int_{\mathbb{R}^2} w^3 = \frac{3}{2}\int_{\mathbb{R}^2} w^2. \tag{5.63}$$

Substituting (5.63) and the expression (5.55) for $f(\tau\lambda)$ into (5.62), we have

$$\begin{aligned}\frac{3}{2}&\left|\eta_0 + K + (\eta_0 - K)\tau\lambda\right|^2 + \mathrm{Re}\left((\eta_0 + K)(1 + \tau\bar{\lambda}_0)\left((\eta_0 + K)\bar{\lambda}_0\right.\right.\\ &\left.\left. + (\eta_0 - K)\tau|\lambda_0|^2\right)\right) \leqslant 0\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{3}{2}&(1 + \mu_0\tau\lambda_R)^2 + \lambda_R + \left(\mu_0\tau + \tau + \mu_0\tau^2|\lambda_0|^2\right)\lambda_R\\ &+ \left(\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau\right)\lambda_I^2 \leqslant 0\end{aligned} \tag{5.64}$$

where we have introduced $\mu_0 := \frac{\eta_0 - K}{\eta_0 + K}$.

If $\eta_0 > K$ (i.e., $\mu_0 > 0$) and τ is large, then

$$\frac{3}{2}\mu_0^2\tau^2 + \mu_0\tau - \tau \geqslant 0. \tag{5.65}$$

So (5.64) does not hold for $\lambda_R \geqslant 0$.

To consider the case when τ is small, we have to obtain an upper bound for λ_I.

From (5.58), we have

$$\lambda_I \int_{\mathbb{R}^2} |\phi|^2 = \mathrm{Im}\left(-f(\tau\lambda_0)\frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2}\int_{\mathbb{R}^2} w^2\bar{\phi}\right).$$

Hence

$$|\lambda_I| \leqslant \left|f(\tau\lambda_0)\right|\sqrt{\frac{\int_{\mathbb{R}^2} w^4}{\int_{\mathbb{R}^2} w^2}} \leqslant C \tag{5.66}$$

where C is independent of λ_0.

Substituting (5.66) into (5.64), we see that (5.64) cannot hold for $\lambda_R \geqslant 0$, if τ is small. □

5.5. *Stability of asymmetric K-spots*

Finally we study the stability or instability of the asymmetric K-peaked solutions constructed in Theorem 5.7.

THEOREM 5.11. *Let (a_ϵ, h_ϵ) be the K-peaked solutions constructed in Theorem* 5.7 *for ϵ sufficiently small, whose peaks are located near $\mathbf{P}^0$. Further assume that*

$(*)$ $\mathbf{P}^0$ *is a nondegenerate local maximum point of $\hat{F}(\mathbf{P})$.*

Then we have:

(a) (*Stability*)

Assume that

$$2\sqrt{k_1k_2} < \eta_0 < K \tag{5.67}$$

and

$$k_1 > k_2, \quad (\rho, \eta) = (\rho_+, \eta_+).$$

Then, for τ small enough, (a_ϵ, h_ϵ) is stable.

(b) (*Instability*)
Assume that either

$$\eta_0 > K$$

or

$$\tau \text{ is large enough.}$$

Then (a_ϵ, h_ϵ) *is linearly unstable.*

A consequence of Theorem 5.11 is *stable* asymmetric patterns can exist in a two-dimensional domain for a very narrow range of D, namely for

$$\frac{1}{2\pi K}\log\frac{\sqrt{|\Omega|}}{\epsilon} < \frac{D}{|\Omega|} < \frac{1}{4\pi\sqrt{k_1k_2}}\log\frac{\sqrt{|\Omega|}}{\epsilon} \tag{5.68}$$

and ϵ small enough, where k_1 and k_2 are two integers satisfying $k_1 + k_2 = K, k_1 \geqslant 1, k_2 \geqslant 1$. In most cases, asymmetric patterns are unstable.

6. High-dimensional case: $N \geqslant 3$

When $N \geqslant 3$, there are very few results on the full Gierer–Meinhardt system. The difference between $N \geqslant 3$ and $N \leqslant 2$ lies on the behavior of the Green function: when $N \leqslant 2$, the Green function is locally *constant* (when $N = 2$, it is locally ∞). The limiting problem is still a single equation (2.8). But when $N \geqslant 3$, the Green function is like $\frac{1}{|x-y|^{N-2}}$. The limiting problem when $N \geqslant 3$ becomes

$$\begin{cases} \Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta h + \frac{a^r}{h^s} = 0 & \text{in } \mathbb{R}^N, \\ a, h > 0, \ a, h \to 0 & \text{as } |y| \to +\infty. \end{cases} \tag{6.1}$$

Problem (6.1) seems out of reach at this moment. We believe that there should a radially symmetric solution to (6.1) which is also stable.

As far as the author knows, the only result in higher-dimensional case is the existence of radially symmetric layer solutions [62].

Let $\Omega = B_R$ be a ball of radius R in $\mathbb{R}^N$. By scaling, we may take $D = 1$ and obtain formally the following elliptic system

$$\begin{cases} \epsilon^2\Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } B_R, \\ \Delta h - h + \frac{a^m}{h^s} = 0 & \text{in } B_R \\ vsa > 0, \ h > 0 & \text{in } B_R, \\ \frac{\partial a}{\partial \nu} = \frac{\partial a}{\partial \nu} = 0 & \text{on } B_R, \end{cases} \tag{6.2}$$

where (p, q, m, s) satisfies

$$p > 1, \quad q > 0, \quad m > 0, \quad s \geqslant 0, \quad \frac{qm}{(p-1)(s+1)} > 1. \tag{6.3}$$

(The case of the whole $\mathbb{R}^N$ is also included here, by taking $R = +\infty$.)

Note that in (6.2), we have replaced a^r by a^m since we will use $r = |x|$ to denote the radial variable.

We first define two functions, to be used later: let $J_1(r)$ be the radially symmetric solutions of the following problem

$$J_1'' + \frac{N-1}{r} J_1' - J_1 = 0, \quad J'(0) = 0, \quad J_1(0) = 1, \quad J_1 > 0. \tag{6.4}$$

The second function, called $J_2(r)$, satisfies

$$J_2'' + \frac{N-1}{r} J_2' - J_2 + \delta_0 = 0, \quad J_2 > 0, \quad J_2(+\infty) = 0, \tag{6.5}$$

where δ_0 is the Dirac measure at 0.

The functions $J_1(r)$ and $J_2(r)$ can be written in terms of modified Bessel's functions. In fact

$$J_1(r) = c_1 r^{\frac{2-N}{2}} I_\nu(r), \qquad J_2(r) = c_2 r^{\frac{2-N}{2}} K_\nu(r), \quad \nu = \frac{N-2}{2} \tag{6.6}$$

where c_1, c_2 are two positive constants and I_ν, K_ν are modified Bessel functions of order ν. In the case of $N = 3$, J_1, J_2 can be computed explicitly:

$$J_1 = \frac{\sinh r}{r}, \qquad J_2(r) = \frac{e^{-r}}{4\pi r}. \tag{6.7}$$

Let $w(y)$ be the unique solution for ODE 2.103. Let $R > 0$ be a fixed constant. We define

$$J_{2,R}(r) = J_2(r) - \frac{J_2'(R)}{J_1'(R)} J_1(r) \tag{6.8}$$

and a Green function $G_R(r; r')$

$$G_R'' + \frac{N-1}{r} G_R' - G_R + \delta_{r'} = 0, \quad G_R'(0; r') = 0, \quad G_R'(R; r') = 0. \tag{6.9}$$

Note that

$$J_{2,R}'(R) = 0, \qquad \lim_{R \to +\infty} J_{2,R}(r) = J_2(r). \tag{6.10}$$

For $t \in (0, R)$, set

$$M_R(t) := \frac{(N-1)(p-1)}{qt} + \frac{J_1'(t)}{J_1(t)} + \frac{J_{2,R}'(t)}{J_{2,R}(t)}. \tag{6.11}$$

When $R = +\infty$, $J_{2,+\infty}(r) = J_2(r)$. We denote $G_{+\infty}(r;r')$ as $G(r;r')$ and $M_{+\infty}(t)$ as $M(t)$. That is,

$$G(r;r') = c_0(r')^{N-1} \begin{cases} J_2(r')J_1(r), & \text{for } r < r', \\ J_1(r')J_2(r), & \text{for } r > r', \end{cases} \tag{6.12}$$

$$M(t) := \frac{(N-1)(p-1)}{qt} + \frac{J_1'(t)}{J_1(t)} + \frac{J_2'(t)}{J_2(t)}. \tag{6.13}$$

Then we have the following existence result on layered solutions.

THEOREM 6.1. (See [62].) *Let $N \geqslant 2$. Assume that there exist two radii $0 < r_1 < r_2 < R$ such that*

$$M_R(r_1)M_R(r_2) < 0. \tag{6.14}$$

Then for ϵ sufficiently small, problem (6.2) *has a solution $(a_{\epsilon,R}, h_{\epsilon,R})$ with the following properties*:

(1) *$a_{\epsilon,R}, h_{\epsilon,R}$ are radially symmetric,*

(2) $a_{\epsilon,R}(r) = \xi_{\epsilon,R}^{\frac{q}{p-1}} w(\frac{r-t_\epsilon}{\epsilon})(1+o(1))$,

(3) $a_{\epsilon,R}(r) = \xi_{\epsilon,R}(G_R(t_\epsilon;t_\epsilon))^{-1}G_R(r;t_\epsilon)(1+o(1))$, *where $G_R(r;t_\epsilon)$ satisfies* (6.9), *$\xi_{\epsilon,R}$ is defined by the following*

$$\xi_{\epsilon,R} = \left(\epsilon\left(\int_{\mathbb{R}} w^m\right)G_R(t_\epsilon;t_\epsilon)\right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{6.15}$$

and $t_\epsilon \in (r_1, r_2)$ satisfies $\lim_{\epsilon\to 0} M_R(t_\epsilon) = 0$.

It remains to check condition (6.14), which can be verified numerically. Under some conditions on p, q, we can obtain the following corollary.

COROLLARY 6.2. *Assume that the following condition holds*:

$$\frac{(N-2)q}{N-1} + 1 < p < q+1. \tag{6.16}$$

Then there exists an $R_0 > 0$ such that for $R > R_0$ and ϵ sufficiently small, problem (6.2) *has* two *radially symmetric solutions $(a_{\epsilon,R}^i, h_{\epsilon,R}^i)$ concentrating on sphere $\{r = t_i\}$ with $M_R(t_i) = 0$, $i = 1, 2$, and $0 < t_1 < t_2 < R$, $i = 1, 2$.*

We remark that Corollary 6.2 is the first rigorous result on the existence to (6.2) of positive solutions in dimension $N \geqslant 3$. Next we consider the existence of bound states. That is, we consider the following elliptic system in $\mathbb{R}^N$:

$$\begin{cases} \epsilon^2 \Delta a - a + \frac{a^p}{h^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta h - h + \frac{a^m}{h^s} = 0 & \text{in } \mathbb{R}^N, \\ a, h > 0, \quad a, h \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{6.17}$$

We have the following result.

THEOREM 6.3. (See [62].) *Let $N \geqslant 2$. Assume that there exist two radii $0 < r_1 < r_2 < +\infty$ such that*

$$M(r_1)M(r_2) < 0. \tag{6.18}$$

Then for ϵ sufficiently small, problem (6.17) *has a solution (a_ϵ, h_ϵ) with the following properties*:

(1) *a_ϵ, h_ϵ are radially symmetric,*

(2) $a_\epsilon(r) = \xi_\epsilon^{\frac{q}{p-1}} w(\frac{r-r_\epsilon}{\epsilon})(1 + o(1))$,

(3) $h_\epsilon(r) = \xi_\epsilon (G(r_\epsilon; r_\epsilon))^{-1} G(r; r_\epsilon)(1 + o(1))$, *where ξ_ϵ is defined at the following*

$$\xi_\epsilon = \left(\epsilon \left(\int_{\mathbb{R}} w^m\right) G(r_\epsilon; r_\epsilon)\right)^{\frac{(1+s)(p-1)-qm}{qm}} \tag{6.19}$$

and $r_\epsilon \in (r_1, r_2)$ satisfying $\lim_{\epsilon \to 0} M(r_\epsilon) = 0$.

Similarly we have the following corollary.

COROLLARY 6.4. *Assume that $N \geqslant 2$ and that the condition* (6.16) *holds. Then for ϵ sufficiently small, problem* (6.2) *has a radially symmetric bound state solution (a_ϵ, h_ϵ) which concentrates on a sphere $\{r = t_0\}$ where $M(t_0) = 0$.*

By using the same method, it is not difficult to generalize the results of Theorem 6.1 to other symmetric domains, such as annulus or the exterior of a ball. We omit the details.

Several interesting questions are left open. First, can multiple layered solutions to (6.2) exist? Second, it would be an interesting question to study the stability of these "ring-like" solutions. Numerical computations in two dimension indicate that the "ring-like" solutions constructed in Theorem 6.1 are unstable and will break into several spots due to angular fluctuations. Third, if we vary R from 0 to $+\infty$, what is the relation between the layered solution constructed in [52] for the single equation (2.4) and the solutions in Theorem 6.1?

7. Conclusions and remarks

In this chapter, I have surveyed the most recent results on the study of Gierer–Meinhardt system.

First, we consider the case $D = +\infty$. In this case, the state-state problem becomes a singularly perturbed elliptic Neumann problem (2.4). Using the LEM, we established various existence results on concentrating solutions. In particular, Theorem 2.5 gives a lower bound on the number of solutions to (2.4). Several interesting questions are associated with (2.4). First, is there a lower bound on the number of boundary spikes? What is the optimal bound on the number of solutions to (2.4)? The followings are just some related conjectures

CONJECTURE 1. *Suppose the mean curvature function $H(P)$ has l local minimum points. Then there is at least*

$$\frac{C}{\epsilon^{l(N-1)}}$$

number of boundary spikes to (2.4).

CONJECTURE 2. *Suppose the distance function $d(P, \partial\Omega)$ has l local maximum points. Then there is at least*

$$\frac{C}{\epsilon^{Nl}}$$

number of interior spikes to (2.4).

CONJECTURE 3. *Suppose we have the energy bound $J_\epsilon[u_\epsilon] \leqslant C\epsilon^m$ for some $m \leqslant N$. Assume that the concentration set $\Gamma_\epsilon = \{u_\epsilon > \frac{1}{2}\}$ is connected. Then the limiting set $\Gamma = \lim_{\epsilon\to 0} \Gamma_\epsilon$ has Hausdorff dimension $N - m$.*

Second, we consider the stability of spike solutions to the shadow system (2.2). By studying both small and large eigenvalues, we have completely characterized the stability (or instability) in the case of $r = 2, 1 < p < 1 + \frac{4}{N}$ or $r = p + 1$. The study of the NLEP (3.52) is not complete yet. Many interesting questions are still open: the case of general r, the case of large τ, the uniqueness of Hopf bifurcation, etc. The nonlinear metastability of interior spike solutions is studied in [6]. The stability of boundary spikes is studied in [32], through a formal approach. It can be proved that when $D > D_0(\epsilon) \gg 1$, the full Gierer–Meinhardt system converges to the shadow system [59,60,76,77]. However, the critical threshold $D_0(\epsilon)$ seems unknown.

Third, we consider the one- and two-dimensional Gierer–Meinhardt systems. For steady states, we established the existence of *symmetric* and *asymmetric* K-peaked spikes. In 1D, the bifurcation of asymmetric K-spikes occur when $D < D_K$. In 2D, the bifurcation of asymmetric K-spikes occur when $D \sim \log\frac{1}{\epsilon}$. We also obtain critical thresholds for the stability of K-peaked solutions: If $\epsilon \ll 1$ there are stability thresholds

$$D_1(\epsilon) > D_2(\epsilon) > D_3(\epsilon) > \cdots > D_K(\epsilon) > \cdots$$

such that if

$$\lim_{\epsilon \to 0} \frac{D_K(\epsilon)}{D} > 1$$

then the K-peaked solution is stable, and if

$$\lim_{\epsilon \to 0} \frac{D_K(\epsilon)}{D} < 1$$

then the K-peaked solution is unstable. In 1D, the critical threshold is $D_K \sim \frac{1}{K^2}$. In 2D, the critical threshold is $\frac{\log \frac{\sqrt{|\Omega|}}{\epsilon}}{2\pi K}$. In 1D, the *small* eigenvalues determine the critical thresholds, while in 2D, the *large* eigenvalues give the critical thresholds. An interesting question is to obtain the next order term in the critical threshold for 2D (which should be $O(1)$ and location-dependent). The dynamics of multiple spikes in 1D and 2D is completely open. In 1D, the dynamical equation for the positions of the spikes is a system of algebraic-differential-equations (ADE). A matched asymptotic analysis is given in [33]. In 2D, the dynamics of two well-separated spots is studied in [20] and it is shown that the two spots will repel each other, provided that the initial distance between the two spots is large enough. In a general two-dimensional domain, the dynamics of multiple spots should be governed by $\nabla F_D(\mathbf{P})$ or $\nabla F_0(\mathbf{P})$.

Finally, it is almost completely open as regards to three-dimensional Gierer–Meinhardt system. The main difficulty is the study of the coupled system (6.1) which requires some new insights. A layered bound state is constructed, but most likely it is unstable. An interesting question is to generalize Theorem 6.1 to general domains.

Although the analysis in this chapter was carried out for the Gierer–Meinhardt system, the results can certainly be generalized to a much wide class of nonlocal reaction diffusion systems that have localized spike solutions.

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